THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF
THE MATHEMATICAL ASSOCIATION OF AMERICA
(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

VOLUME 61
1954

PUBLISHED BY THE ASSOCIATION
MENASHA, WIS., AND SEATTLE, WASH.

THE AMERICAN
MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 61 NUMBER 1

CONTENTS

JANUARY 1954

The AMERICAN MATHEMATICAL MONTHLY

(Founded in 1894 by Benjamin F. Finkel)

EDITORIAL CORRESPONDENCE should be addressed to the Editor, C. B. Allendoerfer, Department of Mathematics, University of Washington, Seattle 5, Washington.

ADVERTISING CORRESPONDENCE should be addressed to J. F. Randolph, Morey Hall, University of Rochester, Rochester 3, N. Y.

NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the Monthly should be sent to the Secretary-Treasurer, H. M. Gehman, University of Buffalo, Buffalo 14, N. Y.

THIS IS THE OFFICIAL JOURNAL OF THE

MATHEMATICAL ASSOCIATION OF AMERICA, INC.

(Devoted to the Interests of Collegiate Mathematics)

Entered as second class matter at the post office at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

Subscription Price: To Members, $4 a Year, To Others, $5 a Year.

Published by the Association at Menasha, Wisconsin, and Seattle, Wash. during the months of January, February, March, April, May, June–July, August–September, October, November, December.

PRINTED IN THE UNITED STATES OF AMERICA

ON THE CONDITION OF RIEMANN INTEGRABILITY

HANS RADEMACHER, University of Pennsylvania

Some time ago Professor M. J. Norris published in this journal a paper* in which he proved the Riemann integrability of continuous functions without taking recourse to the notion of uniform continuity. Of course, the proof requires somewhere a transition from local properties to overall properties, and the author used for this purpose the mean value theorem of differential calculus. Since this theorem is easier for the beginner to grasp than the notion of uniform continuity, Professor Norris' proof is of great didactic merit. It turns out, however, that exactly the same proof exists already in the literature. It was given by Gerhard Kowalewski on pp. 174–176 of his little book *Grundzüge der Differential- und Integralrechnung* (Leipzig 1909).

When I read the Kowalewski-Norris proof for the Riemann integrability of continuous functions again it occurred to me that it can easily be widened so that it gives the Jordan theorem of the necessary and sufficient condition for Riemann integrability, and in a rather simple way, avoiding the Heine-Borel theorem and the Lebesgue integral. This proof I shall give in the following lines.

1. I have first to recall the definition of one-sided upper and lower derivatives. Let $f(x)$ be given as a real function in $a \leqq x \leqq b$, then we define the right upper derivative

$$\overline{D}_r f(x) = \overline{\lim_{x_1 \to x+0}} \frac{f(x_1) - f(x)}{x_1 - x},$$

the right lower derivative

$$\underline{D}_r f(x) = \underline{\lim}_{x_1 \to x+0} \frac{f(x_1) - f(x)}{x_1 - x},$$

and the left upper and the left lower derivatives respectively as

$$\overline{D}_l f(x) = \overline{\lim_{x_1 \to x-0}} \frac{f(x_1) - f(x)}{x_1 - x},$$

$$\underline{D}_l f(x) = \underline{\lim}_{x_1 \to x-0} \frac{f(x_1) - f(x)}{x_1 - x},$$

where $x_1 \to x+0$ is the customary symbol for ($x_1 > x$ and $x_1 \to x$). At a only the right derivatives are defined, at b only the left ones. We have, of course,

$$\overline{D}_r f(x) \geqq \underline{D}_r f(x), \qquad \overline{D}_l f(x) \geqq \underline{D}_l f(x).$$

Otherwise our definition does not exclude that any of the four derivatives is $+\infty$ or $-\infty$. A function is called differentiable at x_0 in the usual sense, if all

* Integrability of continuous functions, this MONTHLY vol. 59, 1952, pp. 244–245.

derivatives at x_0 are equal and finite.

With this definition we state a generalization of the theorem of the mean, given first by W. H. and G. C. Young in 1909.† We start with the generalization of Rolle's theorem, which we enunciate as

LEMMA 1. *If $f(x)$ is a continuous function in $a \leqq x \leqq b$, and if $f(a) = f(b)$, then there exists either a ξ_1, $a < \xi_1 < b$ such that*

$$\underline{D}_l f(\xi_1) \geqq 0 \geqq \overline{D}_r f(\xi_1)$$

or there exists a ξ_2, $a < \xi_2 < b$, such that

$$\overline{D}_l f(\xi_2) \leqq 0 \leqq \underline{D}_r f(\xi_2).$$

Proof: If for all x, $a < x < b$, we have $f(x) = f(a) = f(b)$, we may take $\xi_1 = \frac{1}{2}(a+b)$ and have $a < \xi_1 < b$ and

$$\underline{D}_l f(\xi_1) = \overline{D}_r f(\xi_1) = f'(\xi_1) = 0.$$

Otherwise there will exist an x_0, $a < x_0 < b$, such that $f(x_0) \neq f(a)$. Let us assume $f(x_0) > f(a)$. Now $f(x)$ being continuous in a closed interval attains its maximum there, at ξ_1 say. Since $f(\xi_1) \geqq f(x_0) > f(a)$, $\xi_1 \neq a$ and $\xi_1 \neq b$, so that $a < \xi_1 < b$ follows. Now

$$\frac{f(x_1) - f(\xi_1)}{x_1 - \xi_1} \leqq 0 \qquad \text{for } x_1 > \xi_1$$

and thus

$$\overline{D}_r f(\xi_1) = \overline{\lim_{x_1 \to \xi_1 + 0}} \frac{f(x_1) - f(\xi_1)}{x_1 - \xi_1} \leqq 0,$$

and similarly

$$\underline{D}_l f(\xi_1) \geqq 0.$$

In the case $f(x_0) < f(a)$ we take the minimum $f(\xi_2)$ and obtain the other alternative.

If we now lift the condition $f(a) = f(b)$ and consider in the well-known way the function

$$f_1(x) = f(x) - \frac{f(b) - f(a)}{b - a} x,$$

we obtain from Lemma 1 the generalized mean value theorem which we state as

LEMMA 2. *If $f(x)$ is a continuous function in $a \leqq x \leqq b$ then there exists either a ξ_1, $a < \xi_1 < b$, such that*

---

† On derivatives and the theorem of the mean, Quarterly Journal of Math. vol. 40, 1909, pp. 1–26, esp. p. 10, Theorem 4.

$$\underline{D}_l f(\xi_1) \geqq \frac{f(b) - f(a)}{b - a} \geqq \overline{D}_r f(\xi_1),$$

or there exists a ξ_2, $a<\xi_2<b$, *such that*

$$\overline{D}_l f(\xi_2) \leqq \frac{f(b) - f(a)}{b - a} \leqq \underline{D}_r f(\xi_2).$$

Corollary. If for the continuous function $f(x)$ *the lower derivatives* $\underline{D}_r f(\xi)$ *and* $\underline{D}_l f(\xi)$ *in all interior points* ξ *have the property*

$$\underline{D}_r f(\xi) \leqq B, \qquad \underline{D}_l f(\xi) \leqq B$$

then

$$\frac{f(b) - f(a)}{b - a} \leqq B.$$

2. Whereas the two lemmas dealt with continuous functions only we consider now real functions which need only to be bounded. Let $g(x)$ be defined in the closed interval $[a, b]$. We introduce the following definitions

$$M_g(\alpha, \beta) = \sup_{\alpha \leqq x \leqq \beta} g(x), \tag{2.1}$$

$$m_g(\alpha, \beta) = \inf_{\alpha \leqq x \leqq \beta} g(x), \tag{2.2}$$

and

$$\sigma_g(\alpha, \beta) = M_g(\alpha, \beta) - m_g(\alpha, \beta), \tag{2.3}$$

the "supremum," the "infinum," and the "oscillation" of $g(x)$ in the closed interval $[\alpha, \beta]$. (It is to be understood that if $[\alpha, \beta]$ should partly exceed the interval $[a, b]$ in which $g(x)$ is defined, the instruction "$\alpha \leqq x \leqq \beta$" in the definitions is to be applied only where meaningful, *i.e.* in the common part of $[a, b]$ and $[\alpha, \beta]$.) Since we assumed boundedness of $g(x)$, the numbers defined in (2.1), (2.2), (2.3) are finite.

For $h>0$ the function of h

$$M_g(x_0 - h, x_0 + h)$$

is monotone non-decreasing since a supremum can only grow if the sample of which it is taken is enlarged. Similarly

$$m_g(x_0 - h, x_0 + h)$$

is monotone non-increasing, and

$$\sigma_g(x_0 - h, x_0 + h)$$

monotone non-decreasing as functions of $h>0$.

Since these functions are bounded, the limits

$$\lim_{h\to+0} M_g(x_0 - h, x_0 + h) = M_g(x_0),$$

$$\lim_{h\to+0} m_g(x_0 - h, x_0 + h) = m_g(x_0),$$

$$\lim_{h\to+0} \sigma_g(x_0 - h, x_0 + h) = \sigma_g(x_0),$$

the "local supremum," "local infinum," "local oscillation" of g at x_0 exist. We remark that a function is continuous at x_0 if its local oscillation at x_0 is 0.

LEMMA 3. *$M_g(x)$ and $\sigma_g(x)$ are upper semi-continuous and $m_g(x)$ is lower semi-continuous.*

Proof: Upper semi-continuity of $M_g(x)$ at x_0 means:

$$\overline{\lim_{x_1\to x_0}}\ M_g(x_1) \leqq M_g(x_0).$$

Let $|x_1-x_0|<\delta$, and $0<h<\delta-|x_1-x_0|$. Then the interval $[x_1-h, x_1+h]$ lies in the interior of the interval $[x_0-\delta, x_0+\delta]$, and therefore

$$M_g(x_1) \leqq M_g(x_1 - h, x_1 + h) \leqq M_g(x_0 - \delta, x_0 + \delta);$$

thus

$$\overline{\lim_{x_1\to x_0}}\ M_g(x_1) \leqq M_g(x_0 - \delta, x_0 + \delta).$$

This is true for any $\delta>0$, and thus, for $\delta\to+0$

$$\overline{\lim_{x_1\to x_0}}\ M_g(x_1) \leqq M_g(x_0),$$

as we had to prove. The other statements of the lemma follow in a similar manner.

3. We come now to the main object of our discussion.

THEOREM 1. *A necessary condition for the Riemann integrability of the function $f(x)$ bounded in the interval $a\leqq x\leqq b$ is that, for every $\eta>0$, the set S_η of all x for which $\sigma_f(x)\geqq\eta$ can be covered by finitely many intervals of total length at most η.*

Remark: If the condition of this theorem is fulfilled, then for a *fixed* $\eta>0$ the set S_η can be covered by finitely many intervals of any total length $\epsilon<\eta$. Indeed, if $0<\epsilon<\eta$ then by definition

$$S_\epsilon \supset S_\eta,$$

and thus since S_ϵ can be covered by a finite set of intervals of total length ϵ at most, S_η is in this way also covered.

Proof of THEOREM 1: Suppose the condition is not fulfilled. This would mean that there exists an exceptional $\eta_0>0$ such that any finite interval set covering S_{η_0} has a total length $\geqq\eta_0$. It is convenient to distinguish here two cases:

1) Suppose that S_{η_0} contains a whole interval I of length λ. We take now a partition $a=x_0<x_1<\cdots<x_{n-1}<x_n=b$ of the interval $[a, b]$ and form the difference of the upper and lower Darboux sums for that partition:

$$\begin{aligned}\overline{S}-\underline{S} &= \sum_j M_f(x_{j-1}, x_j)(x_j - x_{j-1}) - \sum_j m_f(x_{j-1}, x_j)(x_j - x_{j-1}) \\ &= \sum_j \sigma_f(x_{j-1}, x_j)(x_j - x_{j-1}).\end{aligned}$$

Of this sum we consider only those summands which belong to intervals $[x_{j-1}, x_j]$ which have a point of I in the interior. Since such a point, say ξ, belongs to S_{η_0}, we can say for the interval $[x_{j-1}, x_j]$ surrounding it that

$$\sigma_f(x_{j-1}, x_f) \geqq \sigma_f(\xi) \geqq \eta_0.$$

On the other hand these intervals together cover I and therefore have a total length λ at least. Thus

$$\overline{S}-\underline{S} \geqq \eta_0\lambda$$

for Darboux sums belonging to any partition, hence the upper and lower Darboux integrals differ by at least $\eta_0\lambda$, and $f(x)$ cannot be integrable.

2) Suppose that S_{η_0} contains no interval, *i.e.*, does not possess an interior point. Assume that we have a partition of (a, b) of fineness $\frac{1}{2}\delta$, *i.e.*,

$$x_j - x_{j-1} \leqq \frac{\delta}{2}\cdot$$

We can see to it that no point of the partition (except possibly a and b) belongs to S_{η_0}. If x_j should belong to S_{η_0}, we can find in a distance less than $\frac{1}{4}\delta$ another point x_j' not belonging to S_{η_0}. The so corrected partition $(a<x_1'<\cdots<x_{n-1}'<b)$ has still the fineness δ. Then each point of S_{η_0} (except possibly a and b) is contained in the *interior* of an interval of the partition. We consider only such intervals $[x_{j-1}', x_j']$ which have a point ξ of S_{η_0} in their interior. For these we have again

$$\sigma_f(x_{j-1}', x_j') \geqq \sigma_f(\xi) \geqq \eta_0.$$

Since these intervals must cover S_{η_0} their total length adds up at least to η_0, and we have

$$\overline{S}-\underline{S} = \sum_j \sigma(x_{j-1}', x_j')(x_j' - x_{j-1}') \geqq \eta_0^2$$

which again excludes Riemann integrability.

4. We come now to that part of our argument which resembles that one given by Kowalewski and Norris, and which leads to the converse of Theorem I.

THEOREM II. *A sufficient condition for Riemann integrability in $a\leqq x\leqq b$ of*

the bounded function $f(x)$ is that for every $\eta>0$ the S_η, defined as in THEOREM I, *can be covered by finitely many intervals of total length η at most.*

Proof: We consider the function

$$F(z) = \overline{\int_a^z} f(x)dx - \underline{\int_a^z} f(x)dx,$$

which certainly fulfills

$$F(z) \geqq 0.$$

We assume $|f(x)| \leqq M$, and have then as consequence of the well-known additivity of Darboux integrals for $h>0$

$$\begin{aligned} 0 \leqq F(z+h) - F(z) &= \overline{\int_a^{z+h}} f(x)dx - \overline{\int_a^z} f(x)dx - \underline{\int_a^{z+h}} f(x)dx + \underline{\int_a^z} f(x)dx \\ &= \overline{\int_z^{z+h}} f(x)dx - \underline{\int_z^{z+h}} f(x)dx \\ &\leqq Mh + Mh = 2Mh, \end{aligned}$$

and similarly for $h<0$

$$|F(z+h) - F(z)| \leqq 2M|h|,$$

which shows the continuity of $F(z)$.

As far as the derivatives are concerned we see first, for $h>0$,

$$\begin{aligned} \frac{F(z+h) - F(z)}{h} &= \frac{1}{h}\left\{\overline{\int_z^{z+h}} f(x)dx - \underline{\int_z^{z+h}} f(x)dx\right\} \\ &\leqq M_f(z, z+h) - m_f(z, z+h) = \sigma_f(z, z+h) \\ &\leqq \sigma_f(z-h, z+h) \end{aligned}$$

and therefore

$$\begin{aligned} \overline{D}_r F(z) &= \overline{\lim_{h\to+0}} \frac{1}{h}\{F(z+h) - F(z)\} \\ &\leqq \lim_{h\to+0} \sigma_f(z-h, z+h) = \sigma_f(z). \end{aligned}$$

In the same manner

$$\overline{D}_l F(z) \leqq \sigma_f(z)$$

follows. Altogether we have

$$\underline{D}_l F(z) \leqq \overline{D}_l F(z) \leqq \sigma_f(z), \tag{4.1}$$

$$\underline{D}_r F(z) \leqq \overline{D}_r F(z) \leqq \sigma_f(z). \tag{4.2}$$

It will be convenient to introduce a function of an interval $I=(\alpha, \beta)$, for $\alpha<\beta$, as follows:

$$F(I) = F(\alpha, \beta) = F(\beta) - F(\alpha).$$

Now let S_η be covered by finitely many intervals $I_j=(\alpha_j, \beta_j)$ of total length η at most. The complementary intervals we call $\overline{I}_j'$. We have then

$$F(b) = F(b) - F(a) = \sum_j F(I_j) + \sum_j F(I_j'). \tag{4.3}$$

In $\overline{I}_j'$ we have throughout

$$\sigma_f(x) < \eta$$

whereas in general $|f(x)| \leqq M$ entails

$$\sigma_f(x) \leqq 2M.$$

The inequalities (4.1) and (4.2) together with the Corollary show therefore

$$F(I_j) \leqq 2M \cdot l(I_j)$$
$$F(I_j') \leqq \eta \cdot l(I_j')$$

where $l(I)$ stands for the length of I. We infer therefore from (4.3)

$$\begin{aligned} F(b) &\leqq 2M \sum_j l(I_j) + \eta \sum_j l(I_j') \\ &\leqq 2M\eta + \eta(b-a) = \eta(2M + b - a). \end{aligned}$$

Since η is any positive number and $F(b) \geqq 0$ we obtain now

$$F(b) = 0,$$

or explicitly,

$$\overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x)dx = 0,$$

which proves our theorem.

5. So far we have remained in the realm of the elementary theory of Riemann-Darboux integrals. But we can now easily connect our two theorems with the Lebesgue theory and prove

THEOREM III. *Necessary and sufficient condition for the Riemann integrability of the bounded function $f(x)$ is that the set S_0 of those x where $\sigma_f(x) > 0$ is of Lebesgue measure zero.*

Proof: It is clear firstly that

$$(5.1) \qquad S_0 = S_1 \cup S_{1/2} \cup S_{1/3} \cup \cdots \cup S_{1/n} \cup \cdots .$$

Now the *necessary* condition of Theorem I implies, in virtue of the "Remark" following it, that all S_η are of Lebesgue measure 0, and then (5.1) shows that

$$(5.2) \qquad m_L S_0 = 0$$

(m_L meaning Lebesgue measure).

Assume now conversely that is true. Then, since

$$S_\eta \subset S_0$$

we have also

$$(5.3) \qquad m_L S_\eta = 0$$

for any $\eta > 0$. This means that S_η can be covered by *denumerably* many intervals of at most, let us say, total length η.

But now, as we shall see in a moment, S_η is closed, and can therefore, in virtue of the Heine-Borel theorem, be covered with only *finitely* many among those infinitely many covering intervals, and thus (5.3) implies that S_η can be covered with *finitely* many intervals of total length η at most, which shows, according to Theorem II, that (5.3) is also *sufficient* for the Riemann integrability of $f(x)$.

The closure of S_η, comprising those points x in which $\sigma_f(x) \geqq \eta$, follows immediately from Lemma 3.

Of course, the arguments of section 5, employing the Heine-Borel theorem, which is essentially equivalent to the theorem of uniform continuity of a continuous function in a closed interval, overstep the limits which this note had set itself. Its main purpose remains the proof given for Theorem II.

THE EQUATION $X^2+PX+Q=0$ IN BINARY MATRICES

H. S. THURSTON and MARY K. ALEXANDER, University of Alabama

1. Introduction. In a series of papers, [**3**], [**4**], [**5**], published in 1884 Sylvester discussed the equation

$$(1) \qquad f(X) = X^2 + PX + Q = 0$$

where P and Q are either binary matrices or quaternions. His technique, presented here in modified form for the matric case, leads to a cubic equation $\phi(\lambda) = 0$ whose roots determine, in general, six solutions of (1). If, however, $\phi(0) = 0$, fewer solutions may exist. Sylvester called this the *irregular* case. While he treated the regular case in considerable detail, he disposed of the

irregular case with the remark that "some of the roots may disappear from the sphere of actuality, or may remain actual but become indeterminate, or these two states of affairs may coexist." The purpose of our paper is to throw some light on this seemingly chaotic situation.

2. The regular case. Any solution $X=X_1$ of (1) satisfies its own characteristic equation

$$x^2+bx+d=0. \tag{2}$$

Thus $X_1^2+BX_1+D=0$, where $B=bI$ and $D=dI$, and subtracting this from (1) we have

$$(P-B)X_1+(Q-D)=0.$$

Conversely, if $X=X_1$ with characteristic equation (2) is a solution of

$$(P-B)X+(Q-D)=0, \tag{3}$$

it is also a solution of (1). The problem of solving (1) is, in effect, that of finding all pairs (b, d) which can exist for a potential solution of (3). If for any such pair $P-B$ is non-singular, then (3) has the unique solution $X=-(P-B)^{-1}(Q-D)$. On the other hand, if $P-B$ is singular, there may or may not exist a solution of (3) with characteristic equation (2). It will be shown that the singularity of $P-B$ is concomitant with the existence of a root $\lambda=0$ of $\phi(\lambda)=0$.

Since any scalar solution of (1) can be found by inspection we shall assume the existence of a non-scalar solution with characteristic equation (2). If

$$P=\begin{bmatrix}\alpha & \beta\\ \gamma & \delta\end{bmatrix}$$

and

$$Q=\begin{bmatrix}\sigma & \xi\\ \eta & \zeta\end{bmatrix},$$

(3) can be written in the form

$$\begin{bmatrix}\alpha-b & \beta\\ \gamma & \delta-b\end{bmatrix}X+\begin{bmatrix}\sigma-d & \xi\\ \eta & \zeta-d\end{bmatrix}=0. \tag{3'}$$

By a well-known theorem [2], any solution of (3′) will satisfy

$$\begin{vmatrix}(\alpha-b)x+\sigma-d & \beta x+\xi\\ \gamma x+\eta & (\delta-b)x+\zeta-d\end{vmatrix}=0. \tag{3''}$$

This takes the form $px^2+qx+r=0$, where

$$\begin{gathered}p=b^2-2\tau_1 b+\Delta_1, \qquad q=2bd-2\tau_2 b-2\tau_1 d+\alpha\zeta+\delta\sigma-\beta\eta-\gamma\xi,\\ r=d^2-2\tau_2 d+\Delta_2,\end{gathered} \tag{4}$$

$2\tau_1$ and $2\tau_2$ being the traces of P and Q, Δ_1 and Δ_2 their determinants, respectively. The assumed solution of (1) being non-scalar, the equations $px^2+qx+r=0$ and $x^2+bx+d=0$ must be equivalent unless $p=q=r=0$, and corresponding coefficients are proportional. Even if p, q, and r are all zero we may still write $p=\lambda$, $q=b\lambda$, $r=d\lambda$. On solving the first and third of these we find respectively

$$b = \tau_1 \pm b_1, \qquad d = \tfrac{1}{2}(\lambda + 2\tau_2 \pm d_1) \tag{5}$$

where b_1 and d_1 are irrational. By observing the correspondence of sign (+ with + and − with −), on substituting these in $q=b\lambda$, the latter reduces to

$$b_1 d_1 = \tau_1\lambda + 2\tau_1\tau_2 - (\alpha\zeta + \delta\sigma - \beta\eta - \gamma\xi)$$

which is rationalized on squaring. We are thus led to a cubic equation $\phi(\lambda)=\lambda^3+u\lambda^2+v\lambda+w=0$, where

$$\begin{aligned} u &= 4\tau_2 - \Delta_1, \qquad v = 4\tau_2^2 - 4\Delta_2 - 4\Delta_1\tau_2 + 2\tau_1(\alpha\zeta + \delta\sigma - \beta\eta - \gamma\xi), \\ w &= \eta\xi(\alpha - \delta)^2 + \beta\gamma(\sigma - \zeta)^2 - (\beta\eta - \gamma\xi)^2 - (\alpha - \delta)(\sigma - \zeta)(\beta\eta + \gamma\xi). \end{aligned} \tag{6}$$

THEOREM 1. *The determinants of $P-B$ and $Q-D$ are respectively $p=\lambda$ and $r=d\lambda$.*

This is obvious from (3′) and (4).

THEOREM 2. *If $\lambda=0$ is not a root of $\phi(\lambda)=0$ then* (1) *has at most six non-scalar solutions.*

It is clear from (5) that corresponding to a root $\lambda_1\neq 0$ there are at most two pairs (b, d). Since for any such pair $P-B$ is non-singular, by Theorem 1, then (3) and hence (1) have a unique solution. If $\phi(\lambda)=0$ has no multiple roots, the six pairs (b, d) are distinct,* and if in addition there is no root $\lambda=0$, six solutions of (1) are obtained.

Example 1. If

$$f(X) = X^2 + \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} X + \begin{bmatrix} -7 & 1 \\ 0 & 0 \end{bmatrix} = 0,$$

$\phi(\lambda)$ is found to be $(\lambda-1)(\lambda-5)(\lambda-10)$. Corresponding to each triple (λ, b, d) we have a solution as follows:

$$(1, 1, -6), \quad X = \begin{bmatrix} 0 & 6 \\ 1 & -1 \end{bmatrix}; \qquad (1, 1, 0), \quad X = \begin{bmatrix} 0 & 0 \\ 7 & -1 \end{bmatrix};$$

$$(5, 3, 0), \quad X = 1/5\begin{bmatrix} -14 & 2 \\ 7 & -1 \end{bmatrix}; \quad (5, -1, -2) \quad X = \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix};$$

$$(10, 4, 3), \quad X = \begin{bmatrix} -3 & 0 \\ 1 & -1 \end{bmatrix}; \qquad (10, -2, 0), \quad X = 1/10\begin{bmatrix} 21 & -3 \\ 7 & -1 \end{bmatrix}.$$

* From the fact, noted by Sylvester, that $\phi(\lambda)=0$ and $F(x)=0$ (defined in section 6) have the same discriminant.

3. The irregular case. If (b, d) is a pair corresponding to a root $\lambda=0$ we may assume a solution

$$X = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$$

of (3). Then (3) becomes

$$\begin{bmatrix} \alpha - b & \beta \\ \gamma & \delta - b \end{bmatrix} \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} + \begin{bmatrix} \sigma - d & \xi \\ \eta & \zeta - d \end{bmatrix} = 0.$$

The solution X can be found if and only if the following system of six equations is consistent, in which case (b, d) is called an *admissible* pair.

$$\begin{aligned} (\alpha - b)x_1 + \beta x_3 + \sigma - d &= 0 \\ (\alpha - b)x_2 + \beta x_4 + \xi &= 0 \\ \gamma x_1 + (\delta - b)x_3 + \eta &= 0 \\ \gamma x_2 + (\delta - b)x_4 + \zeta - d &= 0 \\ x_1 + x_4 = -b, \qquad x_1x_4 - x_2x_3 &= d. \end{aligned} \tag{7}$$

The first four equations are consistent if and only if the matrix

$$N = \begin{bmatrix} \alpha - b & \beta & \sigma - d & \xi \\ \gamma & \delta - b & \eta & \zeta - d \end{bmatrix}$$

is of rank $\rho < 2$.

If $\rho = 0$, $P-B=Q-D=0$, and any matrix

$$X = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$$

for which the last two equations of (7) are satisfied will be a two-parameter solution of (3) and hence of (1). This case arises when P and Q are both scalar matrices.

If, however, $\rho = 1$, the system of four equations will reduce to two equations which with $x_1+x_4=-b$ may be solved for three of the x_i's (either uniquely or with the remaining x_i as a parameter) and the results substituted in the sixth equation as a further test of consistency. Which x_i may serve as parameter depends on which elements of $P-B$ are not zero. In any case it may be chosen as x_2 or x_3, the result of the substitution in the sixth equation of (7) being indicated as follows:

$$\begin{aligned} &\alpha - b \neq 0: && [\xi(\alpha - b) - \beta(\sigma - d)]x_3 = (\alpha - b)^2 d - b(\alpha - b)(\sigma - d) + (\sigma - d)^2 \\ &\beta \neq 0: && [-\xi(\alpha - b) + \beta(\sigma - d)]x_2 = \beta^2 d - b\beta\xi + \xi^2 \\ &\gamma \neq 0: && [-\eta(\delta - b) + \gamma(\zeta - d)]x_3 = \gamma^2 d - b\gamma\eta + \eta^2 \\ &\delta - b \neq 0: && [\eta(\delta - b) - \gamma(\zeta - d)]x_2 = (\delta - b)^2 d - b(\delta - b)(\zeta - d) + (\zeta - d)^2. \end{aligned}$$

If no element of $P-B$ is zero, any x_i may serve as parameter. For example, either x_1 or x_4 may be chosen, substitution in (7) yielding

$$(8) \qquad (\beta\eta - \xi\gamma)x_1 = \beta\gamma d - b\beta\eta + \xi\eta,$$

or

$$(8') \qquad (\xi\gamma - \beta\eta)x_4 = \beta\gamma d - b\gamma\xi + \xi\eta.$$

In any of these cases if the coefficient of the x_i finally exhibited is not zero, we may solve uniquely for this unknown and a single solution of (3) is found. But if the coefficient is zero, there is a solution involving a parameter x_i, or no solution, according as the right member of the corresponding equation is or is not zero. We may therefore state the following theorem.

THEOREM 3. *If $\phi(\lambda)=0$ has a root $\lambda=0$, there may be fewer than six admissible pairs (b, d). To each admissible pair associated with $\lambda=0$ may correspond either a unique solution or a parametric solution of* (1).

On referring to (6) it is seen that, by definition, the vanishing of w is a necessary and sufficient condition for the occurrence of the irregular case. The form in which w is expressed suggests various sufficient conditions. For example, if P or Q is scalar, or if $\eta=\gamma=0$, w will vanish. If $\beta\eta=\gamma\xi$, w will reduce to $\eta[\xi(\alpha-\delta)-\beta(\sigma-\zeta)]^2$, so that $\xi(\alpha-\delta)=\beta(\sigma-\zeta)$ together with $\beta\eta=\gamma\xi$ is sufficient for the occurrence of the irregular case. These conditions hold if P is a linear function of Q, say $P=\mu Q+\nu$, since then $\beta=\mu\xi$, $\gamma=\mu\eta$, $\alpha-\delta=\mu(\sigma-\zeta)$. Thus we have

THEOREM 4. *A sufficient condition that equation* (1) *be irregular is that P be a linear function of Q.*

This case presents some aspects which were not apparent in the general discussion of this section and will now be investigated in detail.

4. The matric equation $f(X, Q)=0$. We shall now consider the equation

$$(9) \qquad f(X, Q) = X^2 + (\mu Q + \nu)X + Q = 0$$

where μ and ν are scalars, $\mu\neq 0$, and Q is a non-scalar matrix. With the trace of Q being denoted by 2τ and the determinant of Q by Δ, $\phi(\lambda)$ is of the form $\lambda^3+k\lambda^2+4mn\lambda$ where

$$(10) \qquad k = 4\tau - \mu^2\Delta - 2\mu\nu\tau - \nu^2, \qquad m = 1 - \mu\nu, \qquad n = \tau^2 - \Delta.$$

For $\lambda=0$ the coefficients b and d given by (5) become $b=\mu d+\nu$, $d=\tau\pm\theta$, where $\theta^2=n$. Equation (3′) takes the form

$$(11) \qquad \mu(Q - D)X + Q - D = 0.$$

Thus the rank of N is clearly less than two, by Theorem 1 and (11), and the first four equations of the system (7) are consistent. The final substitution in

the sixth equation of the system leads to a consistent result if and only if $m=0$. For example, in (8), since $\beta\eta-\gamma\xi=0$, the consistency depends upon the vanishing of the right member. But

$$\begin{aligned}\beta\gamma d - \beta\eta b + \xi\eta &= \mu\xi[\mu\eta d - \eta(\mu d + \nu)] + \xi\eta\\ &= -\mu\nu\xi\eta + \xi\eta\\ &= \xi\eta(1-\mu\nu)\\ &= 0\end{aligned}$$

if and only if $m=0$, ξ and η being in this case different from zero. The same condition can be shown to be necessary and sufficient for a solution in each of the cases that can arise. Thus we have proved

THEOREM 5. *A necessary and sufficient condition that there be an admissible pair* (b, d) *associated with* $\lambda=0$, *and hence a corresponding solution of* (9), *is that* $m=0$.

As pointed out earlier, there will in this case be a parametric solution, the choice of parameter being determined by the non-zero elements of $P-B$. In any case x_2 or x_3 may be chosen as parameter to avoid zero denominators. However, we shall take the case where there are no zero elements as typical, and since the conclusions are the same for all cases, no further reference will be made to the others. We can now express x_2, x_3, and x_4 as follows:

$$x_2 = \frac{-\xi(x_1+\mu d)}{d-\sigma}, \qquad x_3 = \frac{\eta(x_1+\nu)}{d-\zeta}, \qquad x_4 = -(x_1+\nu+\mu d),$$

where neither of the denominators is zero. Since $d=\tau\pm\theta$ is double-valued (unless $n=0$), there are two parametric solutions in x_1 corresponding to a pair (b, d). If $m=n=0$, only one parametric solution is obtained. We have proved

THEOREM 6. *When* $m=0$, *corresponding to* $\lambda=0$ *there are two parametric solutions if* $n\neq 0$, *one such solution if* $n=0$.

If $k=m=n=0$, then $\mu\tau=\nu$, and the single parametric solution is

$$X = \begin{bmatrix} x_1 & \dfrac{\xi(x_1+\nu)}{\sigma-\tau} \\ \dfrac{\eta(x_1+\nu)}{\tau-\zeta} & -(x_1+2\nu)\end{bmatrix}. \tag{12}$$

When $m=0$ it is clear that

$$f(X, Q) = X^2 + (\mu Q + \nu)X + \mu\nu Q = (X+\nu)(X+\mu Q)$$

and $X=-\nu I$, $X=-\mu Q$ are solutions of (9). In case $k\neq 0$ these solutions correspond to $\lambda=-k$. However, if $k=0$, they become particular solutions with the

parameter x_1 taking on the values $-\nu$ and $-\mu\sigma$ respectively. This is obvious from (12) when $n=0$ and easily verified when $n\neq 0$.

Example 2. Let $f(X, Q)=X^2+(Q+1)X+Q=0$ where

$$Q=\begin{bmatrix} 3 & -1 \\ 0 & 2 \end{bmatrix}.$$

Here $m=0$, $k\neq 0$, $n\neq 0$ and $\phi(\lambda)=\lambda^2(\lambda-2)$. Corresponding to $\lambda=2$ are the two solutions $X=-I$, $X=-Q$, while $\lambda=0$ yields the parametric solutions

$$X=\begin{bmatrix} -3 & x_2 \\ 0 & -1 \end{bmatrix}, \qquad X=\begin{bmatrix} x_1 & -x_1-2 \\ x_1+1 & -x_1-3 \end{bmatrix}.$$

Example 3. $f(X, Q)=X^2+(\frac{1}{2}Q+2I)X+Q=0$ where

$$Q=\begin{bmatrix} 6 & 2 \\ -2 & 2 \end{bmatrix}.$$

We now have $m=n=k=0$ and $\phi(\lambda)=\lambda^3$. One parametric solution

$$X=\begin{bmatrix} x_1 & x_1+2 \\ -x_1-2 & -x_1-4 \end{bmatrix}$$

is obtained yielding the particular solutions $X=-2I$, $X=-\frac{1}{2}Q$ when $x_1=-2$ and -3 respectively.

5. Other special cases.

THEOREM 7. *If P is scalar and Q non-scalar there is no solution of* (1) *corresponding to $\lambda=0$; if Q is scalar and P non-scalar there is a solution corresponding to $\lambda=0$ if and only if $Q=0$.*

Under the first hypothesis $P-B=0$, $Q-D\neq 0$, and obviously there is no solution of (3). This is the case $P=\mu Q+\nu$, where $\mu=0$.

If $Q=\sigma I$ and P is non-scalar, the six equations (7) are found to be consistent if and only if $\sigma=0$, in which case parametric solutions are obtained.

The final special case to consider is that in which Q is scalar and $P=\mu Q+\nu$ is also scalar. If $Q=\sigma I$, Sylvester's method leads to $\phi(\lambda)=\lambda^2(\lambda+k)$ where $-k$ is the discriminant of $f(x, \sigma)=0$. To the root $\lambda=-k$ there correspond the scalar solutions $X_1=r_1 I$, $X_2=r_2 I$ where r_1 and r_2 are the roots of $f(x, \sigma)=0$. As noted earlier, for $\lambda=0$ the rank of N is zero and there is a two-parameter solution subject to the conditions $x_1+x_4=-\mu\sigma-\nu$, $x_1x_4-x_2x_3=\sigma$. It is known that a polynomial equation with scalar coefficients has an infinite number of matric solutions of any desired order. Hermann [1] and Thurston [6], [7], have discussed the number of solutions in the ring $R(A)$ of any given matrix A. As in the following example, any such binary solution may be identified with one of the abovementioned scalar or parametric solutions.

Example 4. Consider the equation $X^2+3IX+2I=0$. If

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix},$$

then solutions in $R(A)$ are given by $X=sA+t$ where $s=0$, $t=-1$; $s=0$, $t=-2$; $s=-1/3$, $t=-4/3$; $s=1/3$, $t=-5/3$. These yield respectively the solutions

$$X_1 = -I,\quad X_2 = -2I,\quad X_3 = -1/3\begin{bmatrix} 5 & 2 \\ 1 & 4 \end{bmatrix},\quad X_4 = -1/3\begin{bmatrix} 4 & -2 \\ -1 & 5 \end{bmatrix}.$$

The last two are easily identified as particular cases of the parametric solutions indicated above.

6. The associated quartic equation. The theorem [2] which permitted passing from (3′) to (3″) may be applied to (1) yielding the equation

$$F(x) = \begin{vmatrix} x^2+\alpha x+\sigma & \beta x+\xi \\ \gamma x+\eta & x^2+\delta x+\zeta \end{vmatrix} = 0,$$

or

$$\begin{aligned} F(x) &= x^4 + 2\tau_1 x^3 + (2\tau_2+\Delta_1)x^2 \\ &\quad + (\sigma\delta+\alpha\zeta-\beta\eta-\gamma\xi)x + \Delta_2 = 0, \end{aligned} \tag{13}$$

and any solution of (1) satisfies (13).

THEOREM 8. *If $X=X_1$ is a solution of* (1) *with characteristic equation $g(x) =x^2+bx+d=0$, then $g(x)$ divides $F(x)$.*

Since $f(X_1)=X_1^2+PX_1+Q=0$, we may write

$$f(xI) = f(xI) - f(X_1) = x^2I^2 - X_1^2 + P(xI - X_1) = (xI + X_1 + P)(xI - X_1).$$

Then the determinant of $xI-X_1$ divides that of $f(xI)$. But the determinant of $xI-X_1$ is $g(x)$ and that of $f(xI)$ is $F(x)$.*

This theorem suggests an alternate method of finding pairs (b, d) for potential solutions† of (1). In the regular case the six possible quadratic factors of $F(x)$ are the characteristic functions of the six solutions mentioned in the introduction.

If $P=\mu Q+\nu$, the associated quartic may be written in the factored form $F(x)=F_1(x)F_2(x)$ where

$$\begin{aligned} F_1(x) &= x^2 + (\mu\tau+\nu+\mu\theta)x + (\tau+\theta), \\ F_2(x) &= x^2 + (\mu\tau+\nu-\mu\theta)x + (\tau-\theta). \end{aligned}$$

* This proof is essentially that given by Sylvester [5].

† This theorem also permits the omission of the word "non-scalar" from the enunciation of Theorem 2 since the maximum number of quadratic factors of $F(x)$ is six.

If $m=0$, $\mu\nu=1$, and we may write

$$F_1(x) = x^2 + (\mu\tau + \nu + \mu\theta)x + \mu\nu(\tau + \theta) = (x + \nu)(x + \mu\tau + \mu\theta).$$

Similarly,

$$F_2(x) = (x + \nu)(x + \mu\tau - \mu\theta),$$

so that

(14) $$F(x) = (x + \nu)^2[(x + \mu\tau)^2 - (\mu\theta)^2] = (x + \nu)^2(x^2 + 2\mu\tau x + \mu^2\Delta).$$

If, in addition, $k=0$, then

$$k = 4\tau - \mu^2\Delta - 2\mu\nu\tau - \nu^2 = 2\tau - \mu^2\Delta - \nu^2 = 0,$$

whence

$$2\mu\tau = \mu^3\Delta + \mu\nu\cdot\nu = \mu^3\Delta + \nu.$$

Then

$$x^2 + 2\mu\tau x + \mu^2\Delta = x^2 + (\mu^3\Delta + \nu)x + \mu^3\nu\Delta = (x + \nu)(x + \mu^3\Delta).$$

Thus for $m=k=0$,

(15) $$F(x) = (x + \nu)^3(x + \mu^3\Delta).$$

On the other hand if $m=0$ and $n=\theta^2=0$, we will have from (14),

$$\begin{aligned} F(x) &= (x + \nu)^2(x + \mu\tau)^2 \\ &= [x^2 + (\mu\tau + \nu)x + \mu\nu\tau]^2 \\ &= [x^2 + (\mu\tau + \nu)x + \tau]^2 \end{aligned}$$

(16) $$= [f(x, \tau)]^2.$$

Finally, if $m=n=k=0$, since $\mu\tau=\nu$, $\tau=\nu^2$, we have

(17) $$F(x) = (x + \nu)^4.$$

The following table provides a complete summary of the results of this section as well as those of section 4. It is understood that $m=0$ in all four cases tabulated and the solutions indicated are only those corresponding to $\lambda=0$.

Cases	$\phi(\lambda)$	$F(x)$	Solutions
$k=0$, $n=0$	λ^3	$(x+\nu)^4$	one parametric
$k=0$, $n\neq0$	λ^3	$(x+\nu)^3(x+\mu^3\Delta)$	two parametric
$k\neq0$, $n=0$	$\lambda^2(\lambda+k)$	$[f(x, \tau)]^2$	one parametric
$k\neq0$, $n\neq0$	$\lambda^2(\lambda+k)$	$(x+\nu)^2(x^2+2\mu\tau+\mu^2\Delta)$	two parametric

7. Conclusion. We would not wish the readers of this paper to draw from the discussions of sections 4 and 6 the erroneous conclusion that all solutions corresponding to $\lambda=0$ are parametric. Neither would we wish them to conclude from the examples of these sections that there are solutions corresponding to $\lambda=0$ only when the latter is a multiple root of $\phi(\lambda)=0$. This is equally erroneous, as our final example demonstrates.

Example 5. If

$$f(X) = X^2 + \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} X + \begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix},$$

we find that $\phi(\lambda)=\lambda^3$. Corresponding to $\lambda=0$ there is an inadmissible pair $b=2$, $d=1$, and an admissible pair $b=0$, $d=-1$, yielding a solution

$$X = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}.$$

Example 6. Let

$$f(X) = X^2 + \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix} X + \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}.$$

The root $\lambda=0$ of $\phi(\lambda)=\lambda(\lambda-2)^2=0$ leads to $b=2$, $d=1$, yielding no solution, and $b=-2$, $d=-1$, with the corresponding solution

$$X = \begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix}.$$

In conclusion, the authors wish to express their appreciation for the suggestions and constructive criticisms of the referee in the preparation of this paper.

References

1. Hermann, A., Über Matrixgleichungen und die Zerlegung von Polynomen in Linearfactoren, Compositio Mathematica, 1934, vol. 1, pp. 285–302.

2. MacDuffee, C. C., Theory of Matrices, Springer, 1933, p. 17.

3. Sylvester, J. J., On Hamilton's quadratic equation and the general unilateral equation in matrices, Philosophical Magazine, 1884, vol. 18, p. 454–458.

4. Sylvester, J. J., Sur la Solution Explicite de l'équation Quadratique de Hamilton en Quaternions ou en Matrices du Second Ordre, Comptes Rendus, 1884, vol. 99, pp. 555–558.

5. Sylvester, J. J. Sur les Conditions de l'existence de Racines Égales dans l'équation du Second Degre de Hamilton, etc., Comptes Rendus, 1884, vol. 99, pp. 621–631.

6. Thurston, H. S., On factoring a matric polynomial with scalar coefficients, Compositio Mathematica, 1938, vol. 6, pp. 235–238.

7. Thurston, H. S., On the number of sets conjugate to a matrix with linear elementary divisors, Bull. Amer. Math. Society, 1939, vol. 45, pp. 474–476.

A NOTE ON PRIMITIVE MATRICES†

I. N. HERSTEIN, Cowles Commission and the University of Chicago

Suppose that A is a square matrix consisting of non-negative elements. In certain considerations it is important to know when all the elements of some power of A are strictly positive. Frobenius [2] gave a very simple necessary and sufficient condition for this to happen. In this note we give a simple proof of this result. Our proof is algebraic in nature and avoids the use of the convergence of powers of a matrix.

All matrices considered here will have real elements. For two such matrices (not necessarily square) $B=(b_{ij})$, $C=(c_{ij})$ we define

$$B \geqq C \qquad \text{if } b_{ij} \geqq c_{ij} \text{ for each } i, j,$$

$$B > C \qquad \text{if } b_{ij} > c_{ij} \text{ for each } i, j.$$

A square matrix $A \geqq 0$ (A is then called non-negative) is said to be *indecomposable* if there is no permutation matrix P for which

$$PAP^{-1} = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix},$$

where the A_{ii} are square submatrices

The fundamental result about non-negative, indecomposable matrices is due to Frobenius [2]; this, and other, results have recently been rederived and extended in a greatly simplified manner by Wielandt [3] and Debreu and Herstein [1]. It is the

THEOREM I. *Let $A \geqq 0$ be an indecomposable matrix. Then A has a positive characteristic root r such that*

(1) *r is a simple root;*

(2) *to r can be associated a characteristic vector $x > 0$;*

(3) *if α is any other characteristic root of A, $|\alpha| \leqq r$.*

If $A>0$ then (3) can be sharpened to $|\alpha| < r$ for all characteristic roots $\alpha \neq r$ of A.

If $A \geqq 0$ is indecomposable and if A has no characteristic root other than r of maximal absolute value then A is said to be *primitive.*

In this paper we prove the

THEOREM II (Frobenius). *Let $A \geqq 0$. Then $A^m > 0$ for some integer $m > 0$ if and only if A is primitive.*

Suppose that $A^m > 0$. Then A must be indecomposable; for if

† This paper is a result of the work being done at the Cowles Commission for Research in Economics on the "Theory of Resource Allocation" under subcontract to the RAND Corporation.

$$PAP^{-1} = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix},$$

then

$$PA^mP^{-1} = \begin{pmatrix} B^m & C^m \\ 0 & D^m \end{pmatrix}, \qquad \text{contradicting } A^m > 0.$$

Now suppose that r and $re^{i\phi} \neq r$ are characteristic roots of A of maximal absolute value. Then A^m, A^{m+1} are both positive and have r^m, $r^m e^{im\phi}$, and r^{m+1}, $r^{m+1}e^{i(m+1)\phi}$ respectively as roots of maximal absolute value. Since the largest root of a positive matrix is simple and is actually greater than any other root in absolute value we must have $r^m e^{im\phi} = r^m$, $r^{m+1}e^{i(m+1)\phi} = r^{m+1}$, whence $e^{i\phi} = 1$, a contradiction.

There remains but to show that if A is primitive then $A^m > 0$ for a suitable integer $m > 0$. This will be proved as a consequence of the following few lemmas, which by themselves are of some interest.

LEMMA 1. *If A is primitive then A^m is primitive for every positive integer m.*

Proof. Since r is a simple root of A and is the only root of A of absolute value r, r^m is a simple root of A^m and is the only root of A^m of absolute value r^m. So we need but show that A^m is indecomposable for every integer $m > 0$. Suppose that for some s, A^s is not indecomposable; we can then assume that

$$A^s = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix}.$$

Now $Ax = rx$ for $x > 0$, so $A^s x = r^s x$; partition x according to the partitioning of A^s and we have

$$\begin{pmatrix} B & C \\ 0 & D \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = r^s \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

That is, $Dx_2 = r^s x_2$, and since x_2 is positive, r^s is a characteristic root of D. Since the transpose, A', of A is also indecomposable, we have $A'y = ry$ for $y > 0$. Partitioning as above we obtain that r^s is a characteristic root of B', and so of B. Being a characteristic root of both B and D, r^s must be a multiple root of A^s, which is a contradiction. The lemma is thereby proved.

LEMMA 2. (Wielandt). *Let ϵ be any positive number. Suppose $A \geqq 0$ is an $n \times n$ indecomposable matrix. Then $(\epsilon I + A)^{n-1} > 0$ where I is the identity matrix.*

Proof. It clearly suffices to show that for any vector x, $x \geqq 0$, $(\epsilon I + A)^{n-1}x > 0$. Let

$$x_\nu = (\epsilon I + A)^{\nu-1}x.$$

Then

$$x_{\nu+1} = \epsilon x_\nu + A x_\nu.$$

Hence a zero component can occur in $x_{\nu+1}$ only where a zero component already occurred in x_ν. However, not every such zero component can be preserved in $x_{\nu+1}$. For if so, by a suitable reordering of the coordinates,

$$x_\nu = \begin{pmatrix} p \\ 0 \end{pmatrix}, \qquad p > 0,$$

whence

$$x_{\nu+1} = \epsilon \begin{pmatrix} p \\ 0 \end{pmatrix} + \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} p \\ 0 \end{pmatrix} = \begin{pmatrix} q \\ 0 \end{pmatrix},$$

from which it follows that $A_{21}p = 0$. This together with $p > 0$ forces $A_{21} = 0$, violating the indecomposability of A. So each application of $\epsilon I + A$ to x decreases the number of zero coordinates by at least one. Hence $(\epsilon I + A)^{n-1}x > 0$.

As an easy consequence of Lemma 2 we obtain

LEMMA 3. *If $A = (a_{ij})$ is indecomposable and $a_{ii} > 0$ for each i then $A^{n-1} > 0$.*

For let ϵ be chosen satisfying $0 < \epsilon < \min_i a_{ii}$. Then $A = \epsilon I + B$ where $B \geqq 0$ is indecomposable. Lemma 2 then yields $A^{n-1} > 0$.

Let $A^m = (a_{ij}^{(m)})$. Then we have

LEMMA 4. *Let $A \geqq 0$ be indecomposable. Then for any i, j we can find an $m = m(i, j) > 0$ so that $a_{ij}^{(m)} > 0$.*

Proof. Consider first the case $i \neq j$. Since

$$(I + A)^{n-1} = A^{n-1} + \binom{n-1}{1} A^{n-2} + \cdots + I > 0$$

by Lemma 2, $a_{ij}^{(m)} > 0$ for some $m \leqq n-1$. Now suppose $i = j$. Since A is indecomposable, no column of zeros can occur in A. So there is a k with $a_{ki} > 0$. If $k = i$ then $a_{ii}^{(m)} > 0$ for all m trivially. If, on the other hand, $k \neq i$, then $a_{ik}^{(m)} > 0$ for some m, and since $a_{ii}^{(m+1)} = \sum_r a_{ir}^{(m)} a_{ri} \geqq a_{ik}^{(m)} a_{ki} > 0$, the lemma is proved.

We are now in position to complete the proof of Theorem II. Let A be primitive. Pick m_1 so that in A^{m_1}, $a_{11}^{(m_1)} > 0$. Let $A_1 = A^{m_1} = (a_{ij}(1))$. By Lemma 1, A_1 is primitive, so there is an m_2 such that in $A_1^{m_2}$, $a_{22}^{(m_2)}(1) > 0$. Since $a_{11}(1) = a_{11}^{(m_1)} > 0$, $a_{11}^{(m_2)}(1) > 0$. Let $A_2 = A_1^{m_2}$. Continuing in this way we arrive at an $A_n = A^{m_1 m_2 \cdots m_n}$ which is primitive and whose diagonal elements are all positive. By Lemma 3, $A_n^t > 0$ for some t, hence $A^m > 0$ for some suitably chosen integer m.

Bibliography

1. Debreu, Gerard and Herstein, I. N., Non-negative square matrices (forthcoming).
2. Frobenius, G., Über Matrizen aus nicht negativen Elementen, *Sitzungsberichte der Berliner Akademie,* 1912, p. 456–477.
3. Wielandt, H., Unzerlegbare, nicht negative Matrizen, *Mathematische Zeitschrift,* vol. 52, 1950, p. 642–648.

BOUNDED MODELS OF THE EUCLIDEAN PLANE

Editorial Note. Although the process of graphing $y=f(x)$ in the Euclidean plane is extremely well-known, comparatively little attention has been paid to the deficiencies of this method. The chief objection is that such graphs rarely illustrate the behavior of the function at large values of x and y. Quite independently several authors have submitted papers to the MONTHLY advocating the advantages of graphing such functions in a bounded model of the Euclidean plane. Graphs of this sort show the complete behavior of the function, but naturally involve distortions of the customary graph. As treated in detail below, R. L. Swain has proposed mapping the plane onto the interior of a square, and David Gans has discussed its map onto the interior of a circle. Gans' model has the further advantage of introducing the ideal points needed to convert the Euclidean plane into the plane of projective or elliptic geometry. It also shows the topological properties of these planes. Kenneth May has elaborated on Gans' idea by constructing special graph paper. In order to save space through the elimination of overlapping material, these three papers have been combined into the article below.

I. CONDENSED GRAPHS*

R. L. SWAIN, State University of New York, New Paltz

1. The condensed plane. We map the Euclidean xy-plane onto the interior of the square bounded by the lines $x' = \pm 1$, $y' = \pm 1$ by the relations:

$$x' = \frac{x}{1 + |x|}, \qquad y' = \frac{y}{1+|y|}. \tag{1}$$

Each point of the square region is now regarded as having the coordinates of that point of the original xy-plane which was mapped into it under (1). Each point of the boundary of the square region is also regarded as having coordinates, at least one of which is $+\infty$ or $-\infty$, assigned in the obvious way. The resulting closed square region is called the (rectangular) *condensed plane.*

It will be convenient to regard such a transformation as effected through the aid of a *mapping function* $\phi(t)$, defined for $0 \leqq t < +\infty$, in this case the function

$$\phi(t) = \frac{t}{1+t}. \tag{2}$$

The relations in (1) are given by $x' = \phi(x)$ for $x \geqq 0$, $x' = -\phi(-x)$ for $x \leqq 0$, $y' = \phi(y)$ for $y \geqq 0$, $y' = -\phi(-y)$ for $y \leqq 0$. The essential features of the new scales given by the mapping will therefore be apparent if we discuss its effect in transforming just the non-negative portion of the x-axis.

Under the mapping, the non-negative portion of the original x-axis has been thrown into what was its unit (half-open) interval (0, 1). Each point x' of this interval is now "marked" with a new coordinate x, where $x' = \phi(x)$, that is, $x = x'/(1-x')$. The right end-point, which had the coordinate $x'=1$, is now marked with the symbol $+\infty$. This new scale is shown in Figure 1. The points to be marked $(2^n - 1)$ and $1/(2^n-1)$, *i.e.*, 1, 3, 7, 15, $\cdots$, 1/3, 1/7, 1/15, $\cdots$,

* This material was presented at the International Congress of Mathematicians, Cambridge, Massachusetts, 1950, *Proceedings Int. Congr. Math.*, vol. I, page 760, 1952.

I wish to thank the editor of the MONTHLY for helpful editorial suggestions.

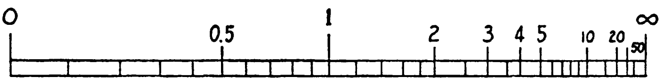


Fig. 1

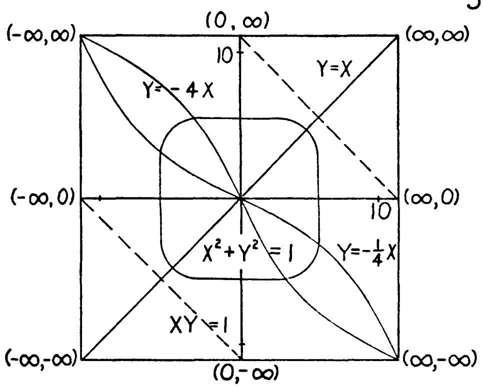


Fig. 2

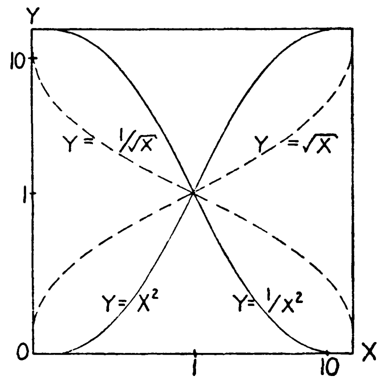


Fig. 3

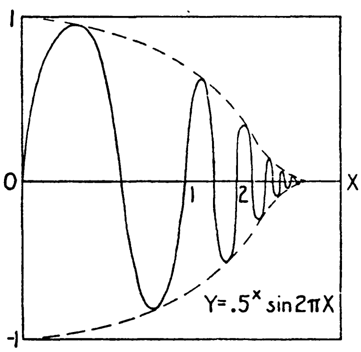


Fig. 4

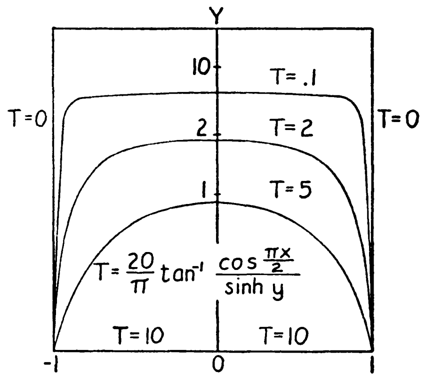


Fig. 5

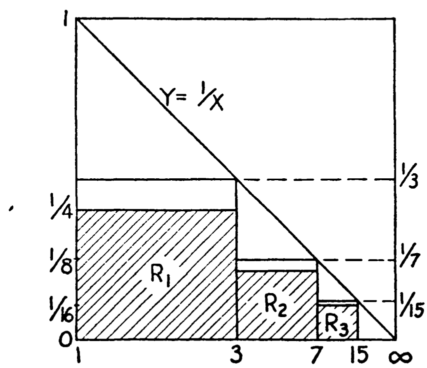


Fig. 6

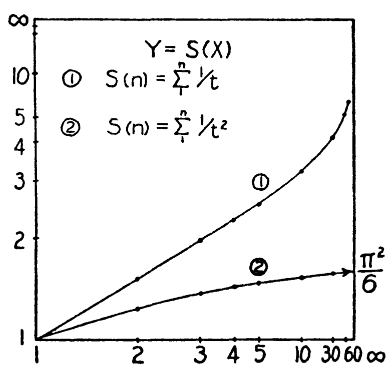


Fig. 7

may be located quickly and accurately in blackboard sketching because these are the "halfway" division points to the right and left from the midpoint, now marked 1, of the interval.

Given any equation $f(x, y)=0$, we may draw its complete graph in the condensed plane directly, by using the scale values marked on the axes. The graph of the line $y=x$ is the straight-line segment joining the points $(-\infty, -\infty)$ and $(+\infty, +\infty)$. The graph of the hyperbola $xy=1$ consists of two straight-line segments, one joining $(0, +\infty)$ and $(+\infty, 0)$, the other joining $(-\infty, 0)$ and $(0, -\infty)$. Besides these, Figure 2 shows the graphs of $x^2+y^2=1$, $y=-4x$ and $y=-(1/4)x$.

2. Drawn slope. The proportionate amount of scale condensation at the point marked x on the condensed x-axis is given by

$$dx' = \phi'(|x|)dx = \frac{1}{(1+|x|)^2}dx. \tag{3}$$

The derivative dy/dx will still be called the "slope" of the graph of an equation $f(x, y)=0$. But the graph in the condensed plane appears to have the slope dy'/dx'. This we call the *drawn slope* of the curve. From (3) we find

$$\frac{dy'}{dx'} = \left(\frac{1+|x|}{1+|y|}\right)^2 \frac{dy}{dx}. \tag{4}$$

Since dy'/dx' and dy/dx always have the same sign, the location of maxima and minima (relative to the marked scale) is not affected by graphing in the condensed plane. However, points of inflection are not preserved. The curve crosses the lines $y=\pm x$ with drawn slope equal to its slope.

The graph of every equation $y=mx+b$ $(m>0)$ terminates at $(+\infty, +\infty)$. We find the drawn slope at this point from (4):

$$\lim_{x\to\infty} \frac{dy'}{dx'} = m \lim_{x\to\infty}\left(\frac{1+x}{1+b+mx}\right)^2 = \frac{1}{m}\cdot \tag{5}$$

3. Symmetries. In the ordinary plane, changing the sign of x or y or interchanging x and y results in a new graph symmetric with respect to the old. In the condensed plane, in addition to these usual symmetries, a new type appears, from taking the reciprocal of either x or y. The graphs of

$$f(x) \text{ and } f(1/x) \text{ are symmetric w.r.t. the lines } \begin{cases} x = +1 & (x \geqq 0) \\ x = -1 & (x \leqq 0), \end{cases}$$

$$f(x) \text{ and } 1/f(x) \text{ are symmetric w.r.t. the lines } \begin{cases} y = +1 & (f(x) \geqq 0) \\ y = -1 & (f(x) \leqq 0). \end{cases}$$

These properties provide a useful teaching feature and may have some practical utility. They arise because the mapping function satisfies the identity

$$\phi(t) + \phi(1/t) = 1. \tag{6}$$

The $f(x)$ and $f(1/x)$ symmetry is established by taking $t=|x|$, the other by $t=|y|$. Some of these symmetries show up in Figures 2 and 3.

4. Asymptotic behavior. The asymptotic properties of a function are of special interest to the technical and engineering student as well as to the mathematician. Figure 4 shows the die-out with time x of the displacement in a damped harmonic oscillation. Figure 5 shows contour curves for a function, the isotherms in a semi-infinite strip with insulated faces whose three edges are maintained at 0°, 10°, and 0°, as marked.

5. Improper integrals. Suppose we are concerned with the convergence of an integral $\int_a^\infty f(x)dx$, where $f(x)$ is continuous and positive for $a<x<\infty$. Suppose that $f(x)$ is decreasing and that $\lim_{x\to\infty}f(x)=0$. In the condensed plane the graph of $y=f(x)$ then terminates at $(+\infty, 0)$. At this point the drawn slope of the curve $xy=k$ $(k>0)$ is $-k$. Consequently, if the drawn slope of the graph of $y=f(x)$ is negative at $(+\infty, 0)$, there will be some k value such that for sufficiently large x, the graph of $y=f(x)$ lies above the graph of $y=k/x$. In this case the integral $\int_a^\infty f(x)dx$ must diverge. It is therefore a necessary (but not a sufficient!) condition for convergence that the graph of $y=f(x)$ be tangent to the axis at $(+\infty, 0)$. Similarly, if $f(0+)=\infty$, $\int_0^b f(x)dx$ cannot converge unless the graph of $f(x)$ is tangent to the y-axis. The form of the graph of $y=1/x^2$, shown in Figure 3, makes it apparent that $\int_0^1(1/x^2)dx$ diverges.

The divergence of the integral $\int_1^\infty(1/x)dx$ may be displayed graphically to the student by blocking in rectangles R_n of (scale) base length 2^n and of (scale) height $1/2^{n+1}$. Each has "true" area $1/2$. R_1 has the x-axis interval $(1, 3)$ for its base, R_2 the interval $(3, 7)$, *etc.* These rectangles lie within the drawn "squares" S_n with the same bases, which inscribe beneath the "line" $xy=1$. This is illustrated in Figure 6.

6. Series. The convergence or divergence of a series may be convincingly portrayed by plotting the partial sums $S(n)$ in the condensed plane. Figure 7 shows such plots for the series $\sum_1^\infty 1/t$ and $\sum_1^\infty 1/t^2$. (The points are joined by a smooth curve as a visual aid.)

It is also feasible to plot the test ratios $r(n)=u_{n+1}/u_n$ for a given series $\sum_1^\infty u_t$, or to plot the curve $r(x)=f(x+1)/f(x)$ in the case of an improper integral of form $\int_a^\infty f(t)dt$. If $|r(n)|$ approaches a limit other than unity, the graph will show this. In case $|r(n)|\to 1$, the following sufficient test, a graphical equivalent of Gauss' test, may aid: Given $\sum_1^\infty u_t$, a series of positive terms, plot the points $P_n=(n, r(n))$ in the condensed plane. If there exists a line l through $(+\infty, 1)$ of drawn slope exceeding $1/4$, such that for some N and all $n>N$, P_n lies below l, then the series converges. (A similar test may be made in the ordinary plane by plotting $P_n=(1/n, 1-r(n))$ and requiring P_n for $n>N$ to lie above a line $y=kx$ with $k>1$.)

7. Mapping functions. We selected $\phi(t)=t/(1+t)$ as our mapping function primarily for its simplicity. Many other choices are possible, such as $\phi(t)=1-e^{-t}$, $\phi(t)=(2/\pi)\tan^{-1}t$, *etc.** The following would appear to be minimum requirements to be met by a mapping function $\phi(t)$: *$\phi(t)$ should be a single-valued, twice differentiable, increasing function, defined for $0\leqq t<+\infty$, with $\phi''(t)<0$ for $t\neq 0$, $\phi(0)=0$, $\phi(+\infty)=1$.*

Given such a mapping function $\phi(t)$, the x-axis (or any distance coordinate) is subjected to the transformation $x'=\phi(x)$ for $x\geqq 0$, $x'=-\phi(-x)$ for $x\leqq 0$. This maps it onto the open interval $(-1, 1)$. Each point x' of this interval is now considered to have the coordinate x in the new scale, and the end-points are given new "coordinates" $\pm\infty$. Since $dx'=\phi'(|x|)dx$, $\phi'(|x|)$ is the measure of scale condensation at the point marked x on the (new) x-axis. Since $\phi''(t)<0$ for $t\neq 0$, $\phi'(|x|)$ decreases smoothly as $|x|$ increases.

A rectangular condensed plane is obtained by shrinking both x- and y-axes according to the same $\phi(t)$. The plane is thus mapped onto the interior of a square of side 2, with center at the origin. Horizontal lines are carried into horizontal lines, vertical into vertical, the origin being the only fixed point. The interior points of the square with original coordinates (x', y') are regarded as having the coordinates (x, y) in the new scale; the boundary of the square is appropriately scaled, at least one coordinate of each of its points being $-\infty$ or $+\infty$, and is adjoined to the interior. The resulting closed region is the rectangular condensed plane generated by the mapping function $\phi(t)$.

We have

$$\frac{dy'}{dx'}=\frac{\phi'(|y|)}{\phi'(|x|)}\frac{dy}{dx}. \tag{7}$$

This formula, the generalization of (4), gives the drawn slope of a curve at the point whose marked coordinates in the condensed plane are (x, y). Since $\phi'>0$, the drawn slope has the same sign as the "true" slope, so that the location of maxima and minima is not affected. The continuity of ϕ' insures that the drawn slope approximates the slope in the neighborhood of the origin. Curves cross the lines $y=\pm x$ with their drawn slope equal to their slope. The graph of $y=x$ is the straight-line segment joining $(-\infty, -\infty)$ to $(+\infty, +\infty)$.

In case the function $g(t)=\phi(1/t)$ has a Maclaurin expansion, it may be shown quite easily that the drawn slope of the line $y=mx+b$ $(m>0)$ at $(+\infty, +\infty)$ is $1/m^p$, where p is the exponent of t in the first non-vanishing term of the expansion of $g(t)$. This generalizes the result in §2.

The complete set of symmetries listed in §3 is obtained only if the mapping

---

* A mapping function $\phi(t)$ may be regarded as derived from a generating function $\psi(t)$ defined for $0\leqq t<1$ and related graphically to $\phi(t)$ as follows: $\phi(t)$ is the abscissa of the point in which the line through the points $(0, 1)$ and $(t, 0)$ in the ordinary plane cuts the graph of $y=\psi(x)$. Taking $\psi(t)=t$ gives the function $\phi(t)=t/(1+t)$ used in our examples.

function satisfies the identity (6). If this identity holds,† the graph of $xy=1$ is still the pair of straight-line segments shown in Figure 2, since $xy=1$ will then transform into $x'+y'=1$ or $x'+y'=-1$ according as x and y are both positive or negative. Furthermore, if the identity holds, it may be proved* that if $y=f(x)$ is any differentiable function such that $\lim_{x\to\infty} f(x)=+\infty$ and $\lim_{x\to\infty} f'(x)=m>0$, then the drawn slope of $y=f(x)$ at $(+\infty, +\infty)$ is $1/m$.

8. Condensed coordinate systems. In the plane or in space, with any of the usual coordinate systems, the distance coordinate scales may be condensed. The hyperbolic paraboloid $z=x^2-y^2$ shows up nicely when drawn in rectangular condensed space. A polar condensed plane is obtained by condensing the distance coordinate while leaving the angular coordinate unchanged. A cylindrical condensed space or a spherical condensed space may also be obtained in this way. The polar condensation and its spatial extensions seem generally unsatisfactory as analytical aids, but may occasionally be useful when the graphs to be displayed have marked polar symmetry. For special kinds of geometrical or analytical illustrations, one distance coordinate may be shrunk while leaving another unchanged.

† Paul Erdös pointed out to me that a $\phi(t)$ defined only on $0\leqq t\leqq 1$ could be selected for which $\phi(0)=0$, $\phi(1)=1/2$, $\phi'(t)>0$, $t^2\phi'(t)$ increasing, and that $\phi(t)$ could then be defined for $1<t<+\infty$ by (6), the result being a mapping function satisfying the identity as well as the earlier stated minimum requirements.

* The proof makes a nice exercise in the application of l'Hospital's Rule, suitable for an advanced student.

II. A CIRCULAR MODEL OF THE EUCLIDEAN PLANE

DAVID GANS, New York University

1. The transformation T. Consider the lower half, *i.e.*, the half for which $z<1$, of the unit-sphere

$$x^2+y^2+z^2-2z=1, \tag{1}$$

with center at $A(0, 0, 1)$ and tangent to the xy-plane at the origin. With A as center, project the xy-plane onto this hemisphere so that an arbitrary point P of the plane goes into the point Q of the hemisphere. In this way the entire xy-plane, is mapped topologically onto the entire hemisphere exclusive of its bounding great circle. Now project this hemisphere orthogonally onto the xy-plane, so that the arbitrary point Q of the hemisphere goes into the point P' of the plane. The aggregate of all the points P' is the interior of the unit-circle $x^2+y^2=1$ in the xy-plane, which circle we denote by γ. The resultant of the two projections is a topological mapping of the entire xy-plane onto the interior of γ such that an arbitrary point P of the plane goes into the point P' interior to γ.

Denote this mapping by T.

In the projection with A as center each straight line of the plane goes into a semi-great-circle on the hemisphere, exclusive of its endpoints, and this semi-great-circle, in turn, projects orthogonally into a semi-ellipse, exclusive of its endpoints, whose major axis is a diameter of γ. Let E denote the aggregate of all such semi-ellipses. The mapping T thus sends the set of all straight lines of the plane into the set E. For this statement to be entirely correct the term "semi-ellipse" must be extended, as we hereby do, to include any diameter of γ, for T sends each straight line $y=mx$ into such a diameter.

It is easy to see that under T each straight line through the origin O goes into the diameter of γ contained in that line, and each circle with center O goes into a smaller circle with center O and interior to γ. Thus the effect of T is to compress the plane radially into the interior of γ.

By using equation (1), together with the fact that the parametric equations of the straight line AP are

$$x = at, \qquad y = bt, \qquad z = -t + 1, \tag{2}$$

we readily obtain

$$x' = \frac{x}{\sqrt{x^2 + y^2 + 1}}, \qquad y' = \frac{y}{\sqrt{x^2 + y^2 + 1}} \tag{3}$$

as the equations of the mapping T, and hence

$$x = \frac{x'}{\sqrt{1 - x'^2 - y'^2}}, \qquad y = \frac{y'}{\sqrt{1 - x'^2 - y'^2}} \tag{4}$$

as the equations of the inverse mapping T^{-1}. All the previously stated properties of T can now be easily verified analytically.

2. The semi-ellipses E. We now wish to consider the problem of locating the semi-ellipse E into which T sends a given straight line l. To locate E exactly we could, of course, assume an equation for l, transform it into the equation of E, and study the latter equation. This procedure, though direct, lacks the geometric appeal and insight of the method we now prefer to follow.

The central projection from A sends l into a semicircle m on the hemisphere, the diameter d of this semicircle going through A. Because of this projection l and d are parallel, being in the same plane through A yet not meeting since they are in the planes $z=0$ and $z=1$, respectively. The semicircle m and its diameter d project orthogonally into the semicircle E and its major axis d', so that d and d' are parallel. Hence l and d' are parallel. That is, the straight line l is parallel to the major axis of the corresponding semi-ellipse E.

Now let B be the foot of the perpendicular from O to l, and B' the point into which T sends B. Then O, B, B' are collinear since T is a radial transformation. Since B is the point of l nearest O it follows from next to the last paragraph of

section 1 that B' is the point of E nearest O, in which case B' must be the extremity of the minor axis of E, and $|\overline{OB'}|$ the length of this axis.* To determine the length of this axis, let p and p' denote the distances $|\overline{OB}|$ and $|\overline{OB'}|$, respectively, let (x, y) and (x', y') be the coordinates of B and B', respectively, and let $ax+by+c=0$ be the equation of l. Since p is the length of the normal from O to l we have

$$p = \frac{|c|}{\sqrt{a^2 + b^2}},$$

so that

$$\frac{p^2}{p^2 + 1} = \frac{c^2}{a^2 + b^2 + c^2}. \tag{5}$$

Also

$$\begin{cases} p^2 = \overline{OB}^2 = x^2 + y^2 \\ p'^2 = \overline{OB'}^2 = x'^2 + y'^2, \end{cases} \tag{6}$$

while the equations of T^{-1} permit us to write

$$x'^2 + y'^2 = \frac{x^2 + y^2}{x^2 + y^2 + 1}. \tag{7}$$

Substitution of (6) in (7) gives

$$p'^2 = \frac{p^2}{p^2 + 1},$$

and substitution of (5) in this gives

$$p'^2 = \frac{c^2}{a^2 + b^2 + c^2}.$$

Hence the length of the minor axis of E is given by

$$p' = \frac{|c|}{\sqrt{a^2 + b^2 + c^2}}.$$

We still need to determine the quadrant in which the minor axis lies. Since B' is between O and B, this axis and segment OB lie in the same quadrant, which is readily determined from the normal form of l, or merely from inspection of the equation $ax+by+c=0$. Thus, if c is always taken negative, the following table gives the proper quadrant:

* By "minor axis of E" is meant the semi-minor axis of the complete ellipse of which E is a part.

a	b	quadrant
+	+	1
−	+	2
−	−	3
+	−	4

Using the preceding results we can draw the major and minor axes and thus provide ourselves with a good idea of the shape and location of E. Of course, knowing these axes, as many points of E as may be desired can be constructed by the familiar straightedge and compass method which utilizes two concentric circles. Should one wish to sketch E only by using the major axis, minor axis, and latera recta, it may be noted that the distance from O and the length of each latus rectum† are, respectively,

$$\sqrt{\frac{a^2+b^2}{a^2+b^2+c^2}} \quad \text{and} \quad \frac{c^2}{a^2+b^2+c^2}.$$

3. A topological model of the Euclidean plane. Since T is a topological mapping, the points and semi-ellipses E within γ have the same topological properties as do the points and straight lines of Euclidean plane geometry. Thus any two points within γ determine a unique semi-ellipse E, any two semi-ellipses E meet in one point or not at all, a semi-ellipse E is a continuous curve that does not intersect itself, and so forth. The interior of γ viewed in this way is therefore a topological model of the Euclidean plane.

The general value of this model lies in the fact that, being bounded, it puts the entire Euclidean plane before our eyes in miniature, as it were, giving us a topological view of the plane in the large that is impossible with the usual unbounded Cartesian model. A straight line, *i.e.*, a semi-ellipse E, can be visualized in its entirety, and the same is true of any other unbounded curve. A family of parallel straight lines appears to us completely as a family of semi-ellipses E no two of which meet since they all have the same major axis, *i.e.*, the same diameter of γ. Clearly this model can serve as a handy means of exhibiting in miniature the intersection properties of straight lines, as well as their other topological properties, and it offers the possibility that these things may also be done to advantage with non-linear unbounded curves. It is worth noting that this model is somewhat analogous to a certain topological model of the hyperbolic plane in which the points interior to a circle and the chords of this circle are defined as hyperbolic points and straight lines, respectively.*

Over and above this general value our model possesses the special advantage

† By "latus rectum" of E is meant half of the latus rectum of the complete ellipse of which E is a part.

* See, for example, Richard Baldus, *Nichteuklidische Geometrie*, chapter IV.

that it can be enlarged in a simple and concrete manner so as to produce a topological model of the plane of elliptic or projective geometry. This is done merely by adding to our model the points on the circle γ, with the understanding that each pair of antipodal points is to be regarded as a single point, and that each semi-ellipse E thus acquires a single new point. This vivid and natural way of introducing ideal points contrasts strongly with the usual method employing the unbounded Cartesian model. In this latter method, as usually presented in books on modern geometry, we are expected to believe that the ideal point on a straight line is a definite point that is reached by going infinitely far either to the left or to the right along the unbounded line! We are furthermore expected to believe that there is exactly one ideal point corresponding to each slope in the plane, and that all these ideal points constitute an ideal straight line. The unbounded Cartesian model is certainly worthless as a pictorial basis for such beliefs.

4. An isometric model of the Euclidean plane. By a very simple device our topological model can be made to possess all the metric properties of the Euclidean plane. First, instead of thinking of the points interior to the circle γ as possessing coordinates (x', y'), where $x'^2+y'^2<1$, we can think of them as having coordinates (x, y), where x and y are any real numbers, merely by regarding the equations for T and T^{-1} as equations for changing coordinates. Thus each point (x', y') interior to γ acquires new coordinates (x, y) in accordance with these equations, so that, for example, the point $x'=0$, $y'=1/2$ acquires the new coordinates $x=0$, $y=1/\sqrt{3}$. Second, having thus changed coordinates, we continue to use the same distance formula $ds^2=dx^2+dy^2$, so that, for example, the distance between the points with new coordinates (2, 0) and (5, 4) is 5.

It follows that our model possesses all the metric properties of Euclidean plane geometry, except, of course, that we cannot continue to visualize things in the conventional manner. We must visualize our straight lines as semi-ellipses E, although they still have equations of the form $ax+by+c=0$. To draw the locus of such an equation we simply draw a certain semi-ellipse E, as explained earlier. The angle between two straight lines must be visualized as the angle between two semi-ellipses E, but the computation of this angle from the equations of these semi-ellipses is exactly the same as in ordinary Cartesian geometry.

An alternative method of converting our topological model of the Euclidean plane into an isometric model would be to let the points interior to γ retain their usual (x', y') coordinates, where $x'^2+y'^2<1$, but to adopt a new formula for distance. The new formula, however, turns out to be so complicated as to make this alternative approach quite impractical.

III. THE USE OF CONDENSED GRAPHS IN ANALYTIC GEOMETRY

K. O. MAY, Carleton College

By constructing appropriate graph paper we may use the circular model of the Euclidean plane described in the previous paper to deal with unbounded curves at the elementary level. For cartesian graphs the coordinate curves $x=a$ are semi-ellipses with major axes coincident with the diameter of the circle and semi-minor axes given by $a[1+a^2]^{-1/2}$, and similarly for the curves $y=b$. In Fig. 1 the coordinate ellipses are shown for x, $y=\pm 10$, ± 20. For polar graphs the coordinate curves $r=a$ are circles whose radii are given by the previous formula, and the curves $\theta=b$ are diameters whose slopes are given by tan b. Circles for $r=10$, 20, 30, and 40 are shown in Fig. 2.

Graph paper may be prepared easily by drawing in one quadrant one set of coordinate curves. This may then be used to trace both curves in all quadrants on a spirit duplicator master unit. The result will be serviceable even if rather inelegant. The writer will be glad to send a sample to anyone interested.

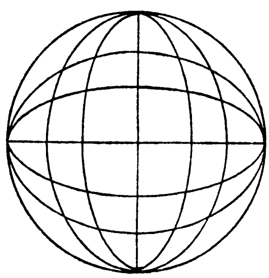


Fig. 1

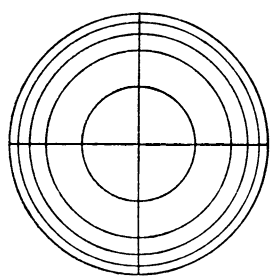


Fig. 2

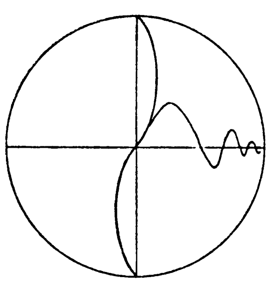


Fig. 3

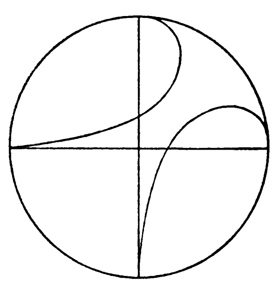


Fig. 4

Once the graph paper is constructed it may be used without reference to its origin in a transformation of the Euclidean plane. Indeed it is a perfectly legitimate independent model of the set of all ordered pairs of real numbers. Of course the numbers assigned to the coordinate ellipses may be changed by any

multiplicative factor. Since there is practically no distortion from the familiar cartesian coordinates in the central region (note the central "square" in Fig. 1), we may choose the scale so as to show as much as we like of a curve in the familiar shape. On the other hand we may choose the scale so as to compress much of the curve near the origin and show greater detail farther out. The graphs of $y=\tan x$ and $y=\sin x$ in Fig. 3 and of $y=e^x$ and $y=\log x$ in Fig. 4 exhibit the familiar forms near the origin.

The following features of this graph paper are easily verified.

1. Symmetries about the origin and about lines through it are undisturbed. In particular, as illustrated in Fig. 4, the relation between the graphs of a function and its inverse remains.

2. The elliptical graph of a linear equation can easily be drawn by rotating the graph paper until one of the coordinate ellipses coincides with it.

3. Slopes may be estimated and bend points located with reference to neighboring coordinate curves.

4. Periodic curves appear to recede in perspective as in Fig. 3.

5. Curves asymptotic to straight lines approach the boundary along ellipses as illustrated for $y=\tan x$ in Fig. 3.

6. Curves without asymptotes approach the boundary in a characteristically different way. Thus curves such as $y=e^x$ (Fig. 4) and $y=x^2$ cross every coordinate ellipse and so appear to come in "along the circumference." These properties may be emphasized by change of scale, but we are not suggesting a graphical method to replace analytic studies of curves "at infinity." However, this and the previous property make the graph paper intuitively very satisfactory for dealing with unbounded curves.

CONCERNING THE RECIPROCAL OF A PRIME

DALJIT SINGH, Indian Agricultural Research Institute, New Delhi

1. On evaluating the reciprocal of a prime number, one often comes across a figure in the quotient which gives a corresponding remainder equal to that figure in the quotient. Again for primes like $p=19$ and 29 such pairs of figures in the quotient and corresponding remainder which are equal are exactly 9. It is therefore interesting to study the conditions under which such properties can be noticed.

2. Let $q(a)$ and $r(a)$ be the quotient and its corresponding remainder in the ath place in the period of the recurring decimal for $1/p$. Then, we have:

$$\text{(2.1)} \qquad \frac{10r(a)}{p} = q(a+1) + \frac{r(a+1)}{p}, \qquad 0 \leqq q(a) \leqq q;\ 0 < r(a) < p.$$

For any $q(a)=r(a)$ we must have $p\cdot q(a)+r(a)=10r(a-1)$, *i.e.*,

$$r(a)\cdot(p+1) = 10r(a-1). \tag{2.2}$$

From (2.2) it is possible to deduce the following results.

3. (a) *For primes of the form $p=10n+s$ where $s=1, 3, 7$, $q(a)=r(a)$ only* if $r(a)=5$ *has a solution.*

(b) *For primes of the form $p=10n+9$, $q(a)=r(a)=b$ is possible for all values of b, $(1\leqq b\leqq 9)$ except for those values of b for which $r(a-1)=b(n+1)$ does not have integral solutions.*

(c) COROLLARY: *For primes of the form $p=10n+9$, $q(a)=r(a)=b(1\leqq b\leqq 9)$ where the period of the recurring decimal has $p-1$ figures.*

Indications of proofs.

(a) In (2.2) putting $p=10n+s$ we get

$$r(a)\cdot(10n+s+1) = 10\cdot r(a-1). \tag{3.1}$$

i.e.,

$$r(a)\cdot(5n+d) = 5\cdot r(a-1) \qquad \text{where } 2d = s+1;\ d = 1, 2, 4. \tag{3.2}$$

By taking residues (mod 5) one easily sees that (3.2) has only one solution *viz.*,

$$r(a) = 5, \qquad 9 \geqq q(a) = r(a) > 0, \tag{3.3}$$

under the condition that $q(a)=5$ has a solution. When the period $=p-1$, $r(a)$ runs through all values such that $0<r(a)\leqq p-1$.

(b) In (2.2) putting $p=10n+9$ we get

$$r(a)\cdot(n+1) = r(a-1). \tag{3.4}$$

Therefore, for all values of $r(a-1)$ which are divisible by $(n+1)$, equation (3.4) has integral solutions.

When the period $=p-1$ it is easily seen that such pairs of solutions of (3.4) are exactly equal to 9, and $q(a)=r(a)=b$ where b assumes all values from 1 to 9.

Examples.

Below are given two examples to illustrate the properties mentioned above.

(i) $p=17$; this is of the form $p=10\cdot 1+7$ and period $=16$. From §3 (a) there exists only one pair $q(a)=r(a)=5$. This is shown in the table given below.

a	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$q(a)$	5	8	8	2	3	5	2	9	4	1	1	7	6	4	7	0
$r(a)$	15	14	4	6	9	5	16	7	2	3	13	11	8	12	1	10

(ii) $p=19$; this is of the form $p=10\cdot 1+9$ and period $=18$. From §3(a) there

exist exactly 9 pairs $q(a)=r(a)=b$ where $1 \leqq b \leqq 9$. This is illustrated in the table given below.

a	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$q(a)$	*5*	2	*6*	*3*	1	5	7	8	*9*	4	*7*	3	6	*8*	*4*	*2*	*1*	0
$r(a)$	*5*	12	*6*	*3*	11	15	17	18	*9*	14	*7*	13	16	*8*	*4*	*2*	*1*	10

MATHEMATICAL TEACHING IN UNIVERSITIES

ANDRÉ WEIL, University of Chicago

The following is the outline of a lecture once given by the author at a joint meeting of the Nancago Mathematical Society and of the Poldavian Mathematical Association. It is printed here at the editor's request, as the principles stated there seem to be of general application.

1. Improvements in the mathematical teaching in Poldavian Universities depend largely upon general improvements in the educational system in Poldavia. Mathematicians should devote themselves to the task of making such improvements as lie within their power at present, and thus contributing their share towards general reforms, which in turn will enable them to make further progress.

2. No satisfactory results can be achieved unless reforms are made both in school-teaching and in University teaching. So far as school-teaching is concerned, the efforts of mathematicians in the country should be mainly directed towards necessary changes in the curricula and towards the training of better teachers.

3. University teaching in mathematics should: (a) answer the requirements of all those who need mathematics for practical purposes; (b) train specialists in the subject; (c) give to all students that intellectual and moral training which any University, worthy of the name, has the duty to impart.

These objects are not contradictory but complementary to each other. Thus, a training for practical purposes can be made to play the same part in mathematics as experiments play in physics or chemistry. Thus again, personal and independent thinking cannot be encouraged without at the same time fostering the spirit of research.

4. The study of mathematics, as well as of any other science, consists in the acquisition of useful reflexes and in that of independent habits of thought. The acquisition of useful reflexes should never be separated from the perception of their usefulness.

It follows that problem-solving should never be practised for its own sake; and particularly tricky problems must be excluded altogether. The purpose of problems is twofold; either to drill the student in the application of some method of special importance, or to develop his originality by guiding him along some new path. Drill is essentially a school-method, and ought to become unnecessary at the final stages of University teaching.

5. Rigor is to the mathematician what morality is to man. It does not consist in proving everything, but in maintaining a sharp distinction between what is assumed and what is proved, and in endeavoring to assume as little as possible at every stage.

The student should therefore be gradually accustomed, by means of startling examples, to question the truth of every unproved proposition, until at last he is able to deduce from the ordinary axioms everything that he has learned.

6. Knowledge of a proof means the understanding of its machinery and the ability to reconstruct it. This implies: (a) perfect correctness in the definitions; (b) a faculty of connecting a given question with the general ideas underlying it; (c) a perception of the logical nature of any proof.

The teacher should therefore always follow, not the quickest nor even the most elegant method, but the method which is related to the most general principles. He should also point out everywhere the relation between the various elements of the hypothesis and the conclusion; students must be accustomed to draw a sharp distinction between premises and conclusion, between necessary and sufficient conditions, between a theorem and its converse.

7. The teaching of mathematics must be a source of intellectual excitement. This can be achieved, at the higher stages, by taking the student to the brink of the unknown; at earlier stages, by making him solve for himself questions of theoretical or practical importance.

This is the method followed in the "seminars" of the German Universities, first organized by Jacobi a century ago, and even now the most prominent feature of the German system; division of labor between students in the study of a given group of questions is a common practice in these seminars, and proves to be a powerful incentive to work.

8. Theoretical lectures should neither be a reproduction of nor a comment upon any text-book, however satisfactory. The student's notebook should be his principal text-book.

In fact, taking down notes intelligently (not under dictation) and working them out carefully at home should be considered as an essential part of the student's work; and experience shows that it is not the least useful part of it.

9. The right of any topic to form part of any curriculum is to be tested according to: (a) its importance for modern mathematics or for the applications of mathematics to modern science or technique; (b) its relations with other branches of the curriculum; (c) the intrinsic difficulty of the ideas underlying it.

This involves a revision of the present curriculum. For instance, the idea of

function, the process of differentiation and integration, should appear at an early stage, because of their enormous importance both for the theory and for the most ordinary practice. Because of its practical importance, numerical calculation, and all the devices connected with it, would seem to deserve a far more prominent place in elementary teaching than they receive at present.

MATHEMATICAL NOTES

EDITED BY F. A. FICKEN, University of Tennessee

Material for this department should be sent to F. A. Ficken, University of Tennessee, Knoxville 16, Tenn.

AN EXPRESSION FOR THE EULER ϕ-FUNCTION

N. C. SCHOLOMITI, University of Illinois, Chicago

Let a and x be non-negative integers, $x>0$. If we divide a by x we have:

$$a = x \cdot Q + r$$

where $Q = [a/x]$ is the integral quotient and r is the remainder in the division. Then:

$$r = a - x \cdot [a/x].$$

(The symbol [ ] will represent the bracket function throughout this article and not a parenthesis. By definition:

$[u]$ = the greatest integer not greater than u.)

For r we have the inequalities:

$$0 \leqq r = a - x \cdot [a/x] < x.$$

1. We construct the function:

$$G(a, x) = \left[\frac{1}{a - x \cdot [a/x] + 1}\right]. \tag{1}$$

Obviously $G(a, x) = 1$ if, and only if, x divides a exactly; otherwise $G(a, x) = 0$; a, x are non-negative integers, $x>0$.

2. Next we construct the function:

$$G(a, b, x) = \left[\frac{1}{a - a \cdot [a/x] + b - b \cdot [b/x] + 1}\right]; \tag{2}$$

a, b, x are non-negative integers, $x>0$. Comparing (2) with (1) it is easily seen that $G(a, b, x) = 1$ if, and only if, x divides exactly both a and b. Otherwise

$G(a, b, x)=0$.

3. Continuing, we consider now the expression:

$$L(a, b) = \sum_{x=2}^{a+b} G(a, b, x).$$

Examining the individual terms of the sum on the right we see that each term has the value 1 if x is a common divisor of a and b and otherwise the term has the value 0. Clearly, then, $L(a, b)$ represents the number of common divisors of a and b exclusive of unity. Notice the elasticity of the upper limit over the summation sign which can be taken as any integer not less than the smaller of the two numbers a and b.

We denote the greatest common divisor of a and b, as usual, by (a, b). Notice now that if $(a, b)=1$, *i.e.*, if a and b are relatively prime they have no common divisor greater than unity and hence $L(a, b)=0$. If, however, $(a, b)>1$ and a and b are not relatively prime then clearly, $L(a, b)>0$.

4. It follows then that the function:

$$L'(a, b) = \left[\frac{1}{1 + L(a, b)}\right] = \left[\frac{1}{(a, b)}\right]$$

is zero when $(a, b)>1$, *i.e.*, when $L>0$; while $L'(a, b)=1$ when, and only when, $(a, b)=1$, *i.e.*, when $L=0$.

5. Finally we arrive at the result:

$$\phi(a) = \sum_{b=1}^{a-1} L'(a, b)$$

where a is a positive integer >1. The truthfulness of this becomes at once obvious when we examine the terms of the sum on the right. A term of the sum is zero when a and b have a common divisor. If there is no common divisor the term has the value one and in this case a and b are relatively prime. Thus the sum counts up for us all integers less than a and relatively prime to it thus yielding Euler's totient.

THE NUMBER OF MULTINOMIAL COEFFICIENTS

PAUL ERDÖS, National Bureau of Standards, and IVAN NIVEN, University of Oregon

The problem is to find the number of multinomial coefficients

$$(1) \qquad \frac{n!}{i_1!i_2! \cdots i_r!(n-k)!}, \qquad \sum_{j=1}^{r} i_j = k,$$

which are less than x, excluding the cases $r=1=i_1$ and $r=1=k$ for which (1) assumes the value n. The values of (1) are thus restricted by

sufficiently large. Maximizing each part of (6) gives

$$f_2(x) < (c \log x)(c \log x)x^{1/3}x^{\epsilon} = o(x^{1/2}).$$

Class 3. Every value (1) in this class is clearly

$$\geqq \frac{n!}{\left[\frac{n}{2}\right]!\left\{n - \left[\frac{n}{2}\right]\right\}!}.$$

Thus each admissible value of n satisfies $n \leqq 2h+1$ where h is chosen so that $\binom{2h}{h}$ exceeds x. Replacing $\binom{2h}{h}$ by 2^h as previously we see that $n < c \log x$. For any fixed n the number of values of (1) is maximized by $p(n)$. Thus $f_3(x) < c \log x \cdot p(c \log x) = o(x^{1/2})$, and the proof of (3) is complete.

A more careful analysis would improve the theorem to yield the estimate

$$(1 + \sqrt{2})x^{1/2} + c_3 x^{1/3} + \cdots + c_m x^{1/m} + o(x^{1/m})$$

for every m. This could be proved by isolation of the cases $k=2, 3, \cdots, m$ for special treatment, where here we stopped at $k=2$.

References

1. Ivan Niven, The asymptotic density of sequences, Bull. Amer. Math. Soc., vol. 57, 1951, pp. 420–434, Theorem 2.

2. G. H. Hardy and S. Ramanujan, Asymptotic formulae in combinatory analysis, Proc. London Math. Soc. (2), vol. 17, 1918, pp. 75–115: or, Paul Erdös, On an elementary proof of some asymptotic formulas in the theory of partitions, Annals of Math. (2), vol. 43, 1942, pp. 437–450.

3. Nagell, T., Introduction to Number Theory, John Wiley (1952), Theorem 104, p. 197.

EINSTEIN NUMBERS

G. A. BAKER, JR., California Institute of Technology

H. T. Davis [1] suggests that we define an operation $\oplus$ as follows:

$$a \oplus b = \frac{a+b}{1+\frac{ab}{c^2}}.$$

Then c has two of the three properties ($A+A=A$; $A+n=A$, n any finite number; and $A^p=A$, p any integer) ascribed to infinity by Cantor, namely

$$c \oplus c = \frac{c+c}{1+\frac{c^2}{c^2}} = \frac{c+c}{2} = c$$

and

$$c \oplus n = \frac{c+n}{1+\dfrac{cn}{c^2}} = c\left(\frac{n+c}{n+c}\right) = c.$$

Einstein makes use of this operation in his theory of relativity. He identifies a and b with relative velocities. It will be interesting to investigate further properties of his numbers.

If R is the set of all real numbers and S is the set of all Einstein numbers (real numbers whose absolute value is less than c), then the function $c \tanh x$ has the following properties:

(1) the domain of $c \tanh x$ is R
(2) the range of $c \tanh x$ is S
(3) $c \tanh x$ is bi-unique
(4) $c \tanh (x+y) = c \tanh x \oplus c \tanh y$, where x and y belong to R, for

$$c \tanh (x+y) = \frac{c \tanh x + c \tanh y}{1 + \dfrac{(c \tanh x)(c \tanh y)}{c^2}} = c \tanh x \oplus c \tanh y.$$

S is closed with respect to $\oplus$. If a and b belong to S then by (2) and (4) we can write $a = c \tanh x$, $b = c \tanh y$, $a \oplus b = c \tanh x \oplus c \tanh y = c \tanh (x+y)$. Therefore, since $|c \tanh (x+y)| < c$, it follows that $|a \oplus b| < c$.

Hence the abelian group consisting of the set, R, of all real numbers and the operation, ordinary addition, is isomorphic under the transformation $c \tanh x$ to the algebraic system consisting of the set, S, and the operation $\oplus$. Therefore S and $\oplus$ form an abelian group.

This isomorphism leads us to define a new operation $\odot$ as follows:

$$a \odot b = c \tanh \left[\left(\tanh^{-1} \frac{a}{c}\right)\left(\tanh^{-1} \frac{b}{c}\right)\right].$$

The abelian group consisting of the set, R_1, of all real numbers except zero and the operation, ordinary multiplication, is isomorphic under the transformation $c \tanh x$ to the algebraic system consisting of the set, S_1, of all Einstein numbers except zero and the operation $\odot$. Therefore, S_1 and $\odot$ form an abelian group. For:′

(1) the domain of $c \tanh x$ is R
(2) the range of $c \tanh x$ is S
(3) $c \tanh x$ is bi-unique
(4) $c \tanh (xy) = c \tanh x \odot c \tanh y$
since

$$c \tanh x \odot c \tanh y = c \tanh \left[\left(\tanh^{-1} \frac{c \tanh x}{c}\right)\left(\tanh^{-1} \frac{c \tanh y}{c}\right)\right]$$
$$= c \tanh xy$$

(5) S_1 is closed with respect to $\odot$ since

$$\left| c \tanh \left[\left(\tanh^{-1} \frac{a}{c}\right)\left(\tanh^{-1} \frac{b}{c}\right)\right] \right|$$

is less than c and is not equal to zero. It equals zero only if $\tanh^{-1} a/c$ or $\tanh^{-1} b/c$ equals zero. However, $\tanh^{-1} x$ equals zero only if x equals zero. Therefore, since zero does not belong to S_1,

$$c \tanh \left[\left(\tanh^{-1} \frac{a}{c}\right)\left(\tanh^{-1} \frac{b}{c}\right)\right]$$

does not equal zero.

The operation $\odot$ is distributive with respect to $\oplus$ for if a, b, d belong to S we may write $a = c \tanh u$, $b = c \tanh v$, $d = c \tanh w$. Then,

$$\begin{aligned} a \odot (b \oplus d) &= c \tanh u \odot (c \tanh v \oplus c \tanh w) \\ &= c \tanh u \odot c \tanh (v + w) \\ &= c \tanh (u(v + w)) \\ &= c \tanh (uv + uw) \\ &= c \tanh (uv) \oplus c \tanh (uw) \\ &= (c \tanh u \odot c \tanh v) \oplus (c \tanh u \odot c \tanh w) \\ &= (a \odot b) \oplus (a \odot d). \end{aligned}$$

We note that c also has the third property of infinity, namely $c^p = c$. For

$$c \odot c \odot \cdots \odot c \odot c = (c \odot c \odot \cdots \odot c) \odot (c \odot c)$$

and

$$c \odot c = c \tanh \left[\left(\tanh^{-1} \frac{c}{c}\right)\left(\tanh^{-1} \frac{c}{c}\right)\right].$$

Since $\tanh^{-1} 1 = \infty$, $(\tanh^{-1} 1)^2 = \tanh^{-1} 1$, then $c \odot c = c \tanh \tanh^{-1} 1 = c$. Hence, by induction, c times itself any integral number of times equals c.

The Einstein numbers form a field under the operations $\oplus$ and $\odot$. The number, c, which may be identified with the velocity of light, acts for this field as infinity does for the real field. The real field is isomorphic to this field under the transformation, $c \tanh x$.

Reference

1. H. T. Davis, College Algebra, Prentice-Hall, Inc., N.Y., 1940, p. 351.

ON THE REMAINDER IN FORMULAS OF NUMERICAL INTERPOLATION, DIFFERENTIATION AND QUADRATURE

HARRY GOHEEN, Iowa State College

W. E. Milne* has developed an approach to the determination of errors in numerical quadrature formulas, differentiation formulas and interpolation formulas based on linear operators. It is the purpose of this note to demonstrate a more elementary approach to this problem which yields the same result and may be given to less mature students.

Following Milne, a formula is called an nth degree formula if it has zero remainder for polynomials of the ith degree for $i=0, 1, 2, \cdots, n$, but not for polynomials of higher degree. The form of such formulas is one of the three:

$$\int_a^b f(x)dx = L_1[f(x_1), \cdots, f(x_k), f'(x_1), \cdots, f'(x_k), \cdots, f^{(s)}(x_1), \cdots, f^{(s)}(x_k)] + r_1, \tag{1}$$

$$f^{(i)}(b) = L_2[f(x_1), \cdots, f(x_k), f'(x_1), \cdots, f'(x_k), \cdots, f^{(s)}(x_1), \cdots, f^{(s)}(x_k)] + r_2, \tag{2}$$

$$f(b) = L_3[f(x_1), \cdots, f(x_k), f'(x_1), \cdots, f'(x_k), \cdots, f^{(s)}(x_1), \cdots, f^{(s)}(x_k)] + r_3, \tag{3}$$

in which L_1, L_2, and L_3 are linear functions of the indicated arguments and r_1, r_2, and r_3 are the remainders. It will be assumed that these formulas are nth degree formulas and that $f(x)$ has an integrable derivative of order $n+1$.

The Taylor expansion, with remainder, of $f(x)$ around $x=a$, is

$$\begin{aligned} f(x) &= f(a) + (x-a)f'(a) + \cdots + \frac{(x-a)^n}{n!}f^{(n)}(a) \\ &\quad + \int_0^{x-a} \frac{(x-a-t)^n}{n!} f^{(n+1)}(a+t)dt. \end{aligned} \tag{4}$$

Then since the formulas (1), (2), and (3) are linear and exact for polynomials of degree n, there result on substitution of (4) into (1), (2), and (3) respectively:

$$\begin{aligned} r_1 &= \int_a^b \int_0^{x-a} \frac{(x-a-t)^n}{n!} f^{(n+1)}(a+t)dtdx \\ &\quad - L_1\left[\int_0^{x_1-a} \frac{(x_1-a-t)^n}{n!} f^{(n+1)}(a+t)dt, \cdots, \right. \\ &\qquad \left. \frac{d^s}{dx^s}\left\{\int_0^{x-a} \frac{(x-a-t)^n}{n!} f^{(n+1)}(a+t)dt\right\}_{x=x_k}\right], \end{aligned} \tag{5}$$

* Milne, W. E. *Numerical Calculus*, Princeton, 1949, pp. 108–116.

$$r_2 = \frac{d^i}{dx^i}\left\{\int_0^{x-a} \frac{(x-a-t)^n}{n!} f^{(n+1)}(a+t)dt\right\}_{x=b}$$

$$(6) \qquad - L_2\left[\int_0^{x_1-a} \frac{(x_1-a-t)^n}{n!} f^{(n+1)}(a+t)dt, \cdots, \frac{d^s}{dx^s}\left\{\int_0^{x-a} \frac{(x-a-t)^n}{n!} f^{(n+1)}(a+t)dt\right\}_{x=x_k}\right],$$

$$r_3 = \int_0^{b-a} \frac{(b-a-t)^n}{n!} f^{(n+1)}(a+t)dt$$

$$(7) \qquad - L_3\left[\int_0^{x_1-a} \frac{(x_1-a-t)^n}{n!} f^{(n+1)}(a+t)dt, \cdots, \frac{d^s}{dx^s}\left\{\int_0^{x-a} \frac{(x-a-t)^n}{n!} f^{(n+1)}(a+t)dt\right\}_{x=x_k}\right].$$

Thus in order to evaluate the remainders, the only equipment necessary is the equipment to evaluate the expressions

$$(8) \qquad \int_0^{x_j-a} \frac{(x_j-a-t)^n}{n!} f^{(n+1)}(a+t)dt,$$

$$(9) \qquad \int_a^b \int_0^{x-a} \frac{(x-a-t)^n}{n!} f^{(n+1)}(a+t)dtdx,$$

and

$$(10) \qquad \frac{d^i}{dx^i}\left\{\int_0^{x-a} \frac{(x-a-t)^n}{n!} f^{(n+1)}(a+t)dt\right\}_{x=x_j}.$$

The expression (9) is a double integral over the triangle in the (x, t) plane bounded by $x=a$, $x=b$, $t=0$ and $t=x-a$. Inversion of the order of integration yields for (9) the expression,

$$(11) \qquad \int_0^{b-a} \int_{t+a}^{b} \frac{(x-a-t)^n}{n!} f^{(n+1)}(a+t)dxdt,$$

or

$$(12) \qquad \int_0^{b-a} \frac{(b-a-t)^{n+1}}{(n+1)!} f^{(n+1)}(a+t)dt.$$

In the expression (10) it may be assumed that i is less than n, since a contrary assumption does not yield a practical formula. Then (10) can be reduced by the rules for differentiating a definite integral with respect to a parameter to

$$(13) \qquad \int_0^{x_j-a} \frac{(x_j - a - t)^{n-i}}{(n-i)!} f^{(n+1)}(a + t)dt.$$

When (8), (12) and (13) are substituted into (5), (6), and (7) there result formulas for the remainders. In these formulas are integrals of polynomials in t times the common function $f^{(n+1)}(a+t)$. If c is the minimum of the limits and d is the maximum of the limits these formulas may be written as integrals between c and d of a piecewise polynomial function multiplied by $f^{(n+1)}(a+t)$. Denoting the piecewise functions by $g_1(t)$, $g_2(t)$, and $g_3(t)$ there result the formulas of Milne,

$$(14) \qquad r_i = \int_c^d g_i(t)f^{(n+1)}(a + t)dt, \qquad i = 1, 2, \text{ and } 3.$$

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, State University of New York

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Harpur College, Endicott, New York. This department welcomes problems believed to be new and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1096. *Proposed by L. R. Ford, Jr., University of Illinois*

As is well known, Lower Slobbovia is too poor a country to afford its own mint. There are N coiners engaged in making Rasbuckniks, the local currency, to government specifications. However it is suspected that some of them may be counterfeiting by introducing some base metal into the alloy. Any pair of counterfeits will weigh the same, although slightly different from the weight of a good coin. Each coiner produces either all good coins or all counterfeits. With one guaranteed good coin, a set of infinitely refinable weights, a beam balance, and as many coins from each coiner as may be needed, determine in three weighings whether any of the coiners is dishonest, and which ones.

E 1097. *Proposed by Leon Bankoff, Los Angeles, Calif.*

Arcs AB and CD are quadrants of circles tangent externally at their midpoints, E, and such that AC and BD when extended meet perpendicularly in F. A circle is inscribed in the mixtilinear triangle EDB, touching ED in M. G is the projection of M upon EF. Show that triangle MGF is a 3:4:5 right triangle.

E 1098. *Proposed by L. A. Ringenberg, Eastern Illinois State College*

Construct all geometries satisfying the following axioms:
Axiom I. Space S is a set of n points, n a positive integer.
Axiom II. A line is a non-null subset of S.
Axiom III. Any two distinct lines have exactly one common point.
Axiom IV. Every point lies in exactly two distinct lines.

E 1099. *Proposed by W. R. Van Voorhis, Fenn College*

On a certain gambling table there are N squares marked "2 to 1," "3 to 1," $\cdots$, "$N+1$ to 1.". A sum of money is placed on each square and one of the squares is selected as a winner, paying the player at the odds marked on that square. The player loses the amounts placed on the other squares. What is the maximum N for which it is possible for a player to place money so that he can never suffer a net loss?

E 1100. *Proposed by L. E. Ward, Sr., Naval Ordnance Test Station, China Lake, Calif., and L. E. Ward, Jr., University of Nevada*

For $x \geqq 0$ and $t \geqq 2$, prove that $(1+x^t)^{1/t}-(1+x^t)^{-1/t} \leqq x$.

SOLUTIONS

Inscription of a Trapezoid in a Quadrilateral

E 1066 [1953, 331]. *Proposed by A. Zirakzadeh, Oklahoma A. and M. College*

Inscribe a trapezoid in a given quadrilateral such that the bases of the trapezoid will be parallel to one of the diagonals of the quadrilateral and the other two sides will pass, respectively, through two given points.

I. *Solution by W. B. Carver, Cornell University.* Let $PQRS$ be the given quadrilateral, the sides PQ, QR, RS, SP being segments respectively of the lines a, b, c, d. Let $ABCD$ be the trapezoid to be constructed, with vertices A, B, C, D respectively on the lines a, b, c, d, with BC and DA parallel to QS, and with AB and CD passing respectively through the given points H and K.

Take A_1 any point on line a; let the line A_1H cut b at B_1, the line through B_1 parallel to QS cut c at C_1, the line C_1K cut d at D_1, and the line through D_1 parallel to QS cut a at A_1'. If A_1' should happen to coincide with A_1 the problem is solved. In any case, A_1 is projected into A_1' by four steps of projection and section. Now take a second point A_2 on a and project it by similar steps into B_2 on b, C_2 on c, D_2 on d, and A_2' on a. Then on the line a the pairs A_1, A_1' and A_2, A_2' are corresponding pairs of a projectivity, and it is easily seen that this projectivity sends the point Q into itself. The two corresponding pairs and the fixed point Q determine this projectivity and the other fixed point exists and may be constructed by a simple *linear* construction. Let the lines $A_1'B_1$ and $A_2'B_2$ intersect at O; then the line HO cuts the line a in the other fixed point A of the projectivity. We project this point A by the four steps indicated above

into B on b, C on c, D on d, and finally into A on a; and $ABCD$ is then the required inscribed trapezoid.

Three special cases may be noted.

1. The second fixed point of the projectivity may happen to coincide with the other fixed point Q. In this case A and B both coincide with Q and C and D both coincide with S. We hestitate to call $ABCD$ a trapezoid in this case, and prefer to say that there is no solution.

2. It may happen that when the point A has been determined, the line AH will be parallel to b. In this case the line DK will also be parallel to c, and there will be no solution.

3. It may happen that A_1' coincides with A_1 and A_2' coincides with A_2, and then *every* point on the line a is a fixed point of the projectivity. In this case we may take A as any point on the line a and obtain a solution $ABCD$, and hence there are an infinite number of solutions.

In the above discussion the broadest interpretation has been given to the words "quadrilateral," "trapezoid," and "inscribed." The given quadrilateral and the trapezoid need not be convex, the inscribed trapezoid need not lie entirely inside the quadrilateral, the vertices A, B, C, D may lie anywhere on the lines a, b, c, d, and not necessarily on the *segments* PQ, QR, RS, SP. If a more restricted meaning is given to the words, the above method of solution still applies, but very frequently there will be no solution in the more restricted sense.

II. *Solution by R. H. Marquis, Ohio University.* Using the lettering of the previous solution one may easily establish, by elementary geometry, the following construction. Draw SK to cut PR in X. Draw XQ. Draw KY parallel to SQ, to cut XQ in Y. Draw HY to cut PQ in A, QR in B, and PR in Z. Draw KZ to cut RS in C and SP in D. Then $ABCD$ is the required trapezoid.

Also solved by Harry Furstenberg, B. Martin, and the proposer.

Formula Relating Factorials, Permutations, and Combinations

E 1067 [1953, 331]. *Proposed by Tatuya Sasakawa, Kyota University, Japan*

Show that

$$n! = \sum_{s=0}^{n} \sum_{i=0}^{s} (-1)^{s-i} {}_nC_s \, {}_sP_i.$$

Solution by A. E. Livingston, University of Washington. Let S_n denote the given double sum. Then

$$S_n = n! \sum_{s=0}^{n} \sum_{i=0}^{s} (-1)^{s-i} {}_{n-i}C_{s-i}/(n-i)!$$

$$= n! \sum_{i=0}^{n} \frac{1}{(n-i)!} \sum_{s=i}^{n} (-1)^{s-i} {}_{n-i}C_{s-i}$$

$$= n!\sum_{i=0}^{n} \frac{1}{(n-i)!}\sum_{j=0}^{n-i} (-1)^j \, {}_{n-i}C_j.$$

Since the last written inner sum represents $(1-1)^{n-i}$ or 1 according as $i\neq n$ or $i=n$, it follows that $S_n=n!$, as asserted in the problem.

Also solved by Harry Furstenberg, H. W. Gould, A. R. Hyde, M. S. Klamkin, David Mandelbaum, William Moser, Michael Skalskyj, Chih-yi Wang, J. V. Whittaker, and the proposer.

Triangles with $s^2=2ab$

E 1068 [1953, 331]. *Proposed by W. O. Pennell, Exeter, N. H.*

Given a triangle with sides a, b, c and $s^2=2ab$, where s is the semiperimeter. Show that: (1) $s<2a$, $s<2b$, (2) $a>c$, $b>c$.

Solution by H. M. Gehman, University of Buffalo. If we substitute $s=(a+b+c)/2$ in $4s^2=8ab$, and then subtract $4ab+4bc$ from each side of the resulting equation, we obtain $(a-b+c)^2=4b(a-c)$, from which it follows that $a>c$. Similarly $b>c$.

We shall use now only the weaker hypothesis that $a\geqq b>c$. From the definition of s and from $a<b+c$, we deduce that $s<b+c<2b$ and that $s<3a/2$. These are stronger statements than those given in the problem.

Also solved by Jack Anderson and Orville Goering (jointly), P. M. Anselone, Leon Bankoff, A. J. Bosgang, W. B. Carver, Fred Discepoli, A. L. Epstein, Harry Furstenberg, Russell Godard and Vern Hoggatt and Robert Jamison (jointly), Douglas Holdridge, A. R. Hyde, M. S. Klamkin, S. Leja, A. E. Livingston, D. C. B. Marsh, George Millman, E. A. Nordhaus, Margaret Olmsted, M. J. Pascual, L. L. Pennisi, L. A. Ringenberg, Azriel Rosenfeld, O. E. Stanaitis, A. V. Sylwester, Chih-yi Wang, and the proposer.

Ringenberg pointed out that the triangle $a=1$, $b=1$, $c=2(\sqrt{2}-1)$ is an example of a triangle of the type considered in the problem. Epstein remarked that if $s^2=nab$, $n>1$, then $s<na$, $s<nb$, $c<(n-1)a$, $c<(n-1)b$; the proof given above for the case $n=2$ is easily generalized to cover this broader situation, and yields the strengthened conclusion that $s<(n+1)a/2$.

Scalenity of a Triangle

E 1069 [1953, 331]. *Proposed by Robert Buehler, Arthur Gregory, and J. R. Wilson, Sandia Corporation, Albuquerque, N. M.*

How un-isosceles can a triangle be?

Solution by W. B. Carver, Cornell University. We must first define some measure of, let us say, the *scalenity* of a triangle. Assuming that no two sides of the triangle are to be equal, we let the lengths of the sides be $a>b>c$. Any measure of scalenity should be homogeneous of degree zero in a, b, c, so that two similar triangles should be equally un-isosceles. As a fairly satisfactory definition we shall

say that the scalenity of the triangle is the smaller of the two ratios a/b and b/c. (The ratio a/c is of course greater than either a/b or b/c.)

Let $x=a/b>1$ and $y=b/c>1$. Then $a=bx$ and $c=b/y$, and we must have $bx<b+b/y$, or $(x-1)y<1$. It follows that either x or y must be less than $(\sqrt{5}+1)/2$, and the scalenity of the triangle must be less than this familiar geometric constant.

Suppose we take an arbitrary line segment BC as the side a of the triangle. Let the point D divide this segment in extreme and mean ratio, so that $BC/BD=BD/DC$. Then we take the vertex A close to D but not, if we are to have a triangle, on the line BC. As we take A closer and closer to D, the scalenity of the triangle increases and approaches the limit $(\sqrt{5}+1)/2$. Thus there is no *most* un-isosceles triangle, because if A does not coincide with D we can always take a new A closer to D and giving a more un-isosceles triangle.

Also solved by C. M. Ablow, A. S. Gregory, A. R. Hyde, M. S. Klamkin, D. C. B. Marsh, A. E. Nordhaus, C. S. Ogilvy, L. A. Ringenberg, and Azriel Rosenfeld.

Editorial Note. Other definitions of *scalenity* might have been used, such as: (1) the smaller of $(a-b)/(a+b+c)$ and $(b-c)/(a+b+c)$, (2) the smaller of $A-B$ and $B-C$, (3) the smaller of A/B and B/C.

The problem suggests further investigations, such as an extension to n-sided polygons and a consideration of the analogous problem in more general metric spaces.

Differential Equation of All Conics of Given Eccentricity

E 1070 [1953, 332]. *Proposed by A. W. Walker, University of Toronto*

Find the lowest order differential equation satisfied by all conics with a given eccentricity e.

Solution by O. E. Stanaitis, St. Olaf College. Let the directrix be the y-axis and let the coordinates of the focus be $(p, 0)$. Then we obtain easily the equation of the general conic of given eccentricity e in the form

$$(x-p)^2+y^2=e^2x^2.$$

Differentiation and elimination of the parameter p leads to the first order differential equation

$$(e^2x-yy')^2=e^2x^2-y^2.$$

Also solved by John Jones, Jr., M. J. Pascual, and the proposer.

Editorial Note. The proposer hoped someone might find a Cartesian differential equation satisfied by all conics in the plane and having given eccentricity e. Halphen has given such a beautifully compact form of a Cartesian differential equation of all conics in the plane that one wonders if a similar achievement cannot be attained for the problem in hand. A Cartesian differential equation

of all parabolas in the plane (the case where $e=1$) is

$$(y''^{-2/3})'' = 0.$$

The set of all conics in the plane having given eccentricity e is given by the four-parameter family

$$(x-a)^2 + (y-b)^2 = e^2(x \cos \omega + y \sin \omega - p)^2,$$

where the point (a, b) and the line $x \cos \omega + y \sin \omega - p = 0$ are associated focus and directrix. Elimination of the parameters a, b, ω, p by successive differentiation would lead to the desired fourth order differential equation.

The proposer, using methods developed in his paper, *The differential equation of a conic and its relation to the aberrancy*, this MONTHLY [1952, pp. 531–538], found an *intrinsic* differential equation satisfied by all conics of given eccentricity e. The other solutions, like that of Stanaitis, restricted the family of conics in one way or another.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known text books or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4568. *Proposed by A. W. Walker, University of Toronto*

It can be shown (*e.g.* by taking y as independent variable) that if $y^5 - 5y^3 + 5y - 5x = 0$, then $(4-25x^2)y'' - 25xy' + y = 0$ (*Mathematical Gazette*, vol. 29, 1945, p. 223). Any three distinct roots of the above quintic equation must therefore be linearly related. Investigate this algebraically, and derive the following factored form of the quintic (readily verified by expansion):

$$(y-y_1)(y-y_2)(y-Ay_1-Ay_2)(y+y_1+Ay_2)(y+Ay_1+y_2) = 0,$$

where $A^2 - A - 1 = 0$.

4569. *Proposed by D. J. Newman, Republic Aviation Corporation, Farmingingdale, N. Y.*

Find explicitly a function $f(x)$ such that $f\{f(x)\}$ is of the order of magnitude of e^x. (In other words, find a function intermediate between x^n and e^x.)

4570. *Proposed by G. A. Dirac, King's College, London, England*

(1) If $d \geqq k \geqq 2$, show that there exist regular connected k-chromatic graphs of degree d and of arbitrarily high order.

(2) If $k \geqq 4$, construct a k-chromatic graph which does not contain a complete k-graph as a subgraph, and in which the degree of every node except one is $k-1$.

(See also problem 4526 [1953, 336]).

4571. *Proposed by Robert Kissling, Student, University of California, Berkeley*

Suppose $a^p + b^p = c^p$ in relatively prime integers with p a prime greater than 3. If q is any prime dividing a^2+ab+b^2 but not $a+b-c$, show that $(q-1)/6p$ is an integer.

4572. *Proposed by Paul Erdös, University of Notre Dame*

Let b_k be any sequence of non-negative real numbers such that

$$\overline{\lim}\, \frac{1}{n} \sum_{k=1}^{n} b_k < \infty.$$

Denote by $f(n)$ the number of b_k's with $1 \leqq k \leqq n$ for which $b_k > 0$. If $\underline{\lim}\, f(n)/n = 0$, show that

$$\sum_{k=1}^{\infty} \frac{b_k}{2^k}$$

is irrational.

SOLUTIONS

An Application of a Jacobi Identity

4501 [1952, 469]. *Proposed by C. D. Olds, San Jose State College, California*

Show that

$$\log (1.1010010001 \cdots) = \frac{1}{11} + \frac{1}{2}\,\frac{1}{101} + \frac{1}{3}\,\frac{1}{1001} + \cdots .$$

Solution by M. R. Spiegel, Rensselaer Polytechnic Institute, Troy, N. Y. From a well known identity we have

$$\frac{(1-x^2)(1-x^4)(1-x^6)\cdots}{(1-x)(1-x^3)(1-x^5)\cdots} = 1 + x + x^3 + x^6 + \cdots, \quad |x| < 1,$$

where the nth term on the right is $x^{n(n-1)/2}$. Taking natural logarithms we have

$$\log (1 + x + x^3 + x^6 + \cdots) = \sum_{n=1}^{\infty} (-1)^n \log (1 - x^n)$$

$$
\begin{aligned}
&= \sum_{n=1}^{\infty} (-1)^{n+1} \left\{ \sum_{p=1}^{\infty} \frac{x^{pn}}{p} \right\} \\
&= \sum_{p=1}^{\infty} \sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{x^{pn}}{p} \right) \\
&= \sum_{p=1}^{\infty} \frac{x^p}{p(1 + x^p)},
\end{aligned}
$$

where the operations are justified by the absolute convergence. The proposed formula results from placing $x=0.1$.

Also solved by Leonard Carlitz, F. J. Duarte, H. M. Feldman, N. J. Fine, Harry Furstenberg, A. M. Gleason, H. W. Gould, Vern Hoggatt, M. S. Klamkin, L. L. Pennisi, O. E. Stanaitis, and Chih-yi Wang.

Isomorphic Groups

4502 [1952, 469]. *Proposed by Karl Goldberg, National Bureau of Standards, Washington, D. C.*

Prove that two groups A and B are isomorphic if and only if there exist operations "$\cdot$" and "$*$" defined over the elements of the groups as follows:

$$
\begin{array}{ll}
a \cdot b \in A & \qquad a \cdot (b * a') = (a \cdot b)a' \\
b * a \in B & \qquad b * (a \cdot b') = (b * a)b'
\end{array}
$$

where a, $a' \in A$; b, $b' \in B$.

Solution by A. M. Gleason, Harvard University. If ϕ is an isomorphism of A onto B, then we can define "$\cdot$" and "$*$" by

$$a \cdot b = a\phi^{-1}(b) \qquad b * a = b\phi(a),$$

and the required relations are easily verified.

Conversely, suppose we are given groups A and B with "$\cdot$" and "$*$" defined satisfying the given relations. Let e_a and e_b be the identity elements of A, B, respectively. From

$$(e_a \cdot b)(a \cdot e_b) = e_a \cdot (b * (a \cdot e_b)) = e_a \cdot ((b * a)e_b) = e_a \cdot (b * a) = (e_a \cdot b)a$$

we deduce that $a \cdot e_b = a$ for all $a \in A$. Hence

$$e_a \cdot (e_b * a) = (e_a \cdot e_b)a = e_a a = a.$$

This, together with the analogous formula $e_b * (e_a \cdot b) = b$ proves that ϕ: $a \rightarrow e_b * a$ is a one to one mapping of A onto B whose inverse is $b \rightarrow e_a \cdot b$. It is an isomorphism of the groups because:

$$\phi(a_1)\phi(a_2) = (e_b * a_1)(e_b * a_2) = e_b * ((a_1 \cdot e_b)a_2) = e_b * (a_1 a_2) = \phi(a_1 a_2).$$

Also solved by S. K. Berberian, T. A. Brown, R. M. Conkling, C. Gugenheim, N. G. Gunderson, T. H. Haynes, Jr., B. C. Kenny, C. E. Lemke and

Raphael Miller, D. C. B. Marsh, Yoshiv Nakamura, F. D. Parker, W. M. Perel, A. Jones Rodriguez, W. A. Rutledge, P. M. Treuenfels, L. M. Weiner, J. R. Wesson, J. V. Whittaker, and the Proposer.

An Inequality Concerning Distances

4503 [1952, 554]. *Proposed by H. S. Shapiro, Bell Telephone Laboratories, Murray Hill, N. J.*

Given n points $P_1, \cdots, P_n$ in $-1 \leqq x \leqq 1$, let τ_k denote the product of the distances from P_k to the other points. Prove

$$\sum_{k=1}^{n} \frac{1}{\tau_k} \geqq 2^{n-2},$$

and equality is attained for suitable P_i.

Solution by the Proposer. Denote the points by $x_1, \cdots x_n$ and set

$$f(x) = (x - x_1)(x - x_2) \cdots (x - x_n).$$

Now let $T(x)$ denote the Tchebychef polynomial of order $n-1$

$$T(x) = 2^{n-2}x^{n-1} + \cdots, \ T(\cos\theta) = \cos(n-1)\theta,$$

whence $|T(x)| \leqq 1$ for $-1 \leqq x \leqq 1$. Then by Lagrange interpolation we can write

$$T(x) = \sum_{k=1}^{n} \frac{f(x)}{(x - x_k)f'(x_k)} \cdot T(x_k),$$

and on equating coefficients of x^{n-1} we get

$$2^{n-2} = \sum_{k=1}^{n} \frac{T(x_k)}{f'(x_k)}, \qquad 2^{n-2} \leqq \sum_{k=1}^{n} \frac{1}{|f'(x_k)|}.$$

But $|f'(x_k)|$ is precisely the number τ_k defined above.

It is easily verified that equality holds if and only if the P_i are the points where $T(x)$ attains its maximum,

$$x_i = \cos\frac{(i-1)\pi}{n-1}.$$

Also solved by Robert Breusch and Fritz Herzog.

Postulates for a Group

4504 [1952, 554]. *Proposed by Olga Taussky, National Bureau of Standards Washington, D. C.*

Prove that a set S which is closed under an associative composition law which satisfies the following three axioms is a group:

1. There exists an idempotent e such that $e^2 = e$.

2. Every element has at least one left inverse with respect to e.
3. Every element has at most one right inverse with respect to e.

Solution by D. O. Ellis, University of Florida. Let x be any element of S and let y be some left inverse of x with respect to e. Then $yx=e$ and $e=e^2=(yx)(yx)=y(xyx)$. Hence, by uniqueness of right inverses, $x=xyx=xe$. The result now follows from the well known theorem: *a semigroup with a right identity and left inverses with respect to it is a group.*

To finish the proof independently of this theorem, let $zy=e$. Then $ex=zyx=ze=z$. Also, $exy=zy=e$. But, $e^2=e$, and so $xy=e$. Hence, $ex=xyx=xe=x$ and we have shown that e is an identity element and that left inverses are also right inverses, which conforms to the classic definition of group.

Also solved by Edward Assmus, Felice D. Bateman, T. A. Brown, M.O' N. Campbell, G. E. Collins, R. M. Conkling, F. E. Cothran, Kathryn Ellis, N. J. Fine, D. T. Finkbeiner, W. T. Fishback, Harley Flanders, Henry Furstenberg, J. W. Gaddum, Harry Goheen, E. E. Grace, P. S. Herwitz, J. Horváth, J. M. Hurt, Jack Indritz, P. W. M. John, Philip Johnson, B. C. Kenny, C. E. Lemke and R. Miller, T. C. Littlejohn, A. E. Livingston, D. C. B. Marsh, H. C. Miller, Jr., Kovina Milosevich, J. H. Oppenheim, F. D. Parker, R. F. Pavley, D. K. Pease, W. O. Portmann, G. B. Preston, D. W. Robinson, A. J. Rodriguez, Azriel Rosenfeld, W. A. Rutledge, D. W. Sasser, James Singer, W. A. Small, R. H. Sprague, F. W. Stallard, W. L. Stamey, D. R. Sudborough, G. H. M. Thomas, Peter Truenfels, T. A. Trumpler, L. M. Weiner, J. V. Whittaker, H. S. Wilf, J. E. Wilkins, Jr., A. B. Willcox and J. D. Baum, H. C. Wiser and A. T. Lundell, Jerrold Yos, and the Proposer.

A Trigonometric Series

4505 [1952, 554]. *Proposed by M. R. Spiegel, Rensselaer Polytechnic Institute, Troy, N. Y.*

1. Show that if $\theta_n=\pi/2\cdot 3^n$, then

$$\sum_{n=1}^{\infty}\frac{\theta_n \sin\theta_n \sin 2\theta_n}{\sin 3\theta_n}=\frac{1}{4}\cdot$$

2. Show that if ϕ_p is the pth positive root of $\tan x=x$ and $\theta_n=\phi_p/3^n$, then

$$\sum_{n=1}^{\infty}\frac{\theta_n \sin\theta_n \sin 2\theta_n}{\sin 3\theta_n}=0.$$

Solution by Chih-yi Wang, University of Minnesota. Let S_N be the partial sum of N terms of the given series in both cases. Define $\theta_0=\pi/2$ in case 1, and $\theta_0=\phi_p$ in case 2. By using the identity

$$4\sin y\sin 2y/\sin 3y = \cot y - 3\cot 3y$$

we obtain, for all sequences $\{\theta_n\}$ where $\theta_{n+1}=\theta_n/3$,

$$S_N = \frac{1}{4}\sum_{n=1}^{N}\left[\theta_n \cot\theta_n - \theta_{n-1}\cot\theta_{n-1}\right] = \frac{1}{4}\left[\theta_N \cot\theta_N - \theta_0\cot\theta_0\right].$$

This gives

$$\frac{1}{4}\theta_N \cot\theta_N, \qquad \frac{1}{4}\left[\theta_N\cot\theta_N - 1\right]$$

in the respective cases. Since, in both cases, $\theta_N \cot\theta_N \to 1$ as $N\to\infty$, the stated results follow.

Also solved by N. J. Fine, Harry Furstenberg, Calvin Foreman, O. E. Stanaitis, John Todd, Peter Ungar, and the Proposer.

Numbers Whose Factors Belong to a Given Sequence

4506 [1952, 554]. *Proposed by Paul Erdös, Notre Dame University*

Let $a_1 < a_2 < \cdots$ be an infinite sequence of real numbers, $\sum 1/a_i < \infty$. Denote by $n_0 = 1 < n_1 < n_2 < \cdots$ the numbers of form $\prod a_i^{\alpha_i}$, where α_i are non-negative integers. Assume the number of $n_i \leqq x$ equals

$$\frac{x}{f(x)} + o\left(\frac{x}{f(x)}\right)$$

where $f(x)$ is increasing and $f(x^2)/f(x) < c$, $f(2x)/f(x)\to 1$. Show that the number of $a_i \leqq x$ equals

$$\frac{x}{f(x)\sum_{i=0}^{\infty}\frac{1}{n_i}} + o\left(\frac{x}{f(x)}\right).$$

The conditions of the theorem are satisfied if, for example, $f(x) = (\log x)^c$, $c > 1$.

Solution by the Proposer. Denote by $g(x)$ the number of $n_i \leqq x$. We assume that

$$(1) \qquad g(x) = \frac{x}{f(x)}, \qquad f(x) \text{ increasing}, \qquad \frac{f(2x)}{f(x)} \to 1.$$

$f(2x)/f(x)\to 1$ implies $f(x) = o(x^\epsilon)$, thus

$$(2) \qquad g(x) > x^{1-\epsilon}.$$

Consider

$$h_k(x) = g(x) - \sum_{i=1}^{k} g\left(\frac{x}{a_i}\right) + \sum_{\substack{1\leq i,j\leq k\\ i\neq j}} g\left(\frac{x}{a_ia_j}\right) + \cdots + (-1)^k g\left(\frac{x}{a_1a_2\cdots a_k}\right).$$

$h_k(x)$ clearly equals the number of integers $\leqq x$ composed of $a_{k+1}, a_{k+2}, \cdots$. Denote by $A(x)$ the number of a's $\leqq x$. Clearly $h_k(x)+k \geqq A(x)$, and it follows from $f(2x)/f(x) \to 1$ and $f(x) \uparrow$, that for any c, $f(cx)/f(x) \to 1$. Thus

$$g\left(\frac{x}{d}\right) = \frac{x}{d} \Big/ f\left(\frac{x}{d}\right) = (1 + o(1)) \frac{x}{df(x)}$$

and hence

$$h_k(x) = \left\{ \prod_{i=1}^{k} \left(1 - \frac{1}{a_i}\right) \right\} g(x) \cdot (1 + o(1)) = (1 + o(1)) \frac{x}{f(x)} \prod_{i=1}^{k} \left(1 - \frac{1}{a_i}\right).$$

Letting $k \to \infty$ we obtain from $h_k(x)+k \geqq A(x)$ and the convergence of $\sum 1/a_i$ that

$$A(x) \leqq (1 + o(1)) \frac{x}{f(x)} \prod_{i=1}^{\infty} \left(1 - \frac{1}{a_i}\right) = (1 + o(1)) \frac{x}{f(x) \sum_{i=0}^{\infty} \frac{1}{n_i}} \cdot \tag{3}$$

On the other hand we have

$$A(x) > h_k(x) - \sum{}^* g\left(\frac{x}{a_i}\right) \tag{4}$$

where the * indicates that the summation is extended over the a_i satisfying $a_k < a_i \leqq x^{1/2}$. (4) is clear since the left member is greater than the number of a's in the interval $(x^{1/2}, x)$, that is, all n's divisible by a smaller a have been omitted. Thus we have

$$A(x) > (1 + o(1)) \frac{x}{f(x) \sum_{i=0}^{\infty} \frac{1}{n_i}} - \sum{}^* g\left(\frac{x}{a_i}\right).$$

Now we only have to show that as $k \to \infty$

$$\sum{}^* g\left(\frac{x}{a_i}\right) = o\left(\frac{x}{f(x)}\right).$$

This estimate can be obtained as follows:

$$\sum{}^* g\left(\frac{x}{a_i}\right) = \sum_{a_k < a_i \leqq x^{1/2}} g\left(\frac{x}{a_i}\right) < \frac{1}{f(x^{1/2})} \sum_{a_k < a_i < x^{1/2}} \frac{x}{a_i}$$
$$< c \frac{x}{f(x)} \sum_{i=k+1}^{\infty} \frac{1}{a_i} = o\left(\frac{x}{f(x)}\right).$$

The attempt to dispense with the condition $f(x^2)/f(x) < c$ has not yet succeeded.

RECENT PUBLICATIONS

Edited by E. P. Vance, Oberlin College

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, Oberlin College, Oberlin, Ohio, and not to any of the other editors or officers of the Association.

Fundamentals of College Mathematics. By R. E. Johnson, N. H. McCoy and Anne F. O'Neill. New York, Rinehart and Company, Inc., 1953. xiv+479 pages.

This text is designed for a first year course in college mathematics. It contains more than enough material for such a course. It is written with care and with rigor and contains an adequate number of exercises for a text in which ideas rather than techniques are stressed.

The main objective of the book is the introduction of the calculus. Chapters on analytic geometry and on algebraic topics are presented as they are needed for the study of the calculus. Chapter 1 discusses the number concept and some basic ideas of analytic geometry. Chapter 2 is devoted entirely to elementary trigonometry which may be omitted without destroying the continuity of the book. There is a short chapter (5) on mathematical induction which is followed by two chapters introducing the calculus. Chapter 8 is a study of polynomial functions and equations. Chapters 12 and 13, concerning conic sections and transformations of axes, interrupt the continuity of the treatment, since little or no use is made of the calculus in these chapters.

Proofs of theorems are given when feasible, otherwise it is clearly stated that they are assumed because the proofs are beyond the scope of the book. The proofs of some of the theorems are indicated as optional by being given in small print. The treatments of limit, continuity, and the definite integral as the limit of a sum are more thorough and rigorous than is usual in an elementary text. Despite the fact that many students will find difficulty with chapters 6 and 10 in which these concepts are presented, an experienced instructor should find pleasure in meeting this challenge. The student who uses this book as a text will not be required to revise his ideas concerning fundamental concepts as he continues his study of mathematics even though he does not fully grasp their significance and importance during this first experience with them.

The treatment of the integral calculus differs from that of the usual text in that the antiderivative is not introduced until after a complete discussion of the definite integral as the limit of a sum.

The authors are to be congratulated on writing an excellent book. It should be seriously considered for use as a text by those who are seeking to raise the level of their first year college mathematics course.

V. H. Wells
Williams College

The Teaching of Secondary Mathematics. By C. H. Brown, Harper and Brothers, New York. Cloth, xi+388 pages, 1953. $4.00.

The principal novelty of this new and challenging book on the teaching of mathematics is its emphasis on the problem of the place of mathematics in education. Almost one-half of the book is devoted to this topic whereas in other books in the field the same topic takes up not more than from one-twentieth to one-seventh of the book.

This pre-occupation (almost obsession) of the author with philosophy makes the book more one on the philosophy of the teaching of mathematics than one on methods of teaching it. The author, of course, knows exactly what he is about, for he declares in his preface: "The function of a book on the teaching of mathematics is, therefore, to provide an analysis of educational theory as applied to the teaching of secondary mathematics; to present the considerations which should govern choice of subject matter and of method; and to suggest possible methods of solving typical teaching problems." (p. x)

The analysis presented by the author is superbly done. It is at its best in the discussion of the great controversy between the doctrine of specific objectives and the program of the "progressives." Clearly, the partisans of neither school will like Professor Brown's conclusion that the controversy has served: " . . . to increase rather than diminish the confusion regarding the place and function of mathematics in secondary education." (p. 116)

Professor Brown's own proposed resolution of the conflict is that the function of subject matter in education is to impart information about the world and to stimulate reflection upon the problems presented by the world. Now mathematics has great potentialities for contributing powerfully to both of these objectives. But these are potentialities, not peculiar virtues inherent in the subject. The teacher's function is to realize these potentialities.

It is a consequence of the author's approach to his subject that the remainder of the book may be found lacking in those specific kinds of help which the new teacher, particularly, looks for in a book on methods. Here Professor Brown again anticipates this possible criticism, for he says: "This book is based on the assumption that teaching method is individual; that a teacher who has an adequate knowledge of the subject matter of his field, who knows the needs, aspirations, and potentialities of his students, who understands the nature of the learning process, and whose philosophy of education is appropriate to a democratic society will be able to devise methods and techniques suited to his particular situation." (p. x) (It may be suggested parenthetically that these are no small requirements for teaching. They are, of course, the necessary and sufficient conditions for creative teaching.)

The principles for the guidance of the teacher in selecting subject matter and in devising methods are then presented in a chapter on demonstrative geometry, two chapters on the ninth year (ninth-year algebra and ninth-year general mathematics), and one on the other courses in the normal secondary

program in secondary mathematics. Chapters on general methods, evaluation, and the professional preparation of teachers complete the book.

"The Teaching of Secondary Mathematics" is a valuable addition to the professional library of any teacher of mathematics and it should prove to be a stimulating text-book for teachers in training. The young teacher, however, may find it necessary and advisable to supplement it with a more traditional text on methods. It is noteworthy that in its emphasis on understandings and concepts in the field of mathematical education the book is in harmony with the similar contemporary stress on concepts, meanings and understandings in the teaching of the subject matter of mathematics itself.

J. H. Hlavaty
New York City

Mathematics of Investment. By P. R. Rider and C. H. Fischer. New York, Rinehart. 1951. viii+228 pages+Tables+Index. $5.00.

Mathematics of Finance. By L. L. Smail. New York, McGraw-Hill, 1953. x+189 pages+Appendix+Tables+Answers+Index. $4.50.

Fundamental Procedures of Financial Mathematics. By M. Rassweiler and I. Rassweiler. New York, Macmillan, 1952. vi+236 pages+Appendix +Answers+Index. $3.25.

The first two of these books handle the subject of investment mathematics in the traditional algebraic manner, while the third uses arithmetic computations only.

The usual topics of simple and compound interest and discount, amortization and sinking funds, depreciation, bonds, life annuities and insurance are clearly and completely presented by both Smail and Rider and Fischer. Many problems, with answers to the odd numbered ones, appear in each book, Smail giving answers in the back of the book and Rider and Fischer immediately after the problems.

Rider and Fischer depart from tradition in several respects in the arrangement of the tables. Five compound interest and annuity functions are listed on a page under a given interest rate. This has obvious advantages since problems using several functions usually involve only one interest rate. Tabular values of $s_{\overline{n|}}^{-1}$ are listed instead of the usual $a_{\overline{n|}}^{-1}$; the reasons given for this change are the presence of $s_{\overline{n|}}^{-1}$ in the general annuity formulas and the slight advantage in replacing the subtraction of the interest rate by addition in obtaining the reading for the other reciprocal annuity factor. Rider and Fischer also include lower than usual interest rates in the tables to comply with modern lower market rates. Smail's tables follow the usual pattern for the compound interest and annuity functions. Both books use the Commissioner's 1941 Standard Ordinary Mortality Table based on $2\frac{1}{2}\%$ interest rate. Rider and Fischer supplement this with the 1937 Standard Annuity Table also based on $2\frac{1}{2}\%$, but

with male and female entries. Both texts have the customary six-place logarithms and seven-place logarithms of interest ratios.

Rider and Fischer have made as practical and teachable as possible the chapters on general annuities, bonds, life annuities and insurance. Smail has an appendix devoted to pertinent topics from algebra. In conclusion both texts are quite suitable for the standard course in investment mathematics.

The Rassweiler and Rassweiler text requires no algebraic preparation. The procedure for each topic is thus: a short paragraph on definitions and basic ideas, a well emphasized statement of the main problem, an outline of the method of solution, an example stated and solved with the steps numbered and named in columns parallel to the work, and finally a set of problems, with answers to the odd numbered ones appearing in the back of the book. For the weaker student and one without algebraic preparation this technique seems good as long as the problems fall into one of the type forms. The better student, however, would profit more from a course using a text based upon algebra.

The usual topics of investment mathematics are treated with the exception of depreciation and capitalized cost. In addition there are chapters on taxation (property, income, excise and sales), commission, pricing and profits, fire and automobile insurance. The Commissioner's 1941 Table is included, but for the work on compound interest and annuities certain the student must get his own table.

J. O. Blumberg
University of Pittsburgh

Demand Analysis. By Herman Wold in association with Lars Jureen. New York, John Wiley and Sons, Inc., 1953. xvi+358 pages. $7.00.

This book is concerned with the behaviour of a consumer (or some particular set of consumers) when faced with economic choices. The larger part of it is given to a discussion of statistical demand analysis. Typically, in this type of analysis, it is supposed that the quantity of a particular commodity which consumers buy is determined causally by the economic conditions in which they find themselves. For example, if q_0 is the amount of a particular commodity bought, a useful hypothesis might be that q_0 is determined by the relation

$$q_0 = f(i, p_0, p_1, \cdots, p_n, \epsilon) \tag{1}$$

where i is the consumers' disposable income, $p_0, p_1, \cdots, p_n$ are the prices of the commodity and other commodities which the consumer might consider, and ϵ is a variable intended to represent the haphazardness of behaviour at different times. Usually ϵ is taken to be a random variable. Very often (1) is supposed to be a linear relation between the logarithms of the variables:

$$\log q_0 = a \log i + \sum_{j=0}^{n} b_j \log p_j + \log \epsilon.$$

It is then the task of statistical demand analysis to determine $a, b_0, \cdots, b_n$ from available data. Thus the subject requires the use of much probability and statistical theory.

The present book is divided into five parts. Part I is largely non-mathematical. It gives a defense of the use of classical least squares techniques in demand analysis. In Part II is a fairly extensive mathematical treatment of the deterministic theory of choice. Part III gives a summary of the properties of stationary random processes, and Part IV a summary of some theory of regression. Part V contains empirical results obtained from Swedish data.

Particularly interesting is Part III. This gives a summary of results on stochastic processes, including some recent work, which it is useful to have in such an accessible form. Part IV contains some interesting material on regression. These two parts of the book are more or less self-contained, and will be useful to statisticians generally.

As an exposition of its subject, the book is rather uneven, however. Throughout, it would be heavy going for a reader who was not already fairly well acquainted with the various aspects of its subject matter, and with the controversy about them. Part I, which is supposed to serve as an introduction, contains a particular point of view of what is the appropriate statistical method to use in analysing demand data. Professor Wold's contribution to this controversy is, in fact, a very important one, especially where he emphasizes the use of causal chains in setting up models of an economic system. Even with such models, however, the interactions between some of the variables may be much more rapid than those between others, and this fact leads to statistical difficulties in the analysis of data. Although Wold mentions these, he seems to dismiss them much too lightly.

The treatment of the theory of preference fields given in Part II is excellent. One flaw however, is in the discussion of the integrability condition, which is sketchy and seems rather misleading. The whole subject is an interesting one for mathematicians, and is still developing.

Summing up, the book, presenting as it does Professor Wold's contributions and personal views, is an important contribution to the literature of econometrics and mathematical economics.

C. B. WINSTEN
Cowles Commission and Oxford University

Theorie der Geometrische Konstruktionen. By Ludwig Bieberbach. Verlag Birkhauser, Basel, Switzerland. 1952. vi+162 pages. Bound, Fr. 18.70, unbound, Fr. 15.60.

Part of the author's purpose as stated in his preface is to make known the implications of the theory of geometric constructions to all mathematics. In this he is successful, as the reader will find material from projective and analytic geometry, the theory of algebraic equations, the structure of algebraic numbers, and number theory. For one section, extensive use is made of function theory.

The first twelve sections of the book deal with ruler and compass constructions in various combinations and modifications. Included are the Poncelet-Steiner constructions, and the Mohr-Mascheroni theory. In general, all quadratic constructions can be treated with these instruments.

Sections 13–15 still deal with ruler and compass constructions so far as they apply to the trisection of angles and the construction of regular polygons. In general, an angle cannot be trisected, but it is not always appreciated that there is an infinite set of angles whose trisection yields to ruler and compass. The investigation of regular polygons leads to a study of the cyclotomic equations involved and finally to the conclusion that regular polygons can be constructed if the number of their vertices is a prime of the form 2^p+1, the product of any two such numbers, or the product of any of these by a power of two.

With Section 16 begins the study of instruments which solve construction problems leading to third and fourth degree equations. The insertion-ruler allows the insertion of a fixed segment between two lines or circles while the ruler passes through a fixed point or is tangent to a circle. Trisection becomes simple, the duplication of the cube is achieved, and the construction of a regular heptagon becomes possible. The relation of this instrument to the conchoid and the limacon serves for analytic background. The carpenter's square with or without marked segments solves these problems and, in fact, the solution of the general cubic equation.

From Section 21 on, much use is made of the two pointed compass, involving a trial and error process which seems a far cry from the strict Euclidean constructions. But the author assures the reader that it is common practice among draftsmen and that he intends to use it as a formal, mathematical operation. A whole new field is opened up dealing with conics. The final instrument (Section 25) is the most general of all, namely, a sheet of transparent material on which any configurations may be traced and which may then be moved freely over the drawing board. Certainly a non-classical instrument, but here is Hjelmslev's Theorem: *A transparent sheet marked with a curve, e.g., a pair of perpendicular lines, in connection with the two point compass is sufficient to solve any construction problem which leads to an algebraic equation of any degree.*

The remainder of the book deals with approximate constructions, constructions on a sphere, and also presents Gelfond's proof of the transcendence of π. There are a few annoying misprints in relation to the figures. To this reviewer, there seems to be a curious unevenness in the presentation, in that simple processes are spelled out letter by letter whereas more complicated trains of thought are left to the reader's mathematical experience. All in all, it is a fine book to be recommended to all interested in geometry.

E. S. HAMMOND
Bowdoin College

CLUBS AND ALLIED ACTIVITIES

EDITED BY H. D. LARSEN, Albion College

NEW MAGAZINE FOR HIGH SCHOOL MATHEMATICS STUDENTS

For many years secondary school teachers of mathematics have felt a need for a journal written especially for the high school student. The National Council of Teachers of Mathematics, in cooperation with the Mathematical Association of America, is attempting to meet this need by the publication of a new journal called *The Mathematics Student Journal.*

The Mathematics Student Journal will contain enrichment and recreational material not found in the ordinary text-books. Alert teachers of mathematics have long been aware of the unlimited amount of enriching and stimulating material available on an elementary level in mathematics, but they have found that the press of busy schedules makes it difficult to look up this material and prepare it in a form for presentation to students. The new journal will aid greatly in solving this problem. Moreover, the journal will encourage the gifted student to continue his study of mathematics and will describe the many excellent opportunities awaiting him in the mathematical profession. A special feature will be a problem department to which students may contribute both problems and solutions. Because of its nature the new journal will appeal also to many adults with an interest in mathematics.

The Mathematics Student Journal will be edited by H. D. Larsen of Albion College. He will be assisted by Frank B. Allen of Lyons Township High School and Junior College, La Grange, Illinois. A second associate editor is to be named by the Association. Readers of this MONTHLY are invited and urged to submit material for publication in the new journal.

The Mathematics Student Journal will be issued four times a year during the months of October, December, February, and April. The first issue will be distributed in February, 1954. The subscription price will be 20¢ per year or 15¢ per semester. However, mailing will be done only in bundles of five copies or more, since the low subscription price does not permit the mailing of individual copies. Teachers should obtain subscriptions for their students and submit them in a group, all orders in a group running for the same period of time and being mailed to the same address. Subscriptions should be addressed to: National Council of Teachers of Mathematics, 1201 Sixteenth Street, N. W., Washington 6, D. C.

Editorial Note. Because of his new duties as editor of the above journal, Professor Larsen is retiring as editor of the Clubs Department of this MONTHLY. Since many of the functions of this department will be assumed by the new journal, publication of the Clubs Department will be discontinued after this issue.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

COOPERATIVE GRADUATE SUMMER SESSIONS IN STATISTICS

Beginning in 1954, North Carolina State College, the University of Florida, Virginia Polytechnic Institute and the Southern Regional Education Board will sponsor cooperative Graduate Summer Sessions in Statistics. The summer sessions will be of particular interest to the following: (1) research and professional workers who want intensive instruction in basic statistical concepts and who wish to learn modern statistical methodology; (2) teachers of elementary statistical courses who want some formal training in modern statistics; (3) prospective candidates for graduate degrees in statistics; (4) graduate students in other fields who desire supporting work in statistics; and (5) professional statisticians who wish to keep informed of advanced specialized theory and methods.

The courses are arranged to enable the person to take consecutive work in successive summers. The summer work in statistics may be applied at any one of the cooperating institutions in partial fulfillment of the requirements for a Master's degree. The catalog requirements for the degree must be met at the degree-granting institutions. Each doctoral candidate should consult with the institution from which he desires to obtain the degree regarding the applicability of the summer courses in statistics.

The first session will be conducted at Virginia Polytechnic Institute, June 9–July 17, 1954. Professor Maurice Kendall of the University of London will give a course in multi-variate analysis; Dr. Ralph Comstock of North Carolina State College will give one in quantitative genetics. The staff of the Department of Statistics of Virginia Polytechnic Institute will offer courses in probability and inference, analysis of variance, statistical methods, engineering statistics, education statistics, rank order statistics and the theory of sequential methods.

Inquiries should be addressed to Boyd Harshbarger, Head, Department of Statistics, Virginia Polytechnic Institute, Blacksburg, Virginia.

"MATHEMATICS IN PUBLIC HIGH SCHOOLS"

A new publication entitled *Mathematics in Public High Schools*, Office of Education Bulletin 1953, No. 5, has been prepared by Dr. K. E. Brown. In this pamphlet such questions as the following are discussed. Are enough pupils taking mathematics in high school so our national supply of scientists and engineers can continue to increase? Are there as many pupils enrolled in mathematics in the secondary public schools as there were ten years ago? What mathematics, if any, is required of the pupils in the public high schools? In addition, the publication gives data about field trips, length of class periods, size of classes, number of teachers, and other pertinent information about mathematics in grades seven

to twelve. The pamphlet may be obtained by writing to: Superintendent of Documents, United States Government Printing Office, Washington 25, D. C.; price 20 cents.

PERSONAL ITEMS

Professor H. E. Bray of Rice Institute represented the Association at the inauguration of President Logan Wilson of the University of Texas on October 29, 1953.

Professor Dora McFarland has received a University of Oklahoma Foundation Teaching Award. This award of $500 was made for "extraordinary excellence in student counseling and teaching of freshmen and sophomores."

Agricultural and Mechanical College of Texas announces the following: Mr. M. L. Coffman and Mr. R. E. Collins, formerly students at the College, have been appointed to instructorships; Instructor T. Y. Hicks has resigned and is now an applied scientist with International Business Machines Corporation.

Bowling Green State University reports: Mrs. Florence S. Ogg has been appointed Lecturer; Associate Professor W. F. Cornell is on leave of absence for the year 1953–54 and is attending Ohio State University.

At Carnegie Institute of Technology: Professor A. E. Heins has been awarded a John Simon Guggenheim Fellowship and is studying in Denmark; Dr. H. G. Cohen, formerly a research associate at Hebrew Institute of Technology, Haifa, Israel, has been appointed to an assistant professorship.

At The Catholic University of America: Dr. Benjamin Lepson, formerly a mathematician with the Office of Naval Research, Washington, D. C., has been appointed to an assistant professorship; Dr. N. A. Wiegmann, previously a research mathematician at the National Bureau of Standards, Washington, D. C., has been appointed Visiting Professor.

Central Michigan College of Education announces the following: Assistant Professors D. R. Sudborough and H. W. Zeoli have been promoted to associate professorships; Miss Nikoline A. Bye has been elected Vice-President of the Michigan Council of Teachers of Mathematics for the year 1953–54.

Florida State University makes the following announcements: Dr. B. F. Hadnot, formerly mathematics teacher at Eglin Air Force Base, Florida, Dr. R. N. Tompson, previously a graduate student at Brown University, and Mr. A. V. Fend, recently a graduate student at the University of Illinois, have been appointed to assistant professorships; Miss Josephine Story, previously a graduate student at the University, has been appointed to an instructorship; Instructor J. M. Plant has been promoted to an assistant professorship.

At Illinois Institute of Technology: A series of eight monthly lectures for high school science teachers has been planned for the year 1953–54 to keep teachers informed of latest developments in their fields; Professor Karl Menger has given one of these lectures entitled "Geometry and Scientific Thought"; Professor Gordon Pall will give a lecture, "The Role of Amateurs in Number-Theory."

Massachusetts Institute of Technology announces: Professor Warren Ambrose was on leave during the fall semester of 1953–54 at Tata Institute of Fundamental Research, Bombay, India; Professor Witold Hurewicz and Professor Raphael Salem were also on leave during the fall semester of 1953–54 on Guggenheim Fellowships; Dr. J. F. Nash, Jr., previously C. L. E. Moore instructor, has been promoted to an assistant professorship; Dr. D. L. Wallace, formerly a graduate student at Princeton University, has been appointed C. L. E. Moore instructor.

McGill University reports the following: Assistant Professor R. T. Sharp of the University of Alberta has been appointed to an assistant professorship; Mr. P. J. Lanfer and Mr. C. D. McKay have been appointed Lecturers; Professor W. L. G. Williams has retired.

At Washington Square College, New York University: Dr. W. M. Hirsch, previously lecturer at Columbia University, has been appointed to an assistant professorship; Research Associate Professor Bernard Friedman has been promoted to a research professorship; Assistant Professor David Gans is on leave during 1953–54 on a Ford Foundation Fellowship; Research Associate Professor Keller is on leave during 1953–54 and is Head of the Mathematics Branch, Office of Naval Research.

Princeton University announces the following: Professor Charles Ehresmann of the University of Strasbourg was Visiting Professor of Mathematics during the fall term; Associate Professor Richard Otter of Notre Dame University is Visiting Lecturer during 1953–54; Dr. Felix Haas of Lehigh University has been appointed Henry Burchard Fine Instructor; Dr. L. S. Shapley and Dr. J. L. Snell have been reappointed Fine Instructors for the year 1953–54; D. A. Buchsbaum, Columbia University, and R. H. Crowell, Herbert Forrester, George Hufford and R. J. Semple, formerly graduate students at the University, have been appointed to instructorships.

Rutgers University announces: Assistant Professor Albert Wootton of Champlain College has been appointed Lecturer; Mr. N. R. Stanley, previously mathematician with the Republic Aviation Corporation, has been appointed to an instructorship; Professor R. E. Luce has received a Ford Foundation Grant for research abroad.

Southwestern Louisiana Institute reports: Assistant Professor P. M. Tullier, Jr., of Loyola University, Louisiana, has been appointed to an associate professorship; Mr. R. E. Allan, formerly of Tulane University, and Miss Margaret M. LaSalle, previously of Francis T. Nicholls Junior College, Louisiana State University, have been appointed to assistant professorships.

St. John's College announces the promotion of Assistant Professor A. H. Sarno to an associate professorship and of Instructor E. J. Germino to an assistant professorship.

Stanford University reports the following: Assistant Professor G. E. Latta of the University of British Columbia has been appointed Acting Assistant

Professor; Professor George Polya has retired with the title of Professor Emeritus.

At Stevens Institute of Technology: Professor L. Z. Pollara of Sienna College has been appointed to an associate professorship; Mr. M. E. White, recently with the United States Air Force, has been appointed to an instructorship.

The U. S. Naval Academy announces the following: Assistant Professor O. M. Thomas has been promoted to an associate professorship; Professor J. B Eppes has retired; Professor L. M. Kells has retired with the title of Professor Emeritus.

University of Arizona makes the following announcements: Mr. C. E. Aull, previously a graduate assistant at the University of Oregon, Miss Miriam J. Russell, formerly a graduate student at George Peabody College for Teachers, and Assistant Professor A. H. Steinbrenner of the United States Naval Academy, have been appointed to instructorships; Professor H. B. Leonard has retired with the title of Professor Emeritus.

The University of British Columbia announces: Dr. R. J. Wisner has been appointed to an instructorship; Associate Professor D. C. Murdoch has been promoted to a professorship.

The University of Chicago announces the following appointments to instructorships: Dr. Maurice Auslander, previously at Columbia University, and Dr. Serge Lang, formerly at the Institute for Advanced Study.

University of Cincinnati reports the following: Research Associate Arno Jaeger of the University of Illinois has been appointed to an associate professorship; Assistant Professor J. L. Baker, Jr., and W. E. Restemeyer have been promoted to associate professorships; Professors C. N. Moore and E. S. Smith have retired with the title of Professor Emeritus; during the first semester of 1953–54, Dr. Konrad Knopp of the University of Tübingen and Dr. Wolfgang Jurkat were in residence at the University.

University of Georgia makes the following announcements: Assistant Professor M. K. Fort, Jr., of the University of Illinois has been appointed to an associate professorship; Mr. B. K. Youse of Memphis State College has been appointed to an instructorship; Professor G. B. Huff has been appointed Head of the Department of Mathematics.

The University of Hawaii announces: Mr. M. J. Vitousek of Stanford University has been appointed to an instructorship; Associate Professor Christopher Gregory, who is also Chairman of the Department of Mathematics, has been promoted to a professorship.

The University of Maryland announces the following: Dr. G. C. Cree of Washington University, Dr. Gertrude Ehrlich, University of Tennessee, and Dr. Sol Schwartzman have been appointed to instructorships; Mrs. Eleanor B. P. Spencer has been appointed to a part-time instructorship; Instructor G. L. Spencer has been promoted to an assistant professorship.

The University of Minnesota reports the following: Dr. J. M. Slye of the

University of Texas has been appointed to an instructorship; Assistant Professors M. D. Donsker and B. R. Gelbaum have been promoted to associate professorships.

University of Nebraska announces: Assistant Professor R. M. Kozelka of Tufts College has been appointed to an assistant professorship; Dr. R. A. Moore of Washington University has been appointed to an instructorship; Associate Professor H. B. Ribeiro has been promoted to a professorship; Assistant Professor Lulu L. Runge has retired with the title of Assistant Professor Emeritus.

At the University of New Hampshire: Miss Evelyn J. Jones, formerly a mathematical analyst for the National Security Agency, Washington, D. C., has been appointed Graduate Teaching Assistant; Instructor Frederic Cunningham has been promoted to an assistant professorship.

The University of Richmond announces the following appointments: Associate Professor J. W. Sawyer of the Atlanta Division, University of Georgia, to an associate professorship; Mrs. Marion J. Stokes to an instructorship.

University of South Carolina announces the promotions of Associate Professor J. D. Novak to a professorship and Assistant Professor T. H. Lee to an associate professorship.

University of Southern California makes the following announcements: Associate Professors R. S. Phillips and P. A. White have been promoted to professorships; Professor R. S. Phillips is on leave during the year 1953–54 on a Guggenheim Fellowship and is at Yale University; Professor Ernst Snapper has received a National Science Foundation Fellowship Award for the year 1953–54 and is on leave to study at Harvard University; Associate Professor Albert Whiteman is on leave for the year 1953–54 and is at the Institute for Advanced Study; Dr. Harry Gonshor, previously a graduate student at Harvard University, has been appointed Visiting Instructor.

Dean Emeritus Samuel Beatty has been appointed Chancellor of the University of Toronto.

Reverend Stanley J. Bezuszka, formerly an assistant at Brown University, has been appointed Chairman of the Department of Mathematics of Boston College.

Dr. B. H. Bissinger, formerly vice-president of George Gillis Shoe Corporation, Fitchburg, Massachusetts, has been appointed to an associate professorship at Lebanon Valley College.

Mrs. Shirley A. Blackett, previously with the Educational Testing Service, Princeton, New Jersey, has been appointed to an instructorship at Northeastern University.

Mr. B. L. Blankenship, who has been teaching at San Saba High School, Texas, is now Head of the Department of Mathematics of Ranger Junior College, Texas.

Mrs. Helen L. Brooks has been appointed Lecturer at the University of Toledo.

Dr. D. R. Clutterham is now Senior Aerophysics Engineer with Consolidated

Vultee Aircraft Corporation, Fort Worth, Texas.

Dr. H. K. Crowder of Case Institute of Technology has been promoted to an assistant professorship.

Assistant Professor Mary H. Cummings of the University of Rhode Island has been promoted to an associate professorship.

Mr. G. M. Eddington, formerly dean of men at Tusculum College, is teaching at Santa Monica High School, California.

Assistant Professor Herta T. Freitag has been named Head of the Department of Mathematics of Hollins College.

Mr. Otha Fuller, Jr., of Claflin College has been appointed to an instructorship at Agricultural, Mechanical, and Normal College, Pine Bluff, Arkansas.

Assistant Professor D. C. Gerneth of Mississippi Southern College has accepted a position as Senior Aerophysics Engineer with the Consolidated Vultee Aircraft Corporation, Fort Worth, Texas.

Mr. Isidore Goldman is now Laboratory Supervisor for Greer Hydraulics, Inc., Brooklyn, New York.

Mr. Arnold Grudin, previously a part-time instructor at the University of Colorado, has been appointed to an instructorship at Denison University.

Miss Helen M. Heater has been appointed to an associate professorship at Alma College.

Dr. R. T. Herbst has a position as a mathematician at the Applied Physics Laboratory, Johns Hopkins University.

Assistant Professor I. R. Hershner of the University of North Carolina has been appointed Professor and Chairman of the Department of Mathematics of the University of Vermont.

Associate Professor R. C. Hildner of the University of New Mexico is now a research staff member of Sandia Corporation, Albuquerque, New Mexico.

Dr. T. C. Holyoke of Northwestern University has been promoted to an assistant professorship.

Dr. M. A. Hyman, who has been on leave from the Naval Ordnance Laboratory, White Oak, Maryland, has been awarded the degree of Doctor of Science, Magna Cum Laude, by the Technical University of Delft, Holland.

Mr. R. J. Kohlmeyer of Lafayette College has been appointed to an assistant professorship at Pratt Institute.

Assistant Professor R. D. Larsson of Clarkson College has been promoted to an associate professorship.

Mr. R. J. Lemelin, formerly at the Willow Run Research Center, University of Michigan, has accepted a position as an analyst in the Machine Computation Laboratory, Research Department of the United Aircraft Corporation, East Hartford, Connecticut.

Associate Professor C. B. Lindquist of the University of Minnesota, Duluth Branch, has been promoted to a professorship.

Associate Professor M. St. J. MacPhail of Carleton College, Ottawa, Canada, has been promoted to a professorship.

Assistant Professor Morris Morduchow of Polytechnic Institute of Brooklyn has been promoted to an associate professorship.

Dr. C. V. Newsom, Associate Commissioner for Higher Education of the State of New York, served as Administrator of the Industry-College Conference which was held at White Sulphur Springs, West Virginia, November 12–13, 1953.

Dr. M. H. Protter, previously a member of the Institute for Advanced Study, has been appointed to an assistant professorship at the University of California.

Dr. Mina S. Rees, formerly head of the Mathematics Section, Office of Naval Research, has been appointed Dean of Faculty of Hunter College.

Mr. C. A. Rogers, previously a graduate student at the University of Washington, has been appointed to an assistant professorship at Colorado Agricultural and Mechanical College.

Assistant Professor J. A. Schumaker of MacMurray College for Women has been appointed to an instructorship at Grinnell College.

Associate Professor H. M. Schwartz of the University of Arkansas has been promoted to the position of Professor of Physics.

Professor Guy Stevenson, who is also Head of the Department of Mathematics of the University of Louisville, has been appointed Dean of the Graduate School.

Dr. G. L. Thompson of Princeton University has been appointed to an assistant professorship at Dartmouth College.

Miss Jane Uhrhan of Butler University has been promoted to an assistant professorship.

Mr. J. F. Wampler of York College has been promoted to a professorship.

Assistant Professor J. G. Wendel of Louisiana State University has been promoted to an associate professorship.

Assistant Professor J. R. Wesson of Birmingham Southern College has been promoted to a professorship.

Dr. Clement Winston, formerly with the War Production Board, Washington, D. C., is engaged now as a business economist in the United States Department of Commerce, Washington, D. C.

Professor H. A. Wright, who has served as Head of the Department of Mathematics of Transylvania College, has retired.

Miss Lois J. Younger, previously a student at William Jewell College, is teaching at Haddonfield High School, New Jersey.

The following have been appointed to teaching fellowships: University of Michigan, G. A. Paxson; University of Buffalo, A. H. Blessing, Miriam H. Brown, C. L. Gape, G. E. Neu, E. P. Rozycki, Joseph Siciliano.

The following have been appointed to graduate assistantships: Queen's University, W. A. Beyer; California Institute of Technology, D. L. Elliott; University of Florida, D. E. Thoro; University of Illinois, R. P. C. Caldwell, W. D. James, and E. A. Newburg; Yeshiva University, I. N. Katz.

Dean J. B. Brandeberry of the University of Toledo died on September 23, 1953; he was a charter member of the Association.

Professor Emeritus J. G. Hardy of Williams College died on September 5, 1953.

Senior Professor R. C. Lamb (retired) of the United States Naval Academy, died on July 22, 1953.

Professor Emeritus H. F. MacNeish of Brooklyn College, who was serving as a visiting professor at the University of Miami, died on August 28, 1953; he was a charter member of the Association.

Professor J. C. C. McKinsey of the Department of Philosophy of Stanford University died on October 26, 1953.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 113 persons have been elected to membership by the Board of Governors on applications duly certified.

RUTH J. ABRAMS, B.A.(Chicago) Technical Aide, Bell Telephone Laboratories, Murray Hill, N. J.

C. D. AUCHARD, M.A.(Colorado S. C.) Supervisor, Kansas State Teachers College.

C. E. AULL, M.S.(Oregon) Instr., University of Arizona.

V. E. BACH, B.S.(Brooklyn) Student, Brooklyn College.

J. G. BARON, M.D.(Budapest) 856 Wolfram Street, Chicago 14, Ill.

JAMES BATES, B.A.(Wayne) Grad. Research Mathematician, Visibility Laboratory, Scripps Institute of Oceanography, San Diego, Calif.

J. S. BENDAT, Ph.D.(Southern California) Lecturer, University of Southern California; Research Engineer, Northrup Aircraft, Incorporated, Hawthorne, Calif.

W. J. BERGER, M.S.(Carnegie) Mathematical Assistant, Carnegie Institute of Technology.

REV. S. J. BEZUSZKA, S.J., Ph.D.(Brown) Chairman, Boston College.

J. F. BLANSCHE, Student, University of Toronto.

J. W. BRACE, Ph.D.(Cornell) Instr., University ot Maryland.

WALTER BRENTON, M.E.(Cornell) General Superintendent, Portland General Electric Company, Ore.

SOL BRODER, Student, Brooklyn College.

T. F. BROPHY, B.A.(Champlain) Grad. Student, New York State College for Teachers, Albany, N. Y.

P. S. BULLEN, M.S.(Natal) Lecturer, University of Natal, Durban, South Africa.

J. R. BYRNE, Ph.D. (U. of Washington) Asst. Professor, San Jose State College.

DORIS E. CAHN, B.A.(Newcomb) Mathematician, Redstone Arsenal, Huntsville, Ala.

J. M. CALLOWAY, Ph.D.(Pennsylvania) Asst. Professor, Carleton College.

C. J. CILLAY, B.A.(Texas) Private, United States Army.

G. F. CLANTON, B.S. (Baylor) Asso. Professor, Baylor University.

DEMETRIOS COUNES, M.A. (Johns Hopkins) Tutor, City College of the City of New York.

H. A. DANGEL, M.E. (Cincinnati) Professor, University of Cincinnati.

W. F. DARSOW, Ph.D. (Chicago) Instr., DePaul University.

MRS. MATILDE C. DEMACAGNO, Prof. Math. (La Plata) Professor, Secondary Schools of Cuyo University, San Juan, Argentina.

ALBERT DERIN, M.S. (Chicago) Instr., Kansas State College.

J. S. DORROH, B.A. (Mississippi C.) Superintendent, Springhill High School, Eupora, Miss.

S. S. DRAEGER, M.A. (Texas) Instr., Pan American College.

A. L. DUQUETTE, M.A. (Columbia) Instr., St. John's University.

MARY L. EDWARDS, M.A. (Fisk) Instr., Lincoln University.

H. M. ELDRIDGE, M.A. (Columbia) Asso. Professor, Fayetteville State Teachers College, N. C.

T. G. EVANS, Student, Princeton University.

MANUEL FELICIANO-DODONOFF, M.A. (Columbia) Instr., Catholic University of Puerto Rico.

S. R. FILIPPONE, M.A. (Wisconsin) Lecturer, Southern Illinois University.

R. E. FULLERTON, Ph.D. (Yale) Asst. Professor, University of Wisconsin.

A. E. FULTON, M.S. (Georgia) Asst. Professor, Georgia Institute of Technology.

I. S. GÁL, Ph.D. (Budapest) Instr., Cornell University.

T. A. GALIB, B.A. (Brown) Mathematician, United States Naval Underwater Ordnance Station, Newport, R. I.

D. C. GERNETH, Ph.D. (Oklahoma A. & M.) Senior Aerophysics Engineer, Consolidated Vultee Aircraft Corporation, Fort Worth, Tex.

KARL GOLDBERG, M.A. (Columbia) Mathematician, National Bureau of Standards, Washington, D. C.

D. F. GORMAN, Student, St. John's Preparatory, Brooklyn, N. Y.

H. V. GOSLING, Ph.D. (McGill) Retired, 271 Macdonnell Street, Kingston, Ont., Canada.

R. L. GRAVES, Ph.D. (Harvard) Assistant Head Engineer, Standard Oil Company of Indiana, Whiting, Ind.

SIDNEY GROSS, 48 Lenox Road, Rockville Centre, N. Y.

W. D. GUTSHALL, B.A. (U.C.L.A.) Analyst, Telecomputing Corporation, Burbank, Calif.

F. C. HALL, M.A. (Columbia) Asst. Professor, Manhattan College.

J. R. HATCHER, M.S. (Howard U.) Asst. Professor, Fisk University.

T. J. HEAD, Student, University of Oklahoma.

C. M. HEBBERT, Ph.D. (Illinois) Professor, Polytechnic Institute of Brooklyn.

R. T. HERBST, Ph.D. (Duke) Mathematician, Applied Physics Laboratory, Johns Hopkins University.

I. R. HERSHNER, JR., Ph.D. (Harvard) Professor and Chairman, University of Vermont.

C. S. HERZ, Ph.D. (Princeton) Instr., Cornell University.

J. G. HOCKING, Ph.D. (Michigan) Instr., Michigan State College.

WALTER HOFFMAN, M.A. (Wayne) Instr., Wayne University.

W. A. HORNING, Ph.D. (California) Theoretical Physicist, Hanford Works, Richland, Wash.

T. E. HULL, Ph.D. (Toronto) Asst. Professor, University of British Columbia.

L. K. JACKSON, Ph.D. (U.C.L.A.) Asst. Professor, University of Nebraska.

C. A. JOHNSON, Ph.D. (Kansas) Asst. Prof., Missouri School of Mines and Metallurgy.

D. L. JOHNSON, Student, University of Washington.

MOLLIE A. JONES, B.A. (Buffalo) Head of Department, Kenmore Junior and Senior High School, N. Y.

MRS. CAROL R. KARP, M.A. (Michigan S. C.) Instr., New Mexico College of A. and M. A.

M. A. KIRCHBERG, Student, Eastern Illinois State College.

REV. C. F. KOEHLER, S.J., M.A. (Boston C.) Asst. Professor, Loyola College.

A. V. KOZAK, D.Ed. (Penna. State) Chairman of Mathematics; Director, Kalgometrics Research Project, Concord College.

MRS. EMMA D. KUNDEL, B.S. (Huron) Instr., Sisseton Public School, S. D.

ROLANDO LARA, B.S. (Oklahoma) Seismograph Service Corporation, New Orleans, La.

J. W. LAWSON, Ph.D. (Chicago) Professor and Chairman, University of Manitoba.

C. W. LEININGER, M.S. (Arizona) Grad. Student, Southern Methodist University.

REV. J. J. MACDONNELL, S.J., M.A. (Boston C.) Grad. Student, Catholic University.

B. L. MACKIN, M.S. (Oklahoma) Asso. Professor, East Central State College, Ada, Okla.

CHARLOTTE M. MAIN, M.A. (New York S.T.C.) Head of Department, Baldwin School, Bryn Mawr, Pa.

EUGENE MALEK, Student, University of California.

EMANUEL MEHR, M.A. (Brooklyn) Instr., Brooklyn College.

A. S. MELTZER, B.S. (Syracuse) Student, Syracuse University.

E. A. MICHAEL, Ph.D. (Chicago) Asst. Professor, University of Washington.

P. W. MIELKE, JR., B.A. (Minnesota) Grad. Student, University of Chicago.

J. D. MILLER, Student, Eastern Illinois State College.

A. J. MUCCINO, B.S. (Waynesburg) Chemist, Treasury Department, Bureau of the Mint, Washington, D. C.

J. D. MUNN, M.A. (Alabama) Instr., Mississippi Southern College.

J. B. MUSKAT, Student, Yale University.

R. A. NIEMANN, M.A. (Illinois) Chief, Computation Division, Naval Proving Ground, Dahlgren, Va.

J. A. NOHEL, Ph.D. (M.I.T.) Asst. Professor, Georgia Institute of Technology.

M. C. PALMER, M.A. (Vanderbilt) Instr., Westminster College.

F. A. PHILLIPS, M.A. (Oklahoma) Mathematician, Ballistic Research Laboratory, Aberdeen Proving Ground, Md.

JOEL PITCAIRN, M.A. (Harvard) Asst. Research Engineer, Burroughs Adding Machine Company, Philadelphia, Pa.

R. G. POHRER, M.A. (Duke) Grad. Assistant, Duke University.

W. T. PUTNEY, III, M.A. (Washington S. C.) United States Army.

JOHN J. QUINN, Student, St. John's University.

M. I. ROSE, B.A. (Transylvania) Grad. Assistant, University of Kentucky.

SAM ROSENBERG, B.S. (Oklahoma) Grad. Student, Harvard University.

J. U. RUSSELL, M.A. (Mississippi) Grad. Assistant, University of Illinois.

LEO SARIO, Ph.D. (Helsinki) Research Fellow, Harvard Univ.

P. H. SAVET, Ph.D. (Paris) Head, Research Laboratory, Arma Corporation, Brooklyn, N. Y.; Instr., Brooklyn Polytechnic Institute.

R. H. SCHULZ, Student, University of Texas.

GEORGE SEIFERT, Ph.D. (Cornell) Asst. Professor, University of Nebraska.

S. H. SESSKIN, Student, Hofstra College.

A. L. SHIELDS, Ph.D. (M.I.T.) Instr., Tulane University.

A. J. SILVERMAN, B.A. (Temple) Assistant Instr., University of Kansas.

MARIETTA SMITH, M.A. (Appalachian S.T.C.) Teacher, Mars Hill High School, N. C.

MORRIS SMITH, M.S. (Columbia) Junior High School, Brooklyn, N. Y.

A. H. STEINBRENNER, M.A. (Columbia) Instr., University of Arizona.

ERMEL STEPP, JR., Airman Second Class, United States Air Force.

LOUISE C. STOLLE, M.A. (Minnesota) Instr., Berea College.

D. P. STURGIS, B.A. (American) Student, American University.

G. L. THOMPSON, Ph.D. (Michigan) Asst. Professor, Dartmouth College.

A. L. TRITTER, B.A. (Chicago) Grad. Student, University of Chicago.

H. E. URBAN, Student, University of Buffalo.

C. D. VAN WART, Student, Carnegie Institute of Technology.

G. H. VOSPER, 18 La Bar Row, Bangor, Pa.

J. R. WAGNER, B.S. in Ed. (Bloomsburg S.T.C.) Instr., North-Mont Joint School, Turbotville, Pa.

MRS. ELLA R. WATTS, M.A. (Columbia) Instr., Claflin College.

J. E. WETZEL, Student, Purdue University.

J. A. WILKINSON, B.S. (Oklahoma) Meteorological Officer, U. S. A. F., Florida State University.

JAMES WOLFE, Ph.D. (Harvard) Asso. Professor, University of Utah.

EMPLOYMENT OPPORTUNITIES

Beginning with the February 1954, issue, the MONTHLY will devote this space to paid announcements of employment opportunities for mathematicians. The text of such announcements should be in want-ad form and must be in the hands of the editor (C. B. Allendoerfer, Mathematics Department, University of Washington, Seattle 5, Wash.) before the first day of the month preceding the issue in which the notice is to appear. Announcements should indicate the academic rank or similar description of the opening, but should not mention a specific salary. Blind ads are permissible which direct replies to a specific box number in care of the Mathematical Association of America, Buffalo 14, N. Y. In order to conserve space and achieve uniformity, the privilege is reserved to rearrange advertisements. Advertisers will be billed by the Association at the rate of $1.50 per line. Rates for display advertising may be obtained from the Advertising Manager.

CALENDAR OF FUTURE MEETINGS

Thirty-fifth Summer Meeting, University of Wyoming, Laramie, Wyoming, August 30–31, 1954.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Marshall College, Huntington, West Virginia, May 1, 1954.

ILLINOIS, Knox College, Galesburg, May 14–15, 1954.

INDIANA, Rose Polytechnic Institute, Terre Haute, May, 1954.

IOWA, Iowa State College, Ames, April, 1954.

KANSAS, Baker University, Baldwin City, March 27, 1954.

KENTUCKY

LOUISIANA-MISSISSIPPI, Southwestern Louisiana Institute, Lafayette, February 19–20, 1954.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA

METROPOLITAN NEW YORK

MICHIGAN, University of Michigan, Ann Arbor, March 27, 1954.

MINNESOTA, Hamline University, St. Paul, May 8, 1954.

MISSOURI, University of Missouri, Columbia, Spring, 1954.

NEBRASKA

NORTHERN CALIFORNIA, University of San Francisco.

OHIO, Ohio State University, Columbus, April 17, 1954.

OKLAHOMA

PACIFIC NORTHWEST, Reed College, Portland, Oregon, June 18, 1954.

PHILDELPHIA

ROCKY MOUNTAIN, Colorado Agricultural and Mechanical College, Fort Collins, April, 1954.

SOUTHEASTERN, University of South Carolina, Columbia, March 12–13, 1954.

SOUTHERN CALIFORNIA, George Pepperdine College, Los Angeles, March 13, 1954.

SOUTHWESTERN, Arizona State College, Tempe, April 16–17, 1954.

TEXAS, Texas Technological College, Lubbock, April, 1954.

UPPER NEW YORK STATE, College for Teachers at Albany, May 1, 1954.

WISCONSIN, State Teachers College, Eau Claire, May, 1954.

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 61 NUMBER 2

CONTENTS

FEBRUARY 1954

The AMERICAN MATHEMATICAL MONTHLY

(Founded in 1894 by Benjamin F. Finkel)

EDITORIAL CORRESPONDENCE should be addressed to the Editor, C. B. Allendoerfer, Department of Mathematics, University of Washington, Seattle 5, Washington.

ADVERTISING CORRESPONDENCE should be addressed to J. F. Randolph, Morey Hall, University of Rochester, Rochester 3, N. Y.

NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the Monthly should be sent to the Secretary-Treasurer, H. M. Gehman, University of Buffalo, Buffalo 14, N. Y.

THIS IS THE OFFICIAL JOURNAL OF THE

MATHEMATICAL ASSOCIATION OF AMERICA, INC.

(Devoted to the Interests of Collegiate Mathematics)

Entered as second class matter at the post office at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

Subscription Price: To Members, $4 a Year, To Others, $5 a Year.

Published by the Association at Menasha, Wisconsin, and Seattle, Wash. during the months of January, February, March, April, May, June–July, August–September, October, November, December.

PRINTED IN THE UNITED STATES OF AMERICA

SUMMER CONFERENCE IN COLLEGE MATHEMATICS

B. W. JONES, University of Colorado

The National Science Foundation supported a Summer Conference in Collegiate Mathematics which was held at the University of Colorado from June 15 to August 8, 1953. This was part of an effort to assist colleges and universities at some distance from the big research centers of the nation in their training of future mathematicians. The object of the Conference was not to stimulate research activity on the part of college teachers, but to cultivate through them greater interest and competence in mathematics on the part of their students—this to be done by means of the following auxiliary objectives:

1. To introduce recent mathematical ideas, beginning at the level of understanding of its members, encouraging them by study and by talking mathematics with their fellows to acquire an appreciation of present modes of thought and development.
2. To show and give opportunities for discussion of devices (*e.g.*, comprehensive examinations, honors programs, senior theses, *etc.*) for stimulating undergraduate activity and achieving competence.
3. To revivify old ideas in terms of modern thought.

The two full-time lecturers were Professors Emil Artin of Princeton University and Raymond L. Wilder of the University of Michigan. The former in his lectures developed the fundamental notions of algebra beginning with set theory, groups, rings, fields, vector spaces, valuation theory, finally leading up to the foundations of algebraic geometry. During the question period there were from time to time discussions on curriculum as well as on the subject of the lectures.

The material offered by Professor Wilder under the title "Foundations of Analysis and Geometry" was divided into two parts:

1. Introduction to modern foundational and topological concepts via the notion of curve.
2. "Gap-filling courses"; courses designed to bridge the gap between the elementary courses and the courses in function theory, modern analysis, applied mathematics, *etc.*

Part-time lecturers were Professor Pólya of Stanford University whose lectures bore the title "Great and Small Examples of Problem Solving," Professor E. P. Northrop of the University of Chicago, and Dr. Carroll V. Newsom, Associate Commissioner of Education of the State of New York, who discussed curriculum. There were many formal and informal discussions about mathematics and problems connected with college teaching.

There were about 80 members of the Conference. Eight came from New England, 24 from the rest of the Atlantic seaboard, 16 between the Appalachians and the Mississippi, and 33 from west of the Mississippi. Some members received some assistance from grants included in the plan, others from their own institutions, and many came entirely at their own expense. They were for the

most part housed in one of the residence halls of the University and thus had the opportunity of becoming well acquainted and comparing notes at all times of the day.

Notes of the lectures were prepared for the members by assistants assigned to each lecturer; Dr. Robert R. Stoll of Oberlin College for Professor Artin, Dr. Dan E. Christie of Bowdoin College for Professor Wilder, Dr. Chester Luther of Willamette College for Professor Pólya, Dr. Lloyd Williams of Reed College for Professor Northrop, and Father William C. Doyle of Rockhurst College and Dr. Robert Gordon of Hampton Institute for Dr. Newsom.

The members of the Conference were admitted to all events of the summer recreation program of the University and the city which they wished to attend. They had full library privileges, and a room in the library was assigned for their exclusive use.

The total effect of the Conference will be very hard to assess, but certainly the members worked very hard and conscientiously, were enthusiastic about the Conference and were very alive to the impact of the Conference on their own teaching.

SUMMER CONFERENCES IN 1954

In view of the success of the 1953 Summer Conference at Boulder, the National Science Foundation is sponsoring two similar conferences for college teachers of mathematics during the summer of 1954.

In the East there will be a conference at the University of North Carolina, Chapel Hill, under the direction of Professor E. A. Cameron. The conference will run from June 15 to August 7. The principal speakers will be Professor Tibor Rado of Ohio State University and Professor Emil Artin of Princeton University.

In the West there will be a conference at the University of Oregon, Eugene, under the direction of Professor Ivan Niven. The conference will run from June 21 to August 13. The principal speakers will be Professor Hans Rademacher of the University of Pennsylvania, and Professor D. G. Bourgin of the University of Illinois.

Detailed information concerning these conferences may be obtained from their respective directors.

THE SO-CALLED EULER-DIDEROT INCIDENT

R. J. GILLINGS, Sydney University, Australia

1. Introductory Notes. Peter I or Peter the Great, the first emperor of Russia, died in 1725 and was succeeded by his wife Catherine I. She reigned only two years, being followed by Peter II who died in 1730. Then came in relatively quick succession, Anne, daughter of Peter's half-brother, 1730–40; Ivan VI, grand-nephew of Anne, 1740–41; Elizabeth, daughter of Peter and Catherine I, 1741–61; Peter III, grandson of Peter, 1761–62; and Catherine II who reigned the 34 years from 1762 to 1796.

Catherine II is by some historians referred to as Catherine the Great, while others deny her the right to that title. However she was a woman of culture who respected the arts generally, encouraged men of letters, and who herself displayed ability as a writer.

Denis Diderot (1713–84) was a distinguished philosopher, an encyclopaedist, and an author of many scientific publications, who visited St. Petersburg in 1773–74 since the Empress Catherine II had purchased his library and appointed him its first librarian. The following mathematical memoirs of Diderot published in 1784, among others, are discussed by L. G. Krakuer and R. L. Krueger, [1]: "Sur la tension des cordes," "De la developpante du cercle," "Résistance de l'air au mouvements des pendules"; and in J. L. Coolidge's, *The Mathematics of Great Amateurs*, Oxford, 1949, ch. XIV, Diderot's treatment of "vibrating strings" is discussed, together with those mentioned above. Diderot, who had been in turn, a deist, a pantheist, a sceptic and finally an atheist, was then verging on 60 years of age, Catherine was 44 and Euler was 66.

Leonard Euler (1707–83), famous mathematician, was invited to St. Petersburg in 1727, to accept the chair of mathematics at the Academy recently founded by Peter the Great. Here he remained until 1741, when he was induced by Frederic the Great of Prussia to spend 25 years in Berlin until 1766, when he returned to St. Petersburg at the request of Catherine II, and worked there until the end of his life.

Dieudonné Thiébault (1733–1807) was a French man of letters who for the period from 1765 to 1784, lived in Berlin at the invitation of Frederic the Great. He was a member of the Berlin Royal Academy. The incident to which this paper refers had its origin in Thiébault's reminiscences of his twenty years sojourn in Berlin.

2. De Morgan's Statement. [2]. "The following anecdote is found in Thiébault's *Souvenirs de vingt ans de séjour à Berlin*, published in 1804. Thiébault does not claim personal knowledge of the anecdote, but he vouches for its being received as true all over the north of Europe. Diderot paid a visit to Russia at the invitation of Catherine the Second. At that time he was an atheist or at least talked atheism: it would be easy to prove him one thing or the other from his writings. His lively sallies on this subject much amused the Empress,

and all the younger part of her Court. But some of the older courtiers suggested that it was hardly prudent to allow such unreserved exhibitions. The Empress thought so too, but did not like to muzzle her guest by an express prohibition: so a plot was contrived. The scorner was informed that an eminent mathematician had an algebraical proof of the existence of God, which he would communicate before the whole Court, if agreeable. Diderot gladly consented. The mathematician, who is not named, was Euler. He came to Diderot with the gravest air, and in a tone of perfect conviction said, '*Monsieur*! $(a+b^n)/n=x$, *donc Dieu existe*; *répondez*!' Diderot, to whom algebra was Hebrew, though this is expressed in a very roundabout way by Thiébault—and whom we may suppose to have expected some verbal argument of alleged algebraical closeness, was disconcerted, while peals of laughter sounded on all sides. Next day he asked permission to return to France, which was granted."

3. Cajori's Statement. [3]. "The story goes that when the French philosopher Denis Diderot paid a visit to the Russian Court, he conversed very freely and gave the younger members of the Court circle a good deal of lively atheism. Thereupon Diderot was informed that a learned mathematician was in possession of an algebraical demonstration of the existence of God, and would give it to him before all Court, if he desired to hear it. Diderot consented. Then Euler advanced towards Diderot, and said gravely, and in a tone of perfect conviction: '*Monsieur*, $(a+b^n)/n=x$, *donc Dieu existe*; *répondez*!' Diderot, to whom algebra was Hebrew, was embarrassed and disconcerted, while peals of laughter rose on all sides. He asked permission to return to France at once, which was granted." [A reference is then given to De Morgan's *Budget of Paradoxes.*]

4. Bell's Statement. [4]. "We shall tell once more the famous story of Euler and the atheistic (or perhaps only pantheistic) French philosopher Denis Diderot (1713–1784). Invited by Catherine the Great to visit her Court, Diderot earned his keep by trying to convert the courtiers to atheism. Fed up, Catherine commissioned Euler to muzzle the windy philosopher. This was easy because all mathematics was Chinese to Diderot. De Morgan tells what happened (in his classic *Budget of Paradoxes*, 1872): Diderot was informed that a learned mathematician was in possession of an algebraical demonstration of the existence of God, and would give it before all the Court, if he desired to hear it. Diderot gladly consented . . . Euler advanced towards Diderot, and said gravely, and in a tone of perfect conviction: '*Sir*, $(a+b^n)/n=x$, *hence God exists*; *reply*!' It sounded like sense to Diderot. Humiliated by the unrestrained laughter which greeted his embarrassed silence, the poor man asked Catherine's permission to return at once to France. She graciously gave it. Not content with this masterpiece, Euler in all seriousness painted his lily with solemn proofs, in deadly earnest, that God exists and that the soul is not a material substance. It is reported that both proofs passed into the treatises on theology of his day."

5. Hogben's Statement. [5]. "There is a story about Diderot, the Encyclo-

paedist, and materialist, a foremost figure in the intellectual awakening which immediately preceded the French Revolution. Diderot was staying at the Russian court, where his elegant flippancy was entertaining the nobility. Fearing that the faith of her retainers was at stake, the Tsaritsa commissioned Euler, the most distinguished mathematician of the time, to debate with Diderot in public. Diderot was informed that a mathematician has established a proof of the existence of God. He was summoned to court without being told the name of his opponent. Before the assembled court, Euler accosted him with the following pronouncement, which was uttered with due gravity: '$(a+b^n)/n=x$, *donc Dieu existe, répondez!*' Algebra was Arabic to Diderot. Unfortunately he did not realise that was the trouble. . . . Translated freely into English it may be rendered: 'A number x can be got by first adding a number a to a number b multiplied by itself a certain number of times, and then dividing the whole by the number of b's multiplied together. . . .' Like many of us Diderot had stage-fright when confronted by a sentence in size language. He left the court abruptly amid the titters of the assembly, confined himself to his chambers, demanded a safe conduct, and promptly returned to France."

6. Thiébault's Statement. [6]. An English translation of the relevant passage is as follows: From the moment of his arrival Diderot was well received, all his expenses had been paid by the Empress whom he amused immensely by the fecundity and fire of his imagination, by the abundance and singularity of his ideas, and by the zeal, boldness and eloquence with which he publicly upheld atheism. But several of the older courtiers more experienced and perhaps more easily alarmed, persuaded their autocratic sovereign that teachings of this kind could have unfortunate consequences for the whole court, and especially among the large youthful group, destined for important empire posts, who might embrace this doctrine with more eagerness than careful scrutiny. The Empress then desired that some restraint be put upon Diderot on this subject, provided that she did not appear to play any part in the matter, and provided that no one should show any undue authority about it. It was therefore announced to the French philosopher one evening, that a Russian philosopher, a learned mathematician and a distinguished member of the Academy, was prepared to prove the existence of God to him, algebraically, and before the whole court. Diderot said that he would be happy to listen to such a demonstration, in the validity of which of course, he did not believe, and so an hour and a day were fixed to convince him. The occasion having arrived, with the whole court present, that is to say, the men and more particularly the younger members, the Russian philosopher gravely advanced towards the French philosopher, and speaking in a tone of voice to imply his full conviction, said, "*Monsieur*, $(a+b^n)/z=x$, [7] *therefore God exists: answer that!*" Diderot was willing to show the futility and stupidity of this so-called proof, but felt in spite of himself, the embarrassment that one would, on discovering, (among them), their intention of making a game of it, so that he was not disposed to attempt to admonish them

for the indignities proposed for him. This adventure made him fearful that there might be others in store for him of a like nature, and so sometime afterwards he expressed his desire to return to France. Then the Empress having declared her willingness to pay all his traveling expenses, he was sent on his journey after having received 50,000 francs. Eventually his carriage was wrecked near Riga, but he received from the governor of that town the whole of the cost of the repairs. I do not assert the truth of any one of these facts; I say only, that at that time, they were talked about, and were believed by the inhabitants of the north.

7. Conclusions. Since Thiébault's statement is the only authority for the facts discussed in this incident we may now summarize some of the unwarranted changes made by the authors mentioned. We see that De Morgan's statement makes more than one departure from his quoted authority: "Algebra was Hebrew" . . . (see also the facts given in the Introductory Notes), "Diderot . . . was disconcerted while peals of laughter sounded on all sides," "Next day he asked permission to return to France," and "The mathematician who was not named was Euler." We grant that Thiébault's phrase, "a Russian philosopher, a learned mathematician and a distinguished member of the academy," seems rather definitely to refer to the Swiss Mathematician Euler.

De Morgan's inventions are naturally repeated by Cajori. Bell also repeats them but substitutes "All mathematics was Chinese to Diderot," for "Algebra was Hebrew to Diderot." No authority is given for Bell's statements in his final two sentences, hence we question the adequacy of that authority.

It will be observed that Hogben also alters the form of De Morgan's inventions. Struik has well pointed out [8] the incongruity of the story, which is completely out of character both for the devout Euler and the highly intelligent Diderot. Thiébault himself was not convinced of the truth of it. The extent to which legendary stories of history may be distorted is well illustrated by the so-called Euler-Diderot incident.

References

1. L. G. Krakuer and R. L. Krueger, *Isis*, vol. 33, 1941, p. 219–231.

2. A. De Morgan, *A Budget of Paradoxes*. London, 1872, p. 250 and p. 474. The story appears twice.

3. F. Cajori, *A History of Mathematics*. New York, 1919, p. 233. Second edition revised and enlarged.

4. E. T. Bell, *Men of Mathematics*. New York, 1937, p. 146–147.

5. L. Hogben, *Mathematics for the Millions*. New York, 1951, p. 17. Opening paragraph of chapter I.

6. D. Thiébault, *Mes Souvenirs de Vingt Ans de Séjour à Berlin*. Paris, 1804, 3 vols. The passage is on page 141 of volume 3. The English translation published in 1806 in 2 volumes at Philadelphia, under the title of, *Anecdotes of Frederick the Great of Prussia*, makes no mention of this story.

7. The formula is printed in italics and the *z* of the denominator could possibly be mistaken for a 2. It is however "*z*" clearly enough and not "*n*" as De Morgan miscopied it and following him, Cajori, Bell, Smith, Sanford, Hogben and others.

8. D. J. Struik, *A Concise History of Mathematics*. New York, 1948, p. 182.

THE FORMULA FOR CHANGE IN VARIABLES IN A MULTIPLE INTEGRAL

J. SCHWARTZ, Yale University

A well-known theorem of elementary analysis states:

THEOREM. *Let D_1 and D_2 be open sets in Euclidean n-space E^n, and let h: $D_1 \rightarrow D_2$ be a* 1-1 *mapping of D_1 onto D_2 such that h and its inverse h^{-1} are continuous and have continuous derivatives. Let*

$$J(x) = \left| \det \left(\frac{\partial h^i(x)}{\partial x^j} \right) \right|$$

be the absolute value of the Jacobian determinant of the transformation h. Then a function $f(x)$ is integrable over the domain D if and only if the function $f(h(x))J(x)$ is integrable over D_1, and we have

$$\int_{D_2} f(x)dx = \int_{D_1} f[h(x)]J(x)dx.* \tag{1}$$

Before going over to the formal proof, we shall make some historical remarks. It is easy to see, from the formal multiplicative property of Jacobian determinants, that if our theorem holds for each of two transformations, it holds for their product. All but one of the known proofs make use of this fact, in a greater or lesser degree. The most naive idea would be to divide D_1 into "infinitesimal" rectangular parallelepipeds (shortly: rectangles), to observe that the image by h of such a rectangle is an infinitesimal parallelepiped, to compute the volume of this parallelepiped, to sum all the infinitesimals arising in this way, and thus to arrive at formula (1). Even this proof, however, makes use of the determinant formula for the volume of a parallelepiped in E^n; and this determinant formula can only be proved by an inductive argument which amounts, in essence, to use of the "multiplicative" principle enunciated above.

Moreover, to turn this heuristic approach into a rigorous proof is not entirely trivial. It is done, to my knowledge, only in two places: Jourdan's *Cours de Analyse* and O. Haupt's very interesting three volume work on real variable theory, *Differential-und Integralrechnung unter besondere Berucksichtigung neuere Ergebnisse.* In each of these proofs a small rectangle C is considered, and, by careful construction, two parallelepipeds, the first entirely containing and the

---

* Throughout this paper the symbol x stands for the vector $[x_1, x_2, \cdots, x_n]$. The integrals in equations like (1) are understood to be n-fold multiple integrals. For instance $\int f(x)\, dx = \int \cdots \int f(x_1, x_2, \cdots, x_n)dx_1dx_2 \cdots dx_n$.

second entirely contained within the image $h(C)$ of C are found. In this way, sufficiently exact bounds for the volume of $h(C)$ are found so that formula (1) can be obtained in the limit. The principal difficulty in this proof is that of finding the parallelepiped interior to $h(C)$ since the exterior parallelepiped can be established without difficulty. The formal proof to be given below is essentially a simplified version of this proof which makes use of a convenient device for avoiding the interior parallelogram.

Most authors have, however, preferred to avoid this line altogether, with the result that the most popular proof (Courant, Hobson, Pierpont, Franklin, Goursat, Gibson, Levy, Hadamard, Stewart, Graves, and Humbert are among those who reproduce it) is one which operates much more strongly with the multiplicative principle, than the proof just outlined. The proof is carried through by these authors by making the "change of variables" $x \rightarrow h(x)$ one variable at a time, and thus by reducing the theorem, via the multiplicative principle, to the one variable case. This brings to the fore another difficulty: the intermediate mappings need not be one-to-one "in the large," and hence the domain D_1 must be decomposed into subdomains to make the proof work. All in all, the obstructing details met with along this line of march are rather formidable, and the "non-inductive" proof described in the previous paragraph is consequently simpler in principle.

The third type of proof met in the literature (*c.f.* Widder, de la Vallée Poussin, Osgood) is a sort of dimensional induction which uses Green's theorem rather than the multiplicative principle. This method is, however, open to serious objection. First of all, Green's theorem in the form required cannot even be stated, much less proved, without having recourse to such notions as inside and outside of a Jordan curve, positive and negative orientation of a curve, *etc.*, and hence to the profound ideas involved in the algebraic topology of the plane and of E^n. Secondly, though this proof is "elementary" in the plane, its extension to E^n with $n > 2$ requires that use be made of a part of the formal machinery of exterior differentiation and tensor analysis, and thus the proof loses its elementary character.

These introductory remarks made, let us get to work. First of all, we note that by decomposing an arbitrary function $f(x)$ (which we can suppose continuous) into its positive and negative parts, we can see that it is sufficient to prove (1) when $f(x)$ is nowhere negative. If h is linear, the result then follows easily enough by the multiplicative principle. Indeed, an arbitrary linear transformation can be written as a product of linear transformations of three basic types:

$$\text{(a)} \qquad t_\lambda[x_1, x_2, \cdots, x_n] = [\lambda x_1, x_2, \cdots, x_n]$$

$$\text{(b)} \qquad a[x_1, x_2, \cdots, x_n] = [x_1 + x_2, x_2, \cdots, x_n]$$

$$\text{(c)} \qquad s_{ij}[x_1, \cdots, x_i, \cdots, x_j, \cdots, x_n] = [x_1, \cdots, x_j, \cdots, x_i, \cdots, x_n].$$

We have, however,

$$\text{(a)}\qquad \begin{aligned} |\det (t_\lambda)| \int_{E^n} f(t_\lambda x)dx &= |\lambda| \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} f(\lambda x_1, x_2, \cdots, x_n)dx_1 \cdots dx_n \\ &= \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} f(x_1, \cdots, x_n)dx_1 \cdots dx_n \\ &= \int_{E^n} f(x)dx \end{aligned}$$

by the representation of a multiple integral as a repeated integral (Fubini's theorem) and by the obvious one-variable result:

$$|\lambda| \int_{-\infty}^{+\infty} g(\lambda y)dy = \int_{-\infty}^{+\infty} g(y)dy.$$

$$\text{(b)}\qquad \begin{aligned} \int_{E^n} f(ax)dx &= \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} f(x_1 + x_2, x_2, \cdots, x_n)dx_1 \cdots dx_n \\ &= \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} f(x_1, x_2, \cdots, x_n)dx_1 \cdots dx_n \end{aligned}$$

by Fubini's theorem and the obvious one-variable result:

$$\int_{-\infty}^{+\infty} g(y + z)dy = \int_{-\infty}^{+\infty} g(y)dy.$$

Further

$$\text{(c)}\qquad \begin{aligned} &\int_{E^n} f(s_{ij}x)dx \\ &= \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} f(x_1, \cdots, x_j, \cdots, x_i, \cdots, x_n)dx_1 \cdots dx_i \cdots dx_j \cdots dx_n \\ &= \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} f(x_1, \cdots, x_i, \cdots, x_j, \cdots, x_n)dx_1 \cdots dx_j \cdots dx_i \cdots dx_n \\ &= \int_{E^n} f(x)dx \end{aligned}$$

by Fubini's theorem, which tells us that a multiple integral is the same as the repeated integral in any order of integration.

These three facts give us the desired result for linear maps for functions defined and integrable in all of E^n. However, if $f(x)$ is defined and integrable in a domain D, we can extend its domain of definition to all of E^n by the trivial convention $f(y)=0$ for y not in D, and thus our result on linear maps holds for an arbitrary domain D. What, however, about the most general maps described in the hypotheses ot our theorem? To consider these in an effective way,

we first introduce some notational conventions. If $x \in E^n$, so that $x = [x, \cdots, x_n]$, we put $|x| = \max_{1 \leq i \leq n} |x_i|$. This "norm" has the convenient property that in terms of it, a cube with center p and side-length $2s$ can be characterized by the restriction $|x-p| \leq s$. If A: $E^n \rightarrow E^n$ is the linear transformation represented by the matrix a_{ij}, so that

$$A(x) = A[x_1, \cdots, x_n] = \left[\sum_{j=1}^{n} a_{1j}x_j, \cdots, \sum_{j=1}^{n} a_{nj}x_j \right],$$

we put

$$|A| = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}|.$$

Thus, $|A(x)| \leq |A| |x|$. We also introduce the Jacobian matrix $j(x) = j_{ik}(x)$ of the transformation h: $h(x) = [h_1(x) \cdots h_n(x)]$ by putting

$$j_{ik}(x) = \frac{\partial h_i(x)}{\partial x_k}.$$

Let us agree to write the volume of a set S in E^n as $\mu(S)$. Then if C is a cube in the open set D_1, so that C is the set of all x characterized by a condition of the form $|x-p| \leq s$, we have $\mu(C) = (2s)^n$. We have, by the mean value theorem,

$$h_i(x) - h_i(p) = \sum_{k=1}^{n} j_{ik}[p + \theta_i(x)(x - p)](x_k - p_k),$$

where $0 \leq \theta_i(x) \leq 1$. It follows immediately that

$$|h(x) - h(p)| \leq s \max_{y \in C} |j(y)|;$$

i.e., $h(C)$ is entirely contained in the cube defined by

$$|z - h(p)| \leq s \max_{y \in C} |j(y)|,$$

so we see that

$$\mu(h(C)) \leq \left\{ \max_{y \in C} |j(y)| \right\}^n \mu(C).$$

We apply this formula to the map $A^{-1}h$, where A is an arbitrary non-singular linear map, and make use of the fact that we can show that $\mu(A^{-1}(S)) = |\det(A^{-1})| \mu(S)$ for an arbitrary closed set S. To prove this we can apply our theorem in the special case when h is a linear map to the function f defined by $f(x) = 1$ for x in $A^{-1}(S)$, $f(x) = 0$ for x not in $A^{-1}(S)$. Let us apply this to $S = h(C)$. The set $h(C)$ is closed since C is closed and bounded and h is a

continuous map. Since

$$|\det(A^{-1})|\,\mu(h(C)) \leqq \left\{\max_{y\in C} |A^{-1}j(y)|\right\}^n \mu(C),$$

we obtain

$$\mu(h(C)) \leqq |\det(A)| \left\{\max_{y\in C} |A^{-1}j(y)|\right\}^n \mu(C). \tag{2}$$

Now, let the cube C be subdivided into a finite set $C_1 \cdots C_M$ of non-overlapping cubes with centers $x_1 \cdots x_M$, and suppose that δ is greater than the length of a side of any of them. Apply (2) to each of $C_1 \cdots C_M$, taking, however, $A = j(x_i)$ in applying (2) to C_i. Then add. This gives

$$\mu(h(C)) \leqq \sum_{i=1}^{M} |\det(j(x_i))| \left\{\max_{y\in C_i} |j^{-1}(x_i)j(y)|\right\}^n \mu(C_i).$$

Now, since $j(x)$ is a continuous (matrix-valued) function, we have $j^{-1}(z)j(y)$ approaching the identity matrix δ_{ij} as z approaches y, and hence we have

$$\left\{\max_{y\in C_i} |j^{-1}(x_i)j(y)|\right\}^n \leqq 1 + \eta(\delta),$$

where $\eta(\delta)$ approaches zero with δ. This gives

$$\mu(h(C)) \leqq [1 + \eta(\delta)] \sum_{i=1}^{M} |\det(j(x_i))|\,\mu(C_i);$$

as δ approaches zero, the sum on the right approaches $\int_C J(x)dx$, and the inequality becomes

$$\mu(h(C)) \leqq \int_C J(x)dx.$$

This leads immediately to the formula

$$\int_{D_2} f(x)dx \leqq \int_{D_1} f(h(x))J(x)dx \tag{3}$$

for all non-negative functions. Now, if we apply (3) to the inverse map h^{-1}, we find

$$\int_{D_2} f(x)dx \leqq \int_{D_1} f(h(x))J(x)dx \leqq \int_{D_2} f(h(h^{-1}(x)))J(h^{-1}(x))J(x)dx,$$

where $J'(x)$ is the absolute value of the Jacobian determinant of the map h^{-1}. Now, we have only to apply the multiplicative principle $J(h^{-1}(x))J'(x) = 1$, and our proof of (1) is complete.

WHAT THE COLLEGES ARE DOING ABOUT THE POORLY PREPARED STUDENT*

W. L. WILLIAMS, University of South Carolina

1. Introduction. During the last World War when a Navy V-12 unit with students from all parts of the United States was stationed at the University of South Carolina, it became evident early in the program that many of these students were wholly unprepared for the usual beginner's courses in college mathematics even though a large number of them had had one year of plane geometry and from one to three years of algebra in high school.

To meet this situation remedial non-credit courses in both algebra and geometry were introduced in our curriculum with the intention of dropping them at the end of the war, since we thought that there was some relation between the poor performance of the V-12 trainees in mathematics and the tenseness of the times brought on by the war. However, when the war ended, we found that the problem remained, which seemed to indicate that the war merely served to bring to the surface and emphasize a condition which previously existed; namely, that the high schools in our area of the country were not preparing their graduates to enter our freshman mathematics courses even though these students thought they were prepared when they came to us.

Since we found it necessary to continue our remedial courses, I decided last fall to see to what extent the other institutions of higher learning in the United States were having the same problem and what they were doing about it. To accomplish this, something over five hundred questionnaires were sent out and replies were received from institutions in forty-six states with student bodies ranging in size from one hundred and fifty to nineteen thousand and with mathematics departments from ten to four thousand students. The study, therefore, was national in scope and embraced institutions from the smallest to the largest.

2. A lack of confidence revealed in high school mathematics units. The survey showed that forty-one per cent of the colleges and universities in the country no longer accept students for their freshman mathematics courses on the basis of their high school work in mathematics, but instead require a placement test before registration. The reason for this lack of confidence on the part of the colleges and universities is understandable in view of the fact that the questionnaire returns show that, of the students now taking college algebra in our institutions of higher learning, only twenty per cent are regarded as good and thirty-two per cent as fair, while twenty-nine per cent are regarded as poor and nineteen per cent as totally unprepared. The corresponding percentages for trigonometry are about the same, except that twenty-five per cent are regarded as totally unprepared for the course. A quicker picture of the situation may be

* Presented to the Southeastern Section of the Mathematical Association of America, Alabama Polytechnic Institute, March 13, 1953.

obtained by simply stating that in the opinion of those in charge of the various mathematics departments throughout the country about one-half of the students now trying to take college algebra and trigonometry are either poorly or totally unprepared. The problem becomes more serious when it is realized that the students involved in this study are just out of high school and not veterans to any great extent whose poor showing could, perhaps, be attributed in part to the interruption of their education by military service.

3. Remedial courses. Because of the faulty preparation in high school algebra and geometry of so many students now going to our colleges and universities a large number of these institutions are finding it necessary to give remedial courses in these subjects.

The situation with reference to algebra is very serious, for the survey showed that sixty-two per cent of the institutions of higher learning in the country are now giving remedial courses in this subject. The questionnaire also revealed that, of the students now coming to our colleges and universities who would normally take college algebra, thirty-four per cent are in the remedial algebra course. In other words, our institutions of higher learning are today teaching high school algebra (or its equivalent) to slightly more than one-third of their students who thought they were prepared for college algebra. The percentage of unprepared students for college algebra is much larger than the number in the remedial course would indicate, since many students do not take the remedial course. They either drop out of college during the first few days or change to some curriculum not requiring mathematics.

Seventeen per cent of the colleges and universities are giving remedial courses in geometry, but of those giving this course only thirty-three per cent allow credit. In the algebra remedial course sixty-one per cent allow credit.

4. How the problem is being met. As has already been pointed out a large percentage of the colleges and universities are meeting the problem of the unprepared student for college mathematics by giving remedial courses at the high school level. Some of the institutions which replied to the questionnaire stated that they do not admit students to their mathematics departments if they are deficient in high school mathematics. Those which do admit them and are not giving remedial courses are trying to solve the problem in numerous ways among which are the following: Students are required to make up their deficiencies in night school or through the institution's extension division; tutors are provided with the students paying for this service; a more elementary algebra course is given and is made a terminal course; poorly prepared students are advised to take a course of study which does not require mathematics. A number of the institutions frankly admitted that they are doing nothing about the matter—are letting these problem students "sink or swim," with the implication that few of them swim.

5. A suggested solution. Even though a large percentage of the colleges and

universities are giving remedial courses as a means of meeting the problem of the unprepared student in mathematics this is undoubtedly not the best solution. Our institutions of higher learning should not be teaching high school subjects. It is usually cheaper for the high schools to do it and the chances are that they can do it better. I am fully aware of the problem which the colleges and universities face. The problem is a real one. When the unprepared student comes to our campuses with a high school diploma and the usual units in mathematics he has a right to think that he is ready for college. If he is not, he can be denied admission or accepted and helped to overcome his deficiencies. The remedial courses are being used for this purpose. But their use should be regarded as only a temporary solution until one of a permanent nature can be found. This study tried to ascertain the thinking of the colleges and universities in this direction.

Of those replying to the questionnaire from the forty-six states covered, eighty-eight per cent favored a more fundamental drill on the basic ideas and operations in algebra (with no attempt to treat such topics as progressions, complex numbers, *etc.*, which are given in college algebra), together with a thorough review of algebra in the senior year. Only twelve per cent were in favor of trying to meet the problem of the poorly prepared student for college mathematics by giving more algebra in high school. These percentages are most significant. They show unmistakably that the overwhelming majority of the institutions of higher learning believe that the solution of the problem of the poorly prepared student for college mathematics could be solved by the high schools covering less material in their high school mathematics courses and covering it more thoroughly.

To bring about this revision and to, perhaps, accomplish other desirable ends, there must be a closer relationship between our colleges and universities and our secondary schools in order that our mutual problems can be better understood. One way to achieve this end is for those of us in our colleges and universities to cooperate with our secondary schools through the various state education associations and to actively participate in the programs sponsored by these associations.

Recently in South Carolina through the cooperation of some of our high school, elementary and college teachers, machinery was set in motion looking towards the preparation of a syllabus under the direction of a committee composed of representatives of the above group to be used in all mathematics classes from the elementary grades through the senior year in high school. We believe that this syllabus will improve the teaching of mathematics at all levels in our secondary schools and will ultimately result in these schools sending to our colleges and universities graduates who are prepared for college mathematics.

ON NUMBERS WITH INTEGRAL HARMONIC MEAN

MARIANO GARCIA, University of Puerto Rico

The results given in this paper were developed as part of an undergraduate seminar held under my supervision at the University of Puerto Rico College of Agriculture and Mechanic Arts during the second semester of the year 1951–52. We have obtained all the positive integers less than 10,000,000 which have an integral harmonic mean for their divisors, thus extending the table given by Ore [1, p. 618] of the integers between 1 and 10,000 having the same property. The method used to obtain these numbers will be illustrated for some cases only, as the computations involved all follow a similar pattern and are somewhat long in a few cases. There are some theorems which are helpful in reducing the amount of computation. Before stating these we shall make some preliminary remarks.

It will be recalled [1, p. 615] that the harmonic mean $H(n)$ of the divisors of a number n is defined by the formula

$$\frac{1}{H(n)} = \frac{1}{v(n)} \cdot \sum_{d \mid n} \frac{1}{d}$$

where d stands for a divisor of n and $v(n)$ denotes the number of divisors of n. From this it follows that

$$H(n) = \frac{nv(n)}{\sigma(n)}$$

where $\sigma(n)$ denotes the sum of the divisors of n. We also have that if

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$$

is the decomposition into prime factors of the number n,

$$v(n) = (\alpha_1 + 1)(\alpha_2 + 1) \cdots (\alpha_r + 1)$$

and

$$\sigma(n) = \sum_{d \mid n} d = \frac{p_1^{\alpha_1+1} - 1}{p_1 - 1} \cdot \frac{p_2^{\alpha_2+1} - 1}{p_2 - 1} \cdots \frac{p_r^{\alpha_r+1} - 1}{p_r - 1}.$$

From these formulas it follows that

$$H(a \cdot b) = H(a) \cdot H(b)$$

when a and b are relatively prime.

THEOREM 1 (Ore) [1, p. 616]. *A power of a prime can not have an integral harmonic mean.*

THEOREM 2. *If n is odd and has an integral harmonic mean, the prime factor decomposition of n can not contain a prime of the form $(4k-1)$ raised to an odd power.*

Proof. This result follows from the fact that if p is a prime of the form $s\cdot 2^m-1$ and $\alpha+1=2^v q$, where s, m, v and q are positive integers, s and q odd, then $\sigma(p^\alpha)=2^r t$ where t is an odd integer and $r=v+m-1$.

Thus, if an odd integer n contains a prime of the form $(4k-1)$ raised to an odd power, $\sigma(n)$ will contain 2 as a factor at least once more than $v(n)$ will and consequently $H(n)=(nv(n)/\sigma(n))$ can not be integral.

THEOREM 3. *No odd number having a prime factor decomposition of the form $p_1^{2^{s_1}-1}p_2^{2^{s_2}-1}\cdots p_r^{2^{s_r}-1}$ can have an integral harmonic mean.* (Note that this theorem is a generalization, for odd numbers, of a theorem due to Ore [1, top p. 617].)

Proof. If $n=p_1^{2^{s_1}-1}p_2^{2^{s_2}-1}\cdots p_r^{2^{s_r}-1}$, we may assume $p_1<p_2<\cdots<p_r$. Since

$$H(n)=\frac{p_1^{2^{s_1}-1}\,p_2^{2^{s_2}-1}\cdots p_r^{2^{s_r}-1}}{\dfrac{p_1+1}{2}\cdot\dfrac{p_1^2+1}{2}\cdots\dfrac{p_1^{2^{s_1-1}}+1}{2}\cdots\dfrac{p_r+1}{2}\cdot\dfrac{p_r^2+1}{2}\cdots\dfrac{p_r^{2^{s_r-1}}+1}{2}},$$

$(p_1+1)/2$ can not cancel and $H(n)$ can not be integral.

THEOREM 4. *Except for perfect numbers, no integer with prime factor decomposition $p_1^\alpha p_2$ can have an integral harmonic mean.*

Proof. Let n have the prime factor decomposition $p_1^\alpha p_2$ and suppose that

$$H(n)=\frac{p_1^\alpha\cdot p_2\cdot(\alpha+1)\cdot 2}{(p_1^\alpha+\cdots+p_1+1)(p_2+1)}$$

is integral.

Case i. $p_2=2$.

$$H(n)=\frac{p_1^\alpha\cdot 2^2\cdot(\alpha+1)}{(p_1^\alpha+\cdots+p_1+1)\cdot 3}\cdot$$

Thus $p_1^\alpha+\cdots+p_1+1$ must be a factor of $4(\alpha+1)$. But for $\alpha>1$, $p_1^\alpha+\cdots+p_1+1\geqq 3^\alpha+\cdots+3+1>4(\alpha+1)$. Therefore $\alpha=1$, and

$$H(n)=\frac{p_1\cdot 2^3}{(p_1+1)\cdot 3}\cdot$$

Consequently $p_1=3$ and $n=6$, a perfect number.

Case ii. $p_2 \neq 2$.

$$H(n) = \frac{p_1^{\alpha} \cdot p_2 \cdot (\alpha + 1)}{(p_1^{\alpha} + \cdots + 1) \dfrac{p_2 + 1}{2}} \cdot$$

If $\alpha+1$ is prime, since $\alpha+1 < p_1^{\alpha} + \cdots + 1$, $p_1^{\alpha} + \cdots + 1$ must have p_2 as a factor. Thus $p_1^{\alpha} + \cdots + 1$ is either p_2 or $p_2(\alpha+1)$. Suppose that $p_1^{\alpha} + \cdots + 1 = p_2(\alpha+1)$. Then $(p_2+1)/2$ must be divisible by p_1. But $(p_2+1)/2 = (p_1^{\alpha} + \cdots + p_1 + \alpha + 2)/2(\alpha+1)$. Thus $\alpha+2$ is divisible by p_1 and $\alpha+2 \geqq p_1$. Now, $p_1^{\alpha+1} \equiv p_1 \bmod (\alpha+1)$ and therefore $p_1^{\alpha+1} - 1 \equiv p_1 - 1 \bmod (\alpha+1)$. Since $p_1^{\alpha} + \cdots + 1 \equiv 0 \bmod (\alpha+1)$, $p_1 - 1 \equiv 0 \bmod (\alpha+1)$. Thus $p_1 \geqq \alpha+2$ and consequently $p_1 = \alpha+2$. Since $\alpha+1$ is prime, this is only possible if $\alpha+2 = 3 = p_1$. Thus $(p_2+1)/2 = (3+3)/2(2) = 3/2$, which is impossible for $p_2 \neq 2$.

We then have $p_1^{\alpha} + \cdots + p_1 + 1 = p_2$. If $p_1 \neq 2$,

$$\frac{p_2 + 1}{2} = \frac{p_1^{\alpha} + \cdots + p_1 + 2}{2} \geqq \frac{3^{\alpha} + \cdots + 3 + 2}{2} > \alpha + 1$$

and therefore $(p_2+1)/2$ must be divisible by p_1. This is impossible unless $p_1 = 2$. Thus $p_1 = 2$, $p_2 = 2^{\alpha+1} - 1$, and H is integral, but here n is 2^{α} times the prime $2^{\alpha+1} - 1$ and n is therefore perfect.

If $\alpha+1$ is not prime, let r be its smallest prime factor. Then $\alpha + 1 = rq$, where $q \geqq r$. Here

$$H(n) = \frac{p_1^{\alpha} \cdot p_2 \cdot rq}{\dfrac{p_1^{q} - 1}{p_1 - 1} \cdot \dfrac{p_1^{rq} - 1}{p_1^{q} - 1} \cdot \dfrac{p_2 + 1}{2}} \cdot$$

Now, if $q \geqq 4$, $(p_1^{q} - 1)/(p_1 - 1) > rq$ and consequently $(p_1^{q} - 1)/(p_1 - 1)$ must have p_2 as a factor. Also $(p_1^{rq} - 1)/(p_1^{q} - 1) > rq$ and $(p_1^{\alpha+1} - 1)/(p_1 - 1) > p_2 rq = p_2(\alpha+1)$. Thus $H(n)$ can not be integral.

If $q = 3$ and $p_1 \geqq 3$ the same procedure shows that $H(n)$ can not be integral, as is also true of the case $q = 3$, $p_1 = 2$ and $r = 2$. If $q = 3$ and $p_1 = 2$, r can only be 2 or 3, and if r is 3,

$$H(n) = \frac{2^{8} \cdot p_2 \cdot 3^{2}}{7 \cdot 73 \cdot \dfrac{p_2 + 1}{2}} \cdot$$

Thus $H(n)$ can not be integral.

If $q = 2$, r must be 2, and if p_1 is an odd prime, by Theorem 3, $H(n)$ can not be integral. Thus $p_1 = 2$. But then

$$H(n) = \frac{2^3 \cdot p_2 \cdot 4}{3 \cdot 5 \cdot \dfrac{p_2 + 1}{2}}$$

and therefore $H(n)$ can not be integral. This completes the proof.

To obtain the positive integers less than 10^7 with an integral harmonic mean (abbreviated to "H" from now on), it is convenient to construct a table of the values of $H(n)$ for several powers of the smaller primes. Thus $H(2) = 2^2/3$, $H(2^2) = 2^2 \cdot 3/7$, $H(2^3) = 2^5/3 \cdot 5$, $H(3) = 3/2$, $H(3^2) = 3^3/13$, *etc.*

To find the even numbers less than 10^7 with integral H, we start as follows. Let the even number n have the prime factor decomposition $n = 2^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$. Since we are interested only in the numbers $n < 10^7$ which have an integral H, by virtue of Theorems 1 and 4, $\alpha_1 \leqq 20$. Now, $H(2^{20}) = 2^{20} \cdot 3/7 \cdot 127 \cdot 337$. Thus if n has integral H and has the factor 2 raised to (exactly) the 20th power, n either contains a prime raised to the 336th power or higher or has 337 as a factor. However, either one of these possibilities makes $n > 10^7$.

When $\alpha_1 = 19$, we have $H(2^{19}) = 2^{21}/3 \cdot 5 \cdot 11 \cdot 31 \cdot 41$. Since $n < 10^7$, n can not contain a prime to the 30th power or higher. This makes $n \geqq 2^{19} \cdot 31 \cdot 41 > 10^7$.

Similarly we dispose of the cases $\alpha_1 = 18, 17, 16$, *etc.*, obtaining no numbers $< 10^7$ with integral H until we arrive at the case $\alpha_1 = 9$. Here $H(2^9) = 2^{10} \cdot 5/3 \cdot 11 \cdot 31$. Thus 31 and 11 must be factors of n. Now since $n < 10^7$, 31^3 can not be a factor of n. If 31^2 were a factor of n,

$$H(2^9 \cdot 31^2) = \frac{2^{10} \cdot 5}{3 \cdot 11 \cdot 31} \cdot \frac{31^2}{331} = \frac{2^{10} \cdot 5 \cdot 31}{3 \cdot 11 \cdot 331}$$

would give 331 as a factor of n and then $n \geqq 2^9 \cdot 11 \cdot 31^2 \cdot 331 > 10^7$. The factor 31 must then appear to the first power in n.

Now,

$$H(2^9 \cdot 31) = \frac{2^{10} \cdot 5}{3 \cdot 11 \cdot 31} \cdot \frac{31}{2^4} = \frac{2^6 \cdot 5}{3.11} \cdot$$

Thus 11 must be a factor of n, since otherwise $n \geqq 2^9 \cdot 3^{10} \cdot 31 > 10^7$. If 11^3 were a factor of n, $n \geqq 2^9 \cdot 11^3 \cdot 31 > 10^7$. If n contained 11^2 as a factor,

$$7(2^9 \cdot 31 \cdot 11^2) = \frac{2^6 \cdot 5}{3 \cdot 11} \cdot \frac{11^2 \cdot 3}{7.19} = \frac{2^6 \cdot 5 \cdot 11}{7.19}$$

would give 19 as a factor of n and then $n \geqq 2^9 \cdot 11^2 \cdot 19 \cdot 31 > 10^7$. The factor 11 must then occur to the first power.

Then

$$H(2^9 \cdot 31 \cdot 11) = \frac{2^6 \cdot 5}{3 \cdot 11} \cdot \frac{11}{2 \cdot 3} = \frac{2^5 \cdot 5}{3^2} \cdot$$

If 3 were not a factor of n, n would have to contain two squares or higher powers or an 8th power at least. Each of these possibilities would make $n > 10^7$. Thus 3 must be a factor of n.

If 3^4 were a factor of n, $n \geqq 2^9 \cdot 31 \cdot 11 \cdot 3^4 > 10^7$. Now,

$$H(2^9 \cdot 31 \cdot 11 \cdot 3^3) = \frac{2^5 \cdot 5}{3^2} \cdot \frac{3^3}{2.5} = 2^4 \cdot 3,$$

which is integral. We thus obtain $n = 2^9 \cdot 3^3 \cdot 11 \cdot 31$, a number $< 10^7$ with integral H. Observe that we get no more numbers $< 10^7$ with integral H by continuing this sequence, since $2^9 \cdot 3^3 \cdot 11 \cdot 31 \cdot 5 > 10^7$. Now if 3^2 were a factor of n, but not 3^3, since

$$H(2^9 \cdot 31 \cdot 11 \cdot 3^2) = \frac{2^5 \cdot 5}{3^2} \cdot \frac{3^3}{13} = \frac{2^5 \cdot 3 \cdot 5}{13},$$

13 would have to be a factor of n and then $n \geqq 2^9 \cdot 3^2 \cdot 11 \cdot 13 \cdot 31 > 10^7$. If 3 were a factor of n but not 3^2, since

$$H(2^9 \cdot 31 \cdot 11 \cdot 3) = \frac{2^5 \cdot 5}{3^2} \cdot \frac{3}{2} = \frac{2^4 \cdot 5}{3},$$

n would have to contain a square or a higher power. But then $n \geqq 2^9 \cdot 31 \cdot 11 \cdot 3 \cdot 5^2 > 10^7$. We thus conclude the case $\alpha_1 = 9$.

In a similar manner we consider the cases $\alpha_1 = 8, 7, 6, 5, 4, 3, 2, 1$, thus obtaining all the even numbers $< 10^7$ with integral H and which are tabulated in Table I.

To show that there are no odd numbers $n < 10^7$ with integral H, we begin by proving that 3 can not be a factor of n. First, by virtue of Theorem 2, the factor 3 can not appear to an odd power. Thus if 3 is a factor of n, n must be of the form $3^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_r^{\alpha_r}$, where α_1 is even.

If $\alpha_1 \geqq 14$, $n \geqq 3^{14} \cdot 5 > 10^7$. If $\alpha_1 = 12$, since $H(3^{12}) = 3^{12} \cdot 13/797161$, 797161 must be a factor of n and then $n > 10^7$.

The case $\alpha_1 = 10$ gives $H(3^{10}) = 3^{10} \cdot 11/23 \cdot 3851$. Therefore $(23)^2$ and $(3851)^2$ would have to be factors of n, by Theorem 2, and then $n > 10^7$.

If $\alpha_1 = 8$, from $H(3^8) = 3^{10}/13 \cdot 757$ we have that 757 and 13 are both factors of n and $n \geqq 3^8 \cdot 13 \cdot 757 > 10^7$.

If $\alpha_1 = 6$, $H(3^6) = 3^6 \cdot 7/1093$ shows that 1093 is a factor of n. If $(1093)^2$ were a factor of n, $n \geqq 3^6 \cdot (1093)^2 > 10^7$. Thus 1093 must occur to the first power. Then

$$H(3^6 \cdot 1093) = \frac{3^6 \cdot 7}{1093} \cdot \frac{1093}{547} = \frac{3^6 \cdot 7}{547} \cdot$$

Thus $(547)^2$ is a factor of n and $n > 10^7$.

If $\alpha_1 = 4$, since $H(3^4) = 3^4 \cdot 5/11^2$, 11 must occur to an even power in n. Thus

$n=3^4\cdot 11^{\alpha_2}\cdot p_3^{\alpha_3}\cdots p_r^{\alpha_r}$, where α_2 is even. If $\alpha_2\geqq 6$, $n\geqq 3^4\cdot 11^6>10^7$. If $\alpha_2=4$,

$$H(3^4\cdot 11^4) = \frac{3^4\cdot 5}{11^2}\cdot\frac{11^4}{3221} = \frac{3^4\cdot 5\cdot 11^2}{3221}$$

gives that 3221 is a factor of n and thus $n>10^7$. If $\alpha_2=2$,

$$H(3^4\cdot 11^2) = \frac{3^4\cdot 5}{11^2}\cdot\frac{11^2\cdot 3}{7.19} = \frac{3^5\cdot 5}{7.19}$$

gives that 7^2 and 19^2 are factors of n and thus $n\geqq 3^4\cdot 11^2\cdot 7^2\cdot 19^2>10^7$. Thus the case $\alpha_1=4$ gives us no numbers $<10^7$ with integral H.

If $\alpha_1=2$, from $H(3^2)=3^3/13$ we have that 13 is a factor of n. If 13^6 is a factor of n, $n\geqq 3^2\cdot 13^6>10^7$. If 13^5 is a factor of n but not 13^6,

$$H(3^2\cdot 13^5) = \frac{3^3}{13}\cdot\frac{13^5}{7\cdot 61\cdot 157}$$

gives that 157 is a factor of n and $n>10^7$. If 13^4 is a factor of n but not 13^5,

$$H(3^2\cdot 13^4) = \frac{3^3}{13}\cdot\frac{13^4\cdot 5}{30941}$$

gives that 30941 is a factor of n and $n>10^7$. If 13^3 is a factor of n but not 13^4,

$$H(3^2\cdot 13^3) = \frac{3^3}{13}\cdot\frac{13^3}{5\cdot 7\cdot 17} = \frac{3^3\cdot 13^2}{5\cdot 7\cdot 17}$$

gives that 7 must be a factor of n, since otherwise $n\geqq 3^2\cdot 13^3\cdot 5^6\cdot >10^7$. By Theorem 2, 7^2 is also a factor of n and we also have that 17 and 5 are factors of n. Thus $n\geqq 3^2\cdot 13^3\cdot 5\cdot 7^2\cdot 17>10^7$. If 13^2 is a factor of n but not 13^3,

$$H(3^2\cdot 13^2) = \frac{3^3}{13}\cdot\frac{13^2}{61} = \frac{3^3\cdot 13}{61}$$

gives that 61 is a factor of n and since $n<10^7$, 61 can only occur to the first or second power. If 61 occurs to the second power,

$$H(3^2\cdot 13^2\cdot 61^2) = \frac{3^3\cdot 13}{61}\cdot\frac{61^2}{13\cdot 97} = \frac{3^3\cdot 61}{97}$$

gives that 97 is a factor of n and $n\geqq 3^2\cdot 13^2\cdot 61^2\cdot 97>10^7$. Thus 61 occurs to the first power only. Then

$$H(3^2\cdot 13^2\cdot 61) = \frac{3^3\cdot 13}{61}\cdot\frac{61}{31} = \frac{3^3\cdot 13}{31}$$

gives that 31^2 is a factor of n and $n\geqq 3^2\cdot 13^2\cdot 31^2\cdot 61>10^7$. We have thus shown that 3 can not be a factor of any number $<10^7$ with integral H.

In a similar manner it may be shown that 5 can not be a factor of n, although in this case odd powers of 5 must also be considered. The work is continued showing that 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, and 53 can not be factors of n. This is all that is necessary, for by virtue of Theorems 1, 3 and 4, if n does not have any prime <59 as a factor and has integral H, $n \geqq 59^2 \cdot 61^2 > 10^7$.

Following is a table of the forty-five positive integers less than 10^7 which have integral H:

TABLE I

n	$H(n)$	n	$H(n)$
$6=2\cdot 3$	2	$242060=2^2\cdot 5\cdot 7^2\cdot 13\cdot 19$	26
$28=2^2\cdot 7$	3	$332640=2^5\cdot 3^3\cdot 5\cdot 7\cdot 11$	44
$140=2^2\cdot 5\cdot 7$	5	$360360=2^3\cdot 3^2\cdot 5\cdot 7\cdot 11\cdot 13$	44
$270=2\cdot 3^3\cdot 5$	6	$539400=2^3\cdot 3\cdot 5^2\cdot 29\cdot 31$	29
$496=2^4\cdot 31$	5	$695520=2^5\cdot 3^3\cdot 5\cdot 7\cdot 23$	46
$672=2^5\cdot 3\cdot 7$	8	$726180=2^2\cdot 3\cdot 5\cdot 7^2\cdot 13\cdot 19$	39
$1638=2\cdot 3^2\cdot 7\cdot 13$	9	$753480=2^3\cdot 3^2\cdot 5\cdot 7\cdot 13\cdot 23$	46
$2970=2\cdot 3^3\cdot 5\cdot 11$	11	$950976=2^6\cdot 3^2\cdot 13\cdot 127$	27
$6200=2^3\cdot 5^2\cdot 31$	10	$1089270=2\cdot 3^2\cdot 5\cdot 7^2\cdot 13\cdot 19$	42
$8128=2^6\cdot 127$	7	$1421280=2^5\cdot 3^3\cdot 5\cdot 7\cdot 47$	47
$8190=2\cdot 3^2\cdot 5\cdot 7\cdot 13$	15	$1539720=2^3\cdot 3^2\cdot 5\cdot 7\cdot 13\cdot 47$	47
$18600=2^3\cdot 3\cdot 5^2\cdot 31$	15	$2178540=2^2\cdot 3^2\cdot 5\cdot 7^2\cdot 13\cdot 19$	54
$18620=2^2\cdot 5\cdot 7^2\cdot 19$	14	$2229500=2^2\cdot 5^3\cdot 7^3\cdot 13$	35
$27846=2\cdot 3^2\cdot 7\cdot 13\cdot 17$	17	$2290260=2^2\cdot 3\cdot 5\cdot 7^2\cdot 19\cdot 41$	41
$30240=2^5\cdot 3^3\cdot 5\cdot 7$	24	$2457000=2^3\cdot 3^3\cdot 5^3\cdot 7\cdot 13$	60
$32760=2^3\cdot 3^2\cdot 5\cdot 7\cdot 13$	24	$2845800=2^3\cdot 3^3\cdot 5^2\cdot 17\cdot 31$	51
$55860=2^2\cdot 3\cdot 5\cdot 7^2\cdot 19$	21	$4358600=2^3\cdot 5^2\cdot 19\cdot 31\cdot 37$	37
$105664=2^6\cdot 13\cdot 127$	13	$4713984=2^9\cdot 3^3\cdot 11\cdot 31$	48
$117800=2^3\cdot 5^2\cdot 19\cdot 31$	19	$4754880=2^6\cdot 3^2\cdot 5\cdot 13\cdot 127$	45
$167400=2^3\cdot 3^3\cdot 5^2\cdot 31$	27	$5772200=2^3\cdot 5^2\cdot 7^2\cdot 19\cdot 31$	49
$173600=2^5\cdot 5^2\cdot 7\cdot 31$	25	$6051500=2^2\cdot 5^3\cdot 7^2\cdot 13\cdot 19$	50
$237510=2\cdot 3^2\cdot 5\cdot 7\cdot 13\cdot 29$	29	$8506400=2^5\cdot 5^2\cdot 7^3\cdot 31$	49
		$8872200=2^3\cdot 3^3\cdot 5^2\cdot 31\cdot 53$	53

From a number n with integral H, another number np having integral H may sometimes be obtained when p is an odd prime that does not divide n. A new such number np will be obtained when $H(n)$ is divisible by $\frac{1}{2}(p+1)$. For example, if we let $n=4713984$, we find in the manner just outlined that np also has integral H for the values of $p=5$, 7, 23 and 47.

We can also find some numbers of the form np^2 with integral H when n has integral H and is relatively prime to p. For example, from $n=2^3\cdot 5^2\cdot 7^2\cdot 13\cdot 19\cdot 31$ and $n=2^5\cdot 5^2\cdot 7^3\cdot 13\cdot 31$, both of which have $H(n)=91$, we obtain respectively $2^3\cdot 3^2\cdot 5^2\cdot 7^2\cdot 13\cdot 19\cdot 31$ and $2^5\cdot 3^2\cdot 5^2\cdot 7^3\cdot 13\cdot 31$, whose harmonic mean is 189. In a similar manner, from $n=2^2\cdot 5^3\cdot 7^2\cdot 13\cdot 19$ $(H=50)$ we obtain $2^2\cdot 3^3\cdot 5^3\cdot 7^2\cdot 13\cdot 19$ $(H=135)$ by multiplying n by 3^3.

From Table I one may be led to the conjecture that every positive integer with integral H must necessarily have its largest factor in its prime factor decomposition appear to the first power. However, several numbers were found which show that this conjecture is false, among them

$$n_1 = 2^8\cdot 3^3\cdot 5^4\cdot 7^3\cdot 11\cdot 13\cdot 19\cdot 37\cdot 41\cdot 71\cdot 73^3 \qquad (H = 2\cdot 3\cdot 73^2)$$

$$n_2 = 2^9\cdot 5^2\cdot 7\cdot 11\cdot 13^2\cdot 19\cdot 31^3\cdot 37^3\cdot 61\cdot 73\cdot 137^2 \qquad (H = 2^2\cdot 5\cdot 13\cdot 137)$$

and

$$n_3 = 2^9\cdot 3^2\cdot 5\cdot 7^3\cdot 11\cdot 13^2\cdot 31^3\cdot 37^2\cdot 61\cdot 67^2 \qquad (H = 2^3\cdot 37\cdot 67).$$

It is not difficult to obtain numbers having integral H by means of a table of values of H for powers of primes. Over two hundred numbers with integral H and larger than 10^7 were found, among them the following three:

$$2^{126}\cdot 19^2\cdot(2^{127}-1) \qquad (H = 19^2),$$

$$2^{47}\cdot 3^{15}\cdot 5^8\cdot 7^3\cdot 11^3\cdot 13^4\cdot 17^3\cdot 19^2\cdot 23\cdot 29\cdot 31^2\cdot 41\cdot 43\cdot 53\cdot 61\cdot 83^2\cdot 97^2\cdot 127\cdot 191\cdot 193$$
$$\cdot 241\cdot 257\cdot 317\cdot 331\cdot 337\cdot 367\cdot 673\cdot 829\cdot 3169\cdot 30941 \qquad (H = 2^{29}\cdot 3\cdot 5\cdot 7\cdot 13\cdot 17),$$

$$2^{53}\cdot 3^{19}\cdot 5^{15}\cdot 7^{11}\cdot 11^7\cdot 13^3\cdot 17^3\cdot 19^5\cdot 23\cdot 29\cdot 31\cdot 37\cdot 43\cdot 61^3\cdot 73\cdot 79^3\cdot 127\cdot 131\cdot 157\cdot 181$$
$$\cdot 197\cdot 199\cdot 223\cdot 313\cdot 383\cdot 523\cdot 1181\cdot 1861\cdot 3121\cdot 7321\cdot 11489\cdot 11939\cdot 21803$$
$$\cdot 87211\cdot 262657 \qquad (H = 2^{16}\cdot 3^4\cdot 5^3\cdot 11\cdot 17\cdot 19\cdot 61\cdot 79).$$

Bibliography

1. Oystein Ore, On the averages of the divisors of a number, this MONTHLY, vol. 55, 1948, pp. 615–619.

GEOMETRICAL PROPERTIES OF TWO-DIMENSIONAL WAVE MOTION*

L. A. MacCOLL, Bell Telephone Laboratories

1. Introduction. We consider solutions of the differential equation

$$u_{xx} + u_{yy} = c^{-2}u_{tt} \tag{1}$$

of the form

$$u = \exp\left[\alpha(x, y) + i\beta(x, y) + i\omega t\right]. \tag{2}$$

Here x and y denote rectangular coordinates in a plane, t denotes the time, and c and ω are positive constants. The subscripts indicate partial differentiation as

* Presented to the International Congress of Mathematicians, August 31, 1950.

usual. $\alpha(x, y)$ and $\beta(x, y)$ are real-valued functions, which we assume to be of class C^4 in an open connected domain R to which the considerations are restricted.

Such a solution will be called a sinusoidal wave. The importance of these waves in physics is well known, and requires no discussion.

The function u defined by (2) is a solution of (1) if, and only if, α and β satisfy the system of equations

$$\begin{aligned} \alpha_{xx} + \alpha_{yy} + \alpha_x^2 + \alpha_y^2 - \beta_x^2 - \beta_y^2 + k^2 &= 0, \qquad k^2 = \omega^2 c^{-2}, \\ \beta_{xx} + \beta_{yy} + 2(\alpha_x\beta_x + \alpha_y\beta_y) &= 0. \end{aligned} \tag{3}$$

It is clear that α and β cannot both be constants. There are cases, which will be discussed in the subsequent sections, in which one of the functions is a constant. Ordinarily, however, neither function is a constant, and for the moment we consider only this ordinary case.

The level curves of the function α in the domain R will be called curves of constant amplitude, or briefly α-curves. Similarly, the level curves of β in R will be called curves of constant phase, or briefly β-curves.

The chief problem considered in the paper can now be stated in the following form: To determine a set of properties which are necessary and sufficient in order that two given families of curves in the domain R shall be, respectively, the family of α-curves and the family of β-curves for some sinusoidal wave.

2. Waves of constant amplitude. If α is a constant, we have what we call a sinusoidal wave of constant amplitude. This degenerate case gives rise to the following problem: To determine a set of properties which are necessary and sufficient in order that a given family of curves in the domain R shall be the family of β-curves for some sinusoidal wave of constant amplitude.

The simple solution of this problem is easily obtained by means of the methods developed in the subsequent sections. We shall merely state the result in the form of the following theorem.

THEOREM 1. *In order that a family of curves shall be the family of β-curves for some sinusoidal wave of constant amplitude, it is necessary and sufficient that it be a family of parallel straight lines.*

3. Standing waves. If β is a constant, we have a standing sinusoidal wave. In this degenerate case the system of equations (3) reduces to the single equation,

$$\alpha_{xx} + \alpha_{yy} + \alpha_x^2 + \alpha_y^2 + k^2 = 0, \tag{4}$$

and we have the following problem: To determine a set of properties which are necessary and sufficient in order that a given family of curves in the domain R shall be the family of α-curves for some sinusoidal standing wave.

Before proceeding to the solution of this problem, we shall give some formulae which will be used throughout the rest of the paper.

Consider a family of curves which is to be specialized so that it becomes the family of α-curves for some standing wave. Anticipating the final result, we call the curves α-curves even before the necessary specialization has been imposed. It is understood that through each point of the domain R there passes just one α-curve.

Let the α-curves be oriented so that the positive senses on neighboring curves are concordant. Let $T(x, y)$ denote the positively directed tangent to the typical α-curve at the typical point (x, y). Let $\theta(x, y)$ denote the angle which $T(x, y)$ makes with the positive x-axis. For the sake of the later developments, we assume that $\theta(x, y)$ is of class C^3 in R. The function $\theta(x, y)$ characterizes the family of α-curves, in the sense that the family is defined by the differential equation

$$\frac{dy}{dx} = \tan \theta(x, y).$$

Similarly, any congruence of smooth curves is characterized by a point function having a geometrical significance which is analogous to that of $\theta(x, y)$. In particular, the family of (suitably oriented) orthogonal trajectories of the α-curves is characterized by the function $\theta^*(x, y) = \theta(x, y) + \pi/2$.

Let F be any differentiable point function. The rate of change of F with respect to arc length along an α-curve is

$$\frac{dF}{ds} = F_x \cos \theta + F_y \sin \theta, \tag{5}$$

and the rate of change of F with respect to arc length along an orthogonal trajectory of the α-curves is

$$\frac{dF}{ds^*} = F_x \cos \theta^* + F_y \sin \theta^* = -F_x \sin \theta + F_y \cos \theta. \tag{5'}$$

By (5) and (5′), we have the relations

$$\begin{aligned} F_x &= \frac{dF}{ds} \cos \theta - \frac{dF}{ds^*} \sin \theta, \\ F_y &= \frac{dF}{ds} \sin \theta + \frac{dF}{ds^*} \cos \theta. \end{aligned} \tag{6}$$

In particular, the curvature of the typical α-curve at the typical point is

$$\gamma = \frac{d\theta}{ds} = \theta_x \cos \theta + \theta_y \sin \theta, \tag{7}$$

and the curvature of the typical orthogonal trajectory of the α-curves, at the typical point, is

$$\gamma^* = \frac{d\theta^*}{ds^*} = -\theta_x \sin\theta + \theta_y \cos\theta. \tag{7'}$$

By (7) and (7′), we have the relations

$$\begin{aligned} \theta_x &= \gamma \cos\theta - \gamma^* \sin\theta, \\ \theta_y &= \gamma \sin\theta + \gamma^* \cos\theta. \end{aligned} \tag{8}$$

We now proceed to the solution of the problem stated at the beginning of this section.

If the curves under consideration are the level curves of a function $\alpha(x, y)$, there exists a pair of relations of the form

$$\alpha_x = \lambda \sin\theta, \qquad \alpha_y = -\lambda \cos\theta, \tag{9}$$

where λ is a point function.

The function λ is subject to two conditions. First, we have the integrability condition for the system (9), namely,

$$(\lambda \sin\theta)_y = (-\lambda \cos\theta)_x.$$

In the second place, if α is to satisfy (4), we must have

$$(\lambda \sin\theta)_x + (-\lambda\cos\theta)_y + \lambda^2 + k^2 = 0.$$

Regarded as a system of equations, these two equations are equivalent to the system

$$\begin{aligned} \lambda_x &= -k^2 \sin\theta - \lambda\theta_y - \lambda^2 \sin\theta, \\ \lambda_y &= k^2 \cos\theta + \lambda\theta_x + \lambda^2 \cos\theta. \end{aligned} \tag{10}$$

Using (5), (5′), (7), and (7′), we find that (10) is equivalent to the system

$$\begin{aligned} \frac{d\lambda}{ds} &= -\gamma^*\lambda, \\ \frac{d\lambda}{ds^*} &= k^2 + \gamma\lambda + \lambda^2. \end{aligned} \tag{11}$$

The integrability condition for the system (10), regarded as a system in the unknown λ, can be written in the form

$$(\theta_{xx} + \theta_{yy})\lambda + 2k^2(-\theta_x \sin\theta + \theta_y \cos\theta) = 0, \tag{12}$$

or in the following intrinsic form

$$\left[\frac{d\gamma}{ds} + \frac{d\gamma^*}{ds^*}\right]\lambda + 2k^2\gamma^* = 0. \tag{13}$$

The equations (10) and (12), or equivalently (11) and (13), are the fundamental relations upon which the solution of the problem depends.

We can satisfy (13) by setting

$$\gamma^* \equiv 0, \qquad \frac{d\gamma}{ds} \equiv 0. \tag{14}$$

This pair of identities implies that the α-curves are parallel straight lines or concentric circles. Standing waves of each of these types do exist. In fact, when (12) is satisfied in virtue of (14) the system (10) is complete. The system possesses a unique solution which assumes an arbitrarily prescribed value at an arbitrarily prescribed point. When such a solution has been obtained, the equations (9) determine a standing wave with an amplitude which is constant along each curve of the family.

Since α is not to be a constant in the present case, we cannot have $\lambda \equiv 0$. Hence, by (13), it is impossible to have just one of the quantities

$$\frac{d\gamma}{ds} + \frac{d\gamma^*}{ds^*}, \qquad \gamma^*$$

identically zero. Having disposed of the case in which both of the quantities are identically zero, we now turn to the case in which neither is zero.

In this case we can satisfy (13) by setting

$$\lambda = -2k^2\gamma^* A^{-1}, \tag{15}$$

where

$$A = \frac{d\gamma}{ds} + \frac{d\gamma^*}{ds^*}.$$

In general, the λ determined by (15), for a particular family of curves, will not satisfy the equations (11). However, in the present case the family is still unspecialized, and the problem is precisely that of specializing the family so that the λ given by (15) will satisfy (11). Thus, in order that a family of curves, which is neither a family of parallel straight lines nor a family of concentric circles, shall be the family of α-curves for some standing wave, it is necessary that the family possess the properties expressed by the following equations:

$$\begin{aligned} \frac{d}{ds}(A/\gamma^*) &= A, \\ \frac{d}{ds^*}(A/\gamma^*) &= 2k^2 - A\gamma/\gamma^* + A^2/(2\gamma^{*2}). \end{aligned} \tag{16}$$

Conversely, the set of properties expressed by these equations is easily seen to be also sufficient.

The following theorem summarizes the results which have been obtained.

THEOREM 2. *In order that a family of curves shall be the family of α-curves for some sinusoidal standing wave, it is necessary and sufficient that it be a family of parallel straight lines, a family of concentric circles, or a family possessing the properties expressed by equations* (16).

4. Progressive waves with non-constant amplitude. Proceeding now to the case in which neither α nor β is a constant, we consider two families of curves in the domain R, the one characterized by an angle $\theta(x, y)$, the other characterized by an angle $\phi(x, y)$. It is required to specialize the angles so that the first and second families of curves shall be, respectively, the family of α-curves and the family of β-curves for some progressive sinusoidal wave with non-constant amplitude.

If the curves of the first family are the level curves of a function $\alpha(x, y)$, and the curves of the second family are the level curves of a function $\beta(x, y)$, we have the relations

$$\alpha_x = \lambda \sin \theta \qquad \alpha_y = -\lambda \cos \theta, \tag{17}$$

$$\beta_x = \mu \sin \phi, \qquad \beta_y = -\mu \cos \phi, \tag{18}$$

where λ and μ are point functions.

We must have the integrability conditions

$$(\lambda \sin \theta)_y = (-\lambda \cos \theta)_x, \tag{19}$$

$$(\mu \sin \phi)_y = (-\mu \cos \phi)_x. \tag{20}$$

Also, if the functions α and β are to satisfy equations (3), we must have the relations

$$(\lambda \sin \theta)_x + (-\lambda \cos \theta)_y + \lambda^2 - \mu^2 + k^2 = 0, \tag{21}$$

$$(\mu \sin \phi)_x + (-\mu \cos \phi)_y + 2\lambda\mu \cos(\theta - \phi) = 0. \tag{22}$$

The system composed of equations (19), $\cdots$, (22) is equivalent to the following:

$$\begin{aligned} \lambda_x &= -k^2 \sin \theta - \lambda\theta_y - (\lambda^2 - \mu^2) \sin \theta, \\ \lambda_y &= k^2 \cos \theta + \lambda\theta_x + (\lambda^2 - \mu^2) \cos \theta, \\ \mu_x &= -\mu\phi_y - 2\lambda\mu \cos(\theta - \phi) \sin \phi, \\ \mu_y &= \mu\phi_x + 2\lambda\mu \cos(\theta - \phi) \cos \phi. \end{aligned} \tag{23}$$

The integrability conditions for (23), regarded as a system of partial differential equations in the unknowns λ and μ, can be written in the form

$$\begin{aligned} \lambda\mu^2 \sin 2(\theta - \phi) - \left[(\theta - \phi)_x \sin \theta - (\theta - \phi)_y \cos \theta\right]\mu^2 & \\ - \tfrac{1}{2}(\theta_{xx} + \theta_{yy})\lambda + k^2(\theta_x \sin \theta - \theta_y \cos \theta) &= 0, \end{aligned} \tag{24}$$

$$
\begin{aligned}
(25) \qquad & (\lambda^2 - \mu^2) \sin 2(\theta - \phi) + 2[(\theta - \phi)_x \sin (\theta - 2\phi) \\
& \qquad + (\theta - \phi)_y \cos (\theta - 2\phi)]\lambda - [\phi_{xx} + \phi_{yy} - k^2 \sin 2(\theta - \phi)] = 0.
\end{aligned}
$$

(We have removed a factor μ from the left-hand member of the second condition. This is permissible, since μ cannot be zero in the case under consideration.)

Before proceeding further, we shall put the fundamental equations (23), (24), (25) into their simpler intrinsic forms. We shall write $\theta - \phi = \delta$, so that δ denotes the angle between the α-curve and the β-curve passing through the typical point.

It is easily seen that the system of equations (23) is equivalent to the following:

$$
(26) \qquad
\begin{aligned}
\frac{d\lambda}{ds_\alpha} &= -\lambda\gamma_\alpha^*, & \frac{d\lambda}{ds_\alpha^*} &= k^2 + \lambda\gamma_\alpha + \lambda^2 - \mu^2, \\
\frac{d\mu}{ds_\beta} &= -\mu\gamma_\beta^*, & \frac{d\mu}{ds_\beta^*} &= \mu\gamma_\beta + 2\lambda\mu \cos \delta.
\end{aligned}
$$

The symbols s_α, s_α^*, s_β, and s_β^*, respectively, denote the arc lengths along the α-curve, the orthogonal trajectory of the α-curves, the β-curve, and the orthogonal trajectory of the β-curves passing through the typical point. The symbols γ_α, γ_α^*, γ_β, and γ_β^* denote the curvatures of the several curves in an obvious way.

The integrability conditions (24) and (25) can be written in intrinsic form as follows:

$$
(27) \qquad \lambda\mu^2 \sin 2\delta + \frac{d\delta}{ds_\alpha^*}\mu^2 - \frac{1}{2}\left[\frac{d\gamma_\alpha}{ds_\alpha} + \frac{d\gamma_\alpha^*}{ds_\alpha^*}\right]\lambda - k^2\gamma_\alpha^* = 0,
$$

$$
\begin{aligned}
(28) \qquad & (\lambda^2 - \mu^2) \sin 2\delta + 2\left[\frac{d\delta}{ds_\beta} \sin \delta + \frac{d\delta}{ds_\beta^*} \cos \delta\right]\lambda \\
& \qquad - \left[\frac{d\gamma_\beta}{ds_\beta} + \frac{d\gamma_\beta^*}{ds_\beta^*} - k^2 \sin 2\delta\right] = 0.
\end{aligned}
$$

In considering the consequences of the above equations, let us first assume that $\sin 2\delta$ is not identically zero, and let us restrict our attention to a domain R in which $\sin 2\delta$ is nowhere zero.

The solution of the problem is obtained from equations (26), (27), (28) in a way which is entirely analogous to that in which the solution of the problem of the preceding section was obtained from equations (11), (13). The reasoning is straightforward, and it will suffice to state the result in the form of the following theorem.

THEOREM 3. *In order that two distinct non-orthogonal families of curves in a domain R shall be, respectively, the family of α-curves and the family of β-curves for some progressive sinusoidal wave with non-constant amplitude, it is necessary*

and sufficient that the system of equations (27), (28) *have a real solution* (λ, μ) *satisfying the equations* (26).

The solution of equations (27), (28) for λ and μ involves only rational operations, the extraction of a square root, and the solution of a cubic equation. In principle, therefore, the necessary and sufficient condition, which is stated in an implicit form in the theorem, can be stated in a fully explicit form in terms of the geometrical quantities δ, γ_α, γ_β, $d\gamma_\alpha/ds_\alpha$, *etc.* However, because of the complexity of the explicit condition, it seems not worth while to formulate it here.

5. Waves for which $\sin 2\delta \equiv 0$. We now consider the case in which $\sin 2\delta \equiv 0$, noting that this implies $\delta \equiv 0$, $\delta \equiv \pm\pi/2$, or $\delta \equiv \pi$.

In this case the equations (27), (28) reduce to the forms

$$\begin{aligned} &\left[\frac{d\gamma_\alpha}{ds_\alpha} + \frac{d\gamma_\alpha^*}{ds_\alpha^*}\right]\lambda + 2k^2\gamma_\alpha^* = 0, \\ &\frac{d\gamma_\beta}{ds_\beta} + \frac{d\gamma_\beta^*}{ds_\beta^*} = 0. \end{aligned} \tag{29}$$

The geometrical meaning of the condition $\sin 2\delta \equiv 0$ implies the relation

$$\frac{d\gamma_\alpha}{ds_\alpha} + \frac{d\gamma_\alpha^*}{ds_\alpha^*} \equiv \frac{d\gamma_\beta}{ds_\beta} + \frac{d\gamma_\beta^*}{ds_\beta^*} \cdot \tag{30}$$

By (29) and (30), we have the equations

$$\frac{d\gamma_\alpha}{ds_\alpha} + \frac{d\gamma_\alpha^*}{ds_\alpha^*} = 0, \qquad \gamma_\alpha^* = 0. \tag{31}$$

The equations (31) imply that the α-curves can only be parallel straight lines or concentric circles. We know that there do exist progressive waves for which the α-curves are of these types, and for which the β-curves coincide with the α-curves or are the orthogonal trajectories of the α-curves. Hence, we can state the following theorem.

THEOREM 4. *In order that a family of curves shall be the family of α-curves for some progressive sinusoidal wave with a non-constant amplitude, and with the β-curves either coinciding with the α-curves or being the orthogonal trajectories of the α-curves, it is necessary and sufficient that the family be a family of parallel straight lines or a family of concentric circles.*

Similarly, from Jacobi's identity, we have

$$\int_0^1 q^{\beta+1/8}((1-q)(1-q^2)(1-q^3)\cdots)^3\frac{dq}{q}$$

$$=\frac{1}{\beta+1^2/8}-\frac{3}{\beta+3^2/8}+\frac{5}{\beta+5^2/8}-\cdots$$

$$=\frac{2\pi}{\cosh\pi\sqrt{2\beta}},\qquad (\beta>-\tfrac{1}{8}),$$

which we write as

$$\int_0^1 q^{3\theta}((1-q)(1-q^2)(1-q^3)\cdots)^3dq^{n-\theta}=\frac{2\pi(n-\theta)}{\cosh\pi\sqrt{2n-\frac{1}{4}+4\theta}},\qquad (\theta>0).$$

Multiplying across by $p(n)$ and summing with respect to n from 0 to ∞, we have

$$2\pi\sum_0^\infty\frac{p(n)(n-\theta)}{\cosh\pi\sqrt{2n-\frac{1}{4}+4\theta}}$$

$$=\int_0^1 q^{3\theta}((1-q)(1-q^2)(1-q^3)\cdots)^3d\,\frac{1}{q^\theta(1-q)(1-q^2)(1-q^3)\cdots}$$

$$=-\tfrac{1}{2}\left[q^{2\theta}((1-q)(1-q^2)(1-q^3)\cdots)^2\right]_0^1=0,\qquad (\theta>0),$$

our second result.

NOTE ON THE CYCLOTOMIC POLYNOMIAL

L. CARLITZ, Duke University

1. Let p denote a prime $\geqq 3$. It is well-known that

$$4\frac{x^p-1}{x-1}=Y^2(x)-(-1)^{\frac{1}{2}(p-1)}pZ^2(x),\tag{1.1}$$

where $Y(x)$ and $Z(x)$ are polynomials with integral coefficients, deg $Y(x)=\frac{1}{2}(p-1)$, deg $Z(x)<\frac{1}{2}(p-1)$. For references see [1; §16], [2; Chapter 7]. If we put $x=t+1$, (1) becomes

$$4\frac{(t+1)^p-1}{t}=U^2(t)-\epsilon pV^2(t),\qquad (\epsilon=(-1)^{\frac{1}{2}(p-1)}),\tag{1.2}$$

where $U(t)=Y(t+1)$, $V(t)=Z(t+1)$. Let

$$V(t)=b_0+b_1t+\cdots+b_{m-1}t^{m-1},\qquad (m=\tfrac{1}{2}(p-1)),$$

then we show that the b_r are determined (mod p) by means of

$$(1.3)\qquad 2\eta\left(\sum_0^{m-1}\frac{(-1)^r}{r+1}t^r\right)^{1/2}\equiv V(t)+\text{terms of higher degree}\pmod p,$$

where η is a number such that $\eta^2\equiv-\epsilon(\text{mod } p)$. If we put

$$U(t)=a_0+a_1t+\cdots+a_{m-1}t^{m-1}+2t^m,$$

then we have $p\mid a_r$ and

$$(1.4)\qquad \frac{1}{p}a_r\equiv\frac{(-1)^{r+m}}{r+m+1}+\epsilon b'_{r+m}\pmod p,\qquad (0\leqq r\leqq m-1),$$

where $V^2(t)=\sum b_r' t^r$.

2. In the first place we have

$$\frac{(t+1)^p-1}{t}=\sum_{r=1}^{p-1}\binom{p}{r}t^{r-1}+t^{p-1}.$$

Since

$$\binom{p}{r}=\frac{p}{r}\binom{p-1}{r-1}\equiv(-1)^{r-1}\frac{p}{r}\pmod{p^2},\qquad (1\leqq r\leqq p-1),$$

we see that

$$(2.1)\qquad \frac{(t+1)^p-1}{t}\equiv p\sum_{r=1}^{p-1}\frac{(-1)^{r-1}}{r}t^{r-1}+t^{p-1}\pmod{p^2}.$$

It is evident from (1.2) and (2.1) that $U(t)\equiv 2t^m$ (mod p), so that $a_r\equiv 0$ (mod p). We write $a_r=pc_r$. Thus (1.2) becomes

$$(2.2)\qquad \begin{aligned}&4p\sum_{r=1}^{p-1}\frac{(-1)^{r-1}}{r}t^{r-1}+t^{p-1}\\ &\qquad\equiv\left(p\sum_0^{m-1}c_rt^r+2t^m\right)^2-\epsilon p\left(\sum_0^{m-1}b_rt^r\right)\pmod{p^2}.\end{aligned}$$

We examine the coefficients of x^r, $0\leqq r\leqq m-1$, in both members of (2.2). Clearly the term $U^2(t)$ contributes nothing; consequently (2.2) implies that the congruence

$$(2.3)\qquad 4\sum_{r=0}^{m-1}\frac{(-1)^r}{r+1}t^r\equiv-\epsilon\left(\sum_0^{m-1}b_rt^r\right)^2\pmod p$$

is valid for terms of degree $\leqq m-1$. This is evidently equivalent to (1.3).

Next (2.2) yields

$$4\sum_{r=0}^{p-2} \frac{(-1)^r}{r+1} t^r + \epsilon V^2(t) \equiv 4\sum_{0}^{m-1} c_r t^{r+m} \pmod{p}, \tag{2.4}$$

where now we ignore terms of degree $<m$. It is clear that (2.4) is equivalent to (1.4).

3. By means of (1.3) we can give an explicit formula for the residue of b_r. We recall the formula [**3**; p. 147]

$$\left(\frac{\log(1+t)}{t}\right)^z = z\sum_{m=0}^{\infty} \frac{t^m}{m!} \frac{B_m^{(m+z)}}{m+z}, \tag{3.1}$$

where $B_m^{(z)}$ is the Bernoulli number of order z defined by

$$\left(\frac{t}{e^t - 1}\right)^z = \sum_{m=0}^{\infty} \frac{t^m}{m!} B_m^{(z)}.$$

In (3.1) take $z=\frac{1}{2}$ so that

$$\left(\sum_{r=0}^{\infty} \frac{(-1)^r}{r+1} t^r\right)^{1/2} = \sum_{m=0}^{\infty} \frac{t^m}{r!} \frac{B_r^{(r+1/2)}}{2r+1}.$$

Comparison with (1.3) leads at once to

$$b_r \equiv \frac{2\eta}{r!} \frac{B_r^{(r+1/2)}}{2r+1} \pmod{p}, \qquad (0 \leqq r \leqq \tfrac{1}{2}(p-1)). \tag{3.2}$$

4. To give a numerical illustration of (1.3) we take $p=13$. Then $Z=x+x^3+x^5$ and $V=3+9t+13t^2+11t^3+5t^4+t^5$, from which we get

$$\begin{aligned} V^2 &\equiv 9 + 2t + 3t^2 + t^3 + 7t^4 + 5t^5 + \cdots \\ &\equiv -4(1 - \tfrac{1}{2}t^1 + \tfrac{1}{3}t^2 - \tfrac{1}{5}t^3 - \tfrac{1}{5}t^4 + \tfrac{1}{6}t^5 - \cdots) \pmod{13}. \end{aligned}$$

Similarly for $p=17$, we have $Z=x+x^2+x^3+2x^4+x^5+x^6+x^7$ and $V\equiv 8-2t-6t^2+6t^3+6t^4-6t^5+8t^6+t^7+\cdots \pmod{17}$, so that

$$\begin{aligned} V^2 &\equiv -4 + 2t - 7t^2 + t^3 + 6t^4 - 5t^5 - 3t^6 - 8t^7 + \cdots \\ &\equiv -4(1 - \tfrac{1}{2}t + \tfrac{1}{3}t^2 - \tfrac{1}{4}t^3 + \tfrac{1}{5}t^4 - \tfrac{1}{6}t^5 + \tfrac{1}{7}t^6 - \tfrac{1}{8}t^7 + \cdots) \pmod{17}. \end{aligned}$$

References

1. L. E. Dickson, H. H. Mitchell, H. S. Vandiver, G. E. Wahlin, Algebraic Numbers, Bulletin of the National Research Council, no. 28, 1923.

2. G. B. Mathews, Theory of Numbers, Cambridge, 1892.

3. N. E. Nörlund, Vorlesungen über Differenzenrechnung, Berlin, 1924.

THE FUNDAMENTAL THEOREM OF ALGEBRA

S. STEIN, University of California at Davis

The purpose of this note is to present a simple topological proof of the Fundamental Theorem of Algebra. This proof differs from that of Eilenberg and Niven [1] in that all constructions are carried out within the complex plane.

Notations and definitions. Z = the complex plane, z = arbitrary point of Z, $|z|$ = norm of z, $S = \{z \mid |z| = 1\}$, $E = \{z \mid |z| \leqq 1\}$, and Int $E = \{z \mid |z| < 1\}$. Observe that there is a homeomorphism h: Int $E \to Z$ which preserves amplitude (for example, $h(z) = z/(1-|z|)$).

LEMMA. *There is no continuous map f: $E \to S$ such that $f(z) = z^n$ for $z \in S$ (where n is an integer distinct from zero).*

Proof. Consider the diagram of homology groups and homomorphisms:

$$\begin{array}{ccc} H_2(E, S) & \xrightarrow{\partial} & H_1(S) \\ \downarrow f'_* & & \downarrow f''_* \\ H_2(E, S) & \xrightarrow{\partial} & H_1(S). \end{array}$$

From elementary homology theory it is known that ∂ is an isomorphism, $f'_* = 0$, and f''_* is a homomorphism of degree n. Commutativity of the diagram forces $n = 0$ and thus a contradiction.

THEOREM. *If f: $Z \to Z$ is continuous and satisfies*

$$\lim_{|z| \to \infty} f(z)/z^n = 1 \tag{1}$$

where n is an integer distinct from zero then $f(z) = 0$ has a solution in Z.

Proof. Assume $f(z) \neq 0$ for all $z \in Z$. Define f_1: $Z \to S$ by $f_1(z) = f(z)/|f(z)|$. Define f_2: Int $E \to S$ by $f_2 = f_1 h$. Condition (1) implies that f_2 may be extended to f_2: $E \to S$ by defining $f_2(z) = z^n$ for $z \in S$. This f_2 contradicts the lemma.

Remarks. The assumptions of the theorem are satisfied by a polynomial of degree n with leading coefficient 1; thus the Fundamental Theorem of Algebra is a consequence of this theorem.

The lemma yields for $n = 1$ the classical result that S is not a retract of E. From this it follows in a well known manner that a continuous map f: $E \to E$ must have a fixed point.

References

1. Samuel Eilenberg and Ivan Niven, Bull. Amer. Math. Soc., vol. 50, 1944, pp. 246–248.
2. D. J. Newman, this MONTHLY, 1952, Problem 4475, p. 110.

IDEMPOTENT SEMIGROUPS

DAVID MCLEAN, University of Chicago

An idempotent semigroup is a system of elements closed under an associative multiplication such that, for every element a of the semigroup, $a^2 = a$. In this paper, I deal exclusively with idempotent semigroups.

For every element a of the system I define two sets, S_a and T_a, by the equivalences

$$ax = x \longleftrightarrow x \in S_a, \qquad xa = x \longleftrightarrow x \in T_a.$$

The sets S_a and T_a have the property that $S_a = S_b$ and $T_a = T_b \to a = b$. Thus the two sets characterize the element.

LEMMA 1. *An idempotent semigroup is anticommutative if and only if for every three elements a, b, and c of the semigroup $abc = ac$. A semigroup is said to be anticommutative when no two distinct elements of the semigroup commute.*

Proof: If we have $abc = ac$ for any three elements in the semigroup, then $(ab)(ba) = aba = a$, and $(ba)(ab) = bab = b$. Thus if $ab = ba$, then $a = b$. If the semigroup is anticommutative then, since $a(axa) = (axa)a = (axa)$, we must conclude that $a = axa$ for any x. Thus we have $abc = ab(cac) = [a(bc)a]c = a^2c = ac$.

LEMMA 2. *In an anticommutative idempotent semigroup, $S_a = S_{ab}$ and $T_b = T_{ab}$ for any a and b.*

Proof: Using Lemma 1, we see that since $ax = abx$, $ax = x \longleftrightarrow abx = x$ so $S_a = S_{ab}$. Since $xb = xab$, $xb = x \longleftrightarrow xab = x$ so $T_b = T_{ab}$.

The elements of an anticommutative idempotent semigroup are characterized by the two sets S_a and T_a and are multiplied by the rule that $(S_a, T_a)(S_b, T_b) = (S_a, T_b)$.

LEMMA 3. *In a commutative idempotent semigroup, $S_a = T_a$, and $S_{ab} = S_a \cap S_b$.*

Proof: The first part is obvious. Now if $x \in S_a \cap S_b$, then $abx = ax = x$ so that $x \in S_{ab}$, and so $S_a \cap S_b \subset S_{ab}$. If $x \in S_{ab}$, then $abx = x$. It follows that $ax = a(abx) = a^2bx = abx = x$, and $bx = b(abx) = ab^2x = abx = x$. Therefore $x \in S_a \cap S_b$, and $S_{ab} \subset S_a \cap S_b$. Q.e.d.

From the above lemma, we conclude that each element of a commutative idempotent semigroup is characterized by a single set, and that the multiplication is represented by the intersection.

Next I define two relations, R and L, by the equivalences

$$aRb \longleftrightarrow ab = b \quad \text{and} \quad ba = a$$
$$aLb \longleftrightarrow ab = a \quad \text{and} \quad ba = b.$$

LEMMA 4. *$aRb \longleftrightarrow S_a = S_b$, and $aLb \longleftrightarrow T_a = T_b$.* The proof is left to the reader. From Lemma 4, we see that the R and L relations are equivalence relations.

LEMMA 5. *$aRb \rightarrow caRcb$, and $aLb \rightarrow acLbc$.*

Proof: If aRb, then $ab=b$ and consequently $cab=cb$. It follows that $(ca)(cb) = (ca)cab = (ca)(ca)b = (ca)b = cab = cb$. Similarly $cbca=ca$. Therefore $caRcb$. In the same way, the second implication can be proved.

I define a relation P by the equivalence: $aba=a$ and $bab=b \leftrightarrow aPb$.

LEMMA 6. *aRb and $bLc \rightarrow aPc$.*

Proof: By lemma 5, $caRcb$ and $baLca$. By hypothesis, $ba=a$ and $cb=c$ so $caRc$ and $aLca$. Thus, by definition, $(ca)c=c$ and $a(ca)=a$. Consequently aPc.

Lemmas 7 and 8 are stated without proof.

LEMMA 7. *aLb or $aRb \rightarrow aPb$ and $caPcb$ and $acPbc$.*

LEMMA 8. *$aPb \rightarrow aRab$ and $bLab$ and $bRba$ and $aLba$.*

LEMMA 9. *Transitivity of the relation P: aPb and $bPc \rightarrow aPc$.*

Proof: If aPb and bPc, then by Lemma 8 $bLab$, $bLcb$, $bRba$, and $bRbc$. Since the R and L relations are symmetric and transitive, $abLcb$ and $baRbc$. By Lemma 5, $abaLcba$ and $cbaRcbc$. But $aba=a$ and $cbc=c$ so $aLcba$ and $cbaRc$. Thus by Lemma 6, aPc.

LEMMA 10. *$aPb \rightarrow caPcb$ and $acPbc$.*

Proof: If aPb, then by Lemma 8 $aRab$ and $bLab$. Now by Lemma 7, $caPcab$, $cbPcab$, $acPabc$, and $bcPabc$. By Lemma 9, $caPcb$ and $acPbc$.

LEMMA 11. *Let Q be a relation defining a homomorphism of an idempotent semigroup such that for all a and b in the semigroup $abQba$. Then $aPb \rightarrow aQb$.*

Proof: If aPb, then $(ab)(ba)=aba=a$, and $(ba)(ab)=bab=b$. But $(ab)(ba)Q(ba)(ab)$. Thus aQb.

THEOREM 1.* *Given an idempotent semigroup S, there exists a homomorphic mapping ϕ of S onto a commutative idempotent semigroup T such that the inverse image of any element of T is an anticommutative idempotent semigroup. The homomorphism ϕ is the weakest in the sense that any other commutative homomorphic image of S is also a homomorphic image of T.*

Proof: Lemmas 9 and 10 prove that the relation P defines a homomorphism. One easily verifies that $abPba$ so the homomorphic image is commutative. Clearly a^2Pa, and by Lemmas 7 and 8 $aPb \rightarrow aPab$ so the equivalence classes (inverse images) are idempotent semigroups. Since $aba=a$ for any two elements in the same equivalence class, it follows from the proof of Lemma 1 that the equivalence classes are anticommutative idempotent semigroups. The last part

* This theorem is a special case of a more general theorem proved by A. H. Clifford; Semigroups admitting relative inverses, Ann. of Math., vol. 42, 1941, pp. 1037–1049.

of the theorem follows from Lemma 11.

Now I shall use the theory just developed to analyze finitely generated free idempotent semigroups. Given a set M of symbols (generators), a word is any finite sequence of generators. Under the operation of juxtaposition, the set of all words forms a semigroup F. If the set M has cardinal n, then F is called the free semigroup on n generators, and if n is finite then F is said to be finitely generated. Introducing the congruence relation $w^2 = w$ for all words w, we obtain (defining multiplication in the obvious way) an idempotent semigroup S of congruence classes, each class representable by any word in it, and S is a homomorphic image of F under the mapping carrying each word of F into the element of S containing it; S is called the free idempotent semigroup on n generators. For each word w (element of F) I denote by C_w the set of distinct generators occurring in w. It is clear that if u and v are words lying in the same congruence class then $C_u = C_v$; hence, for each element a of S, there is a well-defined set C_a. It follows from the definition that $C_{ab} = C_a \cup C_b$.

LEMMA 12. *$C_a = C_b \longleftrightarrow aPb$.*

Proof: It is obvious that aPb implies $C_a = C_b$. The converse follows from Lemma 10 and the fact that $xyPyx$ for all x, y in S.

LEMMA 13. *$C_a = C_c$ and $C_b \subset C_a \rightarrow abc = ac$.*

Proof: Since $C_b \subset C_a$, $C_{ab} = C_a \cup C_b = C_a$. Hence $C_c = C_a = C_{ab}$, whence, by Lemma 12, $c(ab)c = c$. Therefore $(aca)bc = a[c(ab)c] = ac$. But, by Lemma 12, the hypothesis $C_a = C_c$ implies that $aca = a$. Hence $abc = ac$.

I define the length of a word to be the number of occurrences of (not necessarily distinct) generators in it; thus x and y being generators, the word x is of length 1, the words xx and xy are each of length 2, *etc.* Each word clearly has positive length; there is no need to introduce (as in the theory of free groups) an empty word. A word will be said to have minimum length if the element of S (congruence class in F) containing it contains no shorter word, and in what follows we shall assume at all times that the elements of S are represented by words of minimum length.

THEOREM 2.* *A finitely generated free idempotent semigroup is of finite order.*

LEMMA. *For any number n there is a length m such that all elements of minimum length m must be a product of at least n distinct generators.*

---

* This theorem is considerably enhanced by the fact that there is an infinite sequence of three distinct symbols such that no two consecutive blocks of symbols are identical. See Marston Morse and Gustav Hedlund, Unending Chess, Symbolic Dynamics, and a Problem in Semigroups, Duke Mathematical Journal vol. 11, 1944, pp. 1–7. See also Green, J. A., and Rees, D., On semigroups in which $x^r = x$, Proc. Camb. Phil. Soc., vol. 48, 1952, pp. 35–40.

Proof: (by induction).

For $n=1$, $m=1$. Suppose the lemma is true for n, and consider the product of minimum length $2m+1$.

$$A = \prod_{i=1}^{2m+1} \alpha_i.$$

$$\text{Let} \quad \prod_{i=1}^{m} \alpha_i = x \quad \text{and} \quad \prod_{i=m+2}^{2m+1} \alpha_i = y. \quad \text{Now} \quad A = x\alpha_{m+1}y.$$

By the induction hypothesis C_x and C_y each contains at least n distinct generators. Clearly C_x and C_y are contained in C_A. Now if C_A contains only n distinct generators, then $C_x = C_y = C_A$. Also $\alpha_{m+1} \in C_x$ and $\alpha_{m+1} \in C_y$. Thus by Lemma 13, $A = x\alpha_{m+1}\, y = xy$. But xy is of minimum length $2m$ or less contrary to the hypothesis that A have minimum length $2m+1$. Thus all expressions of minimum length $2m+1$ must be a product of at least $n+1$ distinct generators. This proves the lemma. It follows from the lemma that all elements of minimum length generated by a finite number of generators must not exceed a certain length, and hence there must be only a finite number of them.

COROLLARY. *All finitely generated idempotent semigroups are of finite order.*

By developing the theory of free idempotent semigroups, one arrives at the following formula for the order I_N of the free idempotent semigroup on N generators.

$$I_N = \sum_{r=1}^{N} C_r^N \prod_{i=1}^{r} (r - i + 1)^{2^i}.$$

CLASSROOM NOTES

EDITED BY G. B. THOMAS, Massachusetts Institute of Technology

All material for this department should be sent to G. B. Thomas, Department of Mathematics, Massachusetts Institute of Technology, Cambridge 39, Mass.

A PAPER MODEL OF THE PROJECTIVE PLANE

DAVID B. DEKKER, University of Washington

The study and classification of two-dimensional closed manifolds which takes place in elementary topology leads to the consideration of the cross-cap or its topological equivalent the Möbius strip. A simple discussion of the fact that the projective plane is topologically equivalent to a sphere with a hole covered

by a cross-cap is found in [1]. Here we give the construction of a paper model of the projective plane.

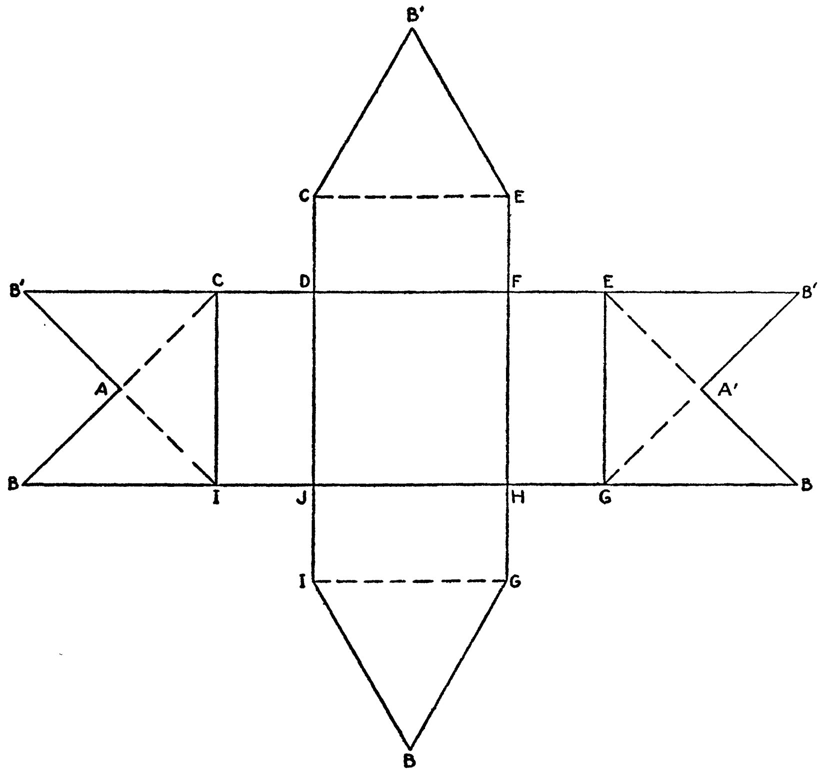


FIG. 1

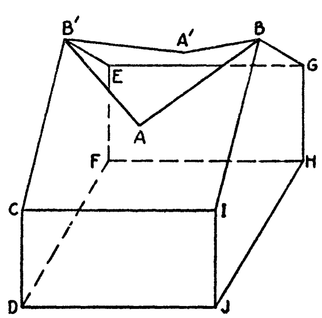


FIG. 2

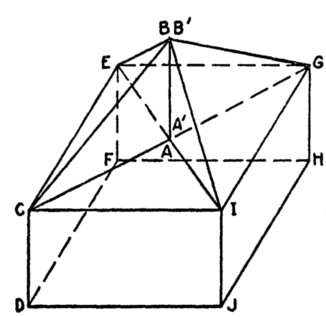


FIG. 3

A diagram of the surface as cut from a sheet of paper is given in Figure 1. The segments BI, BG, $B'E$, $B'C$, CE, EG, GI, IC are each equal to a side of the square $DFHJ$. Also CD, EF, HG, IJ are equal in length. When upward folds

are made along segments CI, DJ, DF, HJ, HF, GE and all the segments with the same labels identified, the surface in Figure 2 is obtained with the hole $ABA'B'$. If also downward folds are made along AC, AI, $A'E$, $A'G$ and upward folds are made along CE and IG in Figure 1, then Figure 2 can be carried over into the form of Figure 3 by identifying B with B' and A with A'. This identification matches the opposite pairs of points of the hole $ABA'B'$ to give the closed surface in Figure 3 with only the segment AB identified with the segment $A'B'$ and the segment BA' identified with $B'A$. Thus Figure 2 gives a surface homeomorphic to a sphere with a hole, and Figure 3 gives a closed surface homeomorphic to a sphere with a cross-cap, that is, homeomorphic to the projective plane. Of course, the surface of Figure 3 appears to cross itself; however, we must think of the segments AB and $A'B$ as distinct sets of points having only the corresponding endpoints identified.

Reference

1. Hilbert, D., and Cohn-Vossen, *Anschauliche Geometrie.* Dover, New York, 1944, pp. 276–283. The English translation is *Geometry and the Imagination.* Chelsea, New York, 1952, pp. 313–321.

ON NUMERICAL SOLUTIONS TO THE ONE-DIMENSIONAL WAVE EQUATION

W. P. Reid, U. S. Naval Ordnance Test Station, China Lake, California

In chapter 3, sections 3.21–3.22 of F. B. Hildebrand's *Methods of Applied Mathematics* one finds a discussion of a method of finite difference approximation for obtaining numerical solutions to the one-dimensional wave equation:

$$v^2 \frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial t^2} = 0. \tag{1}$$

In the discussion, however, it has not been pointed out that Equation 279 on page 326 is not just an approximation, but in fact is exactly true. Indeed a more general equality is

$$\begin{aligned} \phi(x + vh, t + H) = \phi(x + vH, t + h) &+ \phi(x - vH, t - h) \\ &- \phi(x - vh, t - H). \end{aligned} \tag{2}$$

When $H=0$ this reduces to Hildebrand's Equation 279.

The general solution to (1) is

$$\phi(x, t) = f(x + vt) + g(x - vt) \tag{3}$$

where f and g are arbitrary functions whose second partial derivatives with respect to x and t exist. It is easily verified that (2) is an identity if (3) holds.

ON THE MOTION OF A PARTICLE THROUGH A RESISTING MEDIUM OF VARIABLE DENSITY

EUGENE LEIMANIS, University of British Columbia

The motion of a particle through a resisting medium of constant density is treated in most of the well-known textbooks of mechanics, as, for example, Sommerfeld [1]. However, as R. H. Bacon [2] has pointed out, many interesting problems involve motion of a particle through a resisting medium whose density is a function of the position of the particle. He considers, in particular, the fall of a particle dropped from rest from a great height through an atmosphere whose density decreases exponentially with the altitude. Take the positive z-direction as vertically downward, and let the particle be dropped at the instant $t=0$ from the height $z=H$ with initial velocity $dz/dt=0$. Further assume that the air resistance is proportional to the product of the air density and the nth power of the instantaneous speed. Then the equation of motion of the falling particle is

$$\frac{d^2z}{dt^2} = g - ke^{\alpha z}\left(\frac{dz}{dt}\right)^n, \tag{1}$$

where g denotes the acceleration of gravity. Bacon inquires if it is possible to obtain for the cases $n=1$ and $n=2$ solutions of this equation in closed form. This question has been answered in the affirmative by Sexl [3], who gave explicit formulas for the solutions.

An obvious generalization of the equation (1) is

$$\frac{d^2z}{dt^2} = g - f(z)\left(\frac{dz}{dt}\right)^n, \tag{2}$$

where $f(z)$ is any function of z different from zero. The aim of this note is to indicate some values of the exponent n and some forms of the function $f(z)$ for which equation (2) is integrable by quadratures.

Let

$$\frac{dz}{dt} = p,$$

then equation (2) may be written in the form

$$p\frac{dp}{dz} = g - f(z)p^n.$$

By a further substitution $p=u^{-1/2}$, the last equation goes over into

$$\frac{du}{dz} = -2gu^2 + 2f(z)u^{2-n/2}. \tag{3}$$

We will be concerned with the integration of this equation in the following

cases: $2-n/2=0, 1, 3/2, 2, 3$, *i.e.*, $n=4, 2, 1, 0, -2$ respectively.

Case (*i*): $n=4$. Equation (3) then reads

$$\frac{du}{dz}+2gu^2=2f(z),$$

which is a generalized Riccati equation. If $f(z)=kz^m$, we have the original equation of Riccati

$$\frac{du}{dz}+2gu^2=2kz^m \qquad (k \text{ constant}),$$

which is soluble by means of algebraic, exponential, and logarithmic functions for all values of m given by the formula

$$m=-\frac{4s}{2s+1},$$

where s is zero or a positive integer. In the limiting case, when $s\to\infty$, the value to which the exponent tends is -2, and the equation is still soluble by the use of elementary functions.

Case (*ii*): $n=2$. Equation (3) then becomes

$$\frac{du}{dz}=-2gu^2+2f(z)u,$$

which is a Bernoulli equation. By reduction to a linear equation it is easily integrable by quadratures for any given $f(z)$. Sexl [**3**] has considered, in particular, the case $f(z)=ke^{\alpha z}$.

Case (*iii*): $n=1$. Equation (3) is then of the form

$$\frac{du}{dz}=-2gu^2+2f(z)u^{3/2}.$$

When we make the substitution $u=y^2$, the equation becomes

$$\frac{dy}{dz}=f(z)y^2-gy^3,$$

or, by taking $Z=-gz$ and $Y=-y$ as new variables, we obtain

$$\frac{dY}{dZ}=\phi(Z)Y^2+Y^3, \tag{4}$$

where

$$\phi(Z)=\frac{1}{g}f\left(-\frac{Z}{g}\right).$$

According to Appell [4] the simplest cases in which equation (4) can be solved in finite terms are:

$$\text{(a)}\quad \phi(Z) = \frac{k}{\sqrt{Z}}, \qquad \text{(b)}\quad \phi(Z) = ke^Z, \qquad \text{(c)}\quad \phi(Z) = kZ, \qquad \text{(d)}\quad \phi(Z) = \frac{k}{Z^2},$$

where, in general, Z may also be replaced by a linear function of Z. The case $f(z) = ke^{\alpha z}$, considered by Sexl [3], can be easily reduced to the case (b), if one chooses new variables defined by the formulas

$$-\frac{\alpha}{g} Z = Z_1, \qquad Y = \sqrt{-\frac{\alpha}{g}}\, Y_1.$$

The cases (b) and (c) when the density of the atmosphere decreases exponentially or linearly with the altitude may have some interest in practical applications.

Case (*iv*): $n = 0$. Equation (3) is then integrable by separation of variables for any $f(z)$.

Case (*v*): $n = -2$. Equation (3) becomes in this case

$$\frac{du}{dz} = -2gu^2 + 2f(z)u^3. \tag{5}$$

Introduce instead of z a new independent variable Z defined by the formula

$$\frac{dZ}{dz} = 2f(z).$$

Then for $f(z) = z$, $1/z$, $1/\sqrt{z}$, $1/z^2$ the equation (5) goes over into the equation

$$\frac{du}{dZ} = \phi(Z)u^2 + u^3,$$

where $\phi(Z)$ is of the form (a), (b), (c), (d) respectively.

The integration of equation (5) when $g = g(z)$ is a function of z and $f(z) = ke^{\alpha z}$ has been considered by the author in a recent paper [5].

References

1. Sommerfeld, A., Lectures on Theoretical Physics, vol. 1, Mechanics. New York, Academic Press Inc., 1952, p. 21.

2. Bacon, R. H., Motion of a particle through a resisting medium of variable density, Amer. J. of Phys., vol. 19, p. 64, 1951.

3. Sexl, Th., Über die Bewegung eines Massenpunktes in einem widerstehenden Mittel veränderlicher Dichte, Acta Phys. Austriaca, vol. 5, pp. 148–151, 1951.

4. Appell, P., Sur les invariants de quelques équations différentielles, Journal Math. Pures Appl. (4) vol. 5, pp. 361–423, 1889.

5. Leimanis, E. Über die Bewegung eines Massenpunktes in einem widerstehenden Mittel veränderlicher Dichte, Acta Phys. Austriaca, vol. 6, pp. 319–321, 1953.

DIFFERENTIATING THE LOGARITHM

R. C. YATES, U. S. Military Academy, West Point

In differentiating ln x by the "delta process" a point is reached where it is necessary to introduce a carefully selected coefficient—an act which, to the student, seems quite artificial. If, instead, the function $x \cdot \ln x$ be considered, the coefficient enters naturally. Thus, omitting some details,

$$y = x \cdot \ln x$$

$$\frac{\Delta y}{\Delta x} = \frac{x}{\Delta x} \ln\left(1 + \frac{\Delta x}{x}\right) + \ln(x + \Delta x)$$

$$Dy = 1 + \ln x.$$

An important by-product occurs if the function $x \cdot \ln x$ be differentiated formally as a product:

$$Dy = x \cdot D(\ln x) + \ln x$$

and the two results compared. Thus

$$D(\ln x) = \frac{1}{x} \cdot$$

DERIVATIVES OF IMPLICIT FUNCTIONS

M. R. SPIEGEL, Rensselaer Polytechnic Institute

When presenting a definition of the derivative of a function of a single variable, most textbooks and teachers place emphasis on derivatives of explicit functions using the "Δ-process." Thus the student is informed: Given $y = f(x)$, the derivative of y with respect to x is given by

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x},$$

if such a limit exists. After this follows a tremendous number of problems in which the student has to write $dy/dx = \cdots$ a large portion of the time. It seems small wonder then that when presented with an implicit function of x, such as that defined by the relation $4x^2 + y^3 - y = 6x$, the student starts off by writing $dy/dx = \cdots$ and then proceeds to differentiate each term (I trust that many teachers have found the same to be true with their students). In seeking to abolish this sort of thing and to see whether I could instill a better understanding of derivatives of implicit functions, I found that it was instructive to point out to the student, by a few examples, that derivatives of implicit functions need not depend (as many of them believed) on theorems already proved, but that the "Δ-process" could be used directly.

To see how natural such an approach appears to the student, let us consider the following simple example:

If $4x^2+y^3-y=6x$ defines y as a function of x find dy/dx by using the "Δ-process."

Solution: Proceeding according to the "Δ-process" we give x an increment Δx and y "takes on" the increment Δy in such a way that $x+\Delta x$ and $y+\Delta y$ satisfy the given relation. Thus

$$4(x+\Delta x)^2+(y+\Delta y)^3-(y+\Delta y)=6(x+\Delta x)$$

or

$$4x^2+8x\Delta x+4(\Delta x)^2+y^3+3y^2\Delta y+3y(\Delta y)^2+(\Delta y)^3-y-\Delta y = 6x+6\Delta x. \tag{1}$$

Subtracting

$$4x^2+y^3-y=6x$$

from (1) we obtain

$$8x\Delta x+4(\Delta x)^2+3y^2\Delta y+3y(\Delta y)^2+(\Delta y)^3-\Delta y=6\Delta x. \tag{2}$$

Upon division of both sides of equation (2) by Δx we have

$$8x+4\Delta x+3y^2\frac{\Delta y}{\Delta x}+3y\left(\frac{\Delta y}{\Delta x}\right)\Delta y+\left(\frac{\Delta y}{\Delta x}\right)(\Delta y)^2-\frac{\Delta y}{\Delta x}=6.$$

Taking the limit as $\Delta x\to 0$, assuming that when $\Delta x\to 0$, $\Delta y\to 0$ and $(\Delta y/\Delta x)\to(dy/dx)$, we find

$$8x+3y^2\frac{dy}{dx}-\frac{dy}{dx}=6$$

from which finally

$$\frac{dy}{dx}=\frac{6-8x}{3y^2-1}\cdot$$

Similar approaches adopted when dealing with partial derivatives seem to give the student better understanding of explicit and implicit functions of more than one variable. Furthermore the theory of functions of a single variable (explicit and implicit) and the theory of functions of two or more variables seem to become more unified in the minds of the students.

A FUNDAMENTAL THEOREM OF THE CALCULUS

ISRAEL HALPERIN, Queen's University

Instructors of undergraduate calculus courses frequently use a non-rigorous geometric argument to "prove" that a function $f(x)$ must be constant on an interval if it has a derivative equal to zero at each point of the interval. The usual rigorous proof of this theorem uses the theorem of the mean which is itself based on the real variable theorem that a continuous function on a closed finite interval attains its maximum and minimum values. The difficulty, for the instructor, is that these deeper theorems are not known to the student.

However a completely rigorous proof can be derived at once from the following *weakened* form of the theorem of the mean:

THEOREM W. *If $f(x)$ has a derivative $f'(x)$ for each $a \leqq x \leqq b$ then for some $a \leqq x_0 \leqq b$*

$$|f'(x_0)| \geqq \left|\frac{f(b) - f(a)}{b - a}\right|.$$

The following proof of Theorem W is simple enough for any undergraduate calculus course!

Without loss of generality, suppose the interval (a, b) is $(0, 1)$. Denote $|f(1)-f(0)|$ by k and let p_1 be the least of the integers $0, 1, \cdots, 9$ for which

$$\left|\frac{f(0.(p_1 + 1)) - f(0.p_1)}{1/10}\right| \geqq k.$$

There is such a p_1, for if there were not it would follow that

$$|f(1) - f(0)| \leqq |f(1) - f(0.9)| + |f(0.9) - f(0.8)| + \cdots + |f(0.1) - f(0)|$$
$$< \frac{k}{10} + \frac{k}{10} + \cdots + \frac{k}{10} = k,$$

that is, $|f(1)-f(0)| < k$, a contradiction. Designate the interval $(0.p_1, 0.(p_1+1))$ as (a_1, b_1).

Similarly let p_2 be the least of $0, 1, \cdots, 9$ for which

$$\left|\frac{f(0.p_1(p_2 + 1)) - f(0.p_1p_2)}{1/100}\right| \geqq k$$

and designate $(0.p_1p_2, 0.p_1(p_2+1))$ as (a_2, b_2).

Repetition of this procedure yields a number x_0, given by its decimal expansion

$$x_0 = 0.p_1p_2 \cdots$$

and a sequence of intervals (a_n, b_n) such that for every n

$$\left|\frac{f(b_n) - f(a_n)}{b_n - a_n}\right| \geqq k$$

and x_0 is contained in, or is an end point of, the interval (a_n, b_n) and $b_n - a_n \to 0$ as $n \to \infty$. If x_0 is contained in a particular (a_n, b_n) we may preserve the inequality either by replacing a_n by x_0 or by replacing b_n by x_0. Then it follows from the definition of derivative that $|f'(x_0)| \geqq k$, as stated.

The sophisticated student will see that this argument is valid for functions $f(x)$ valued in an arbitrary linear, normed space. He will also use the same argument to prove (as variations of Theorem W) that for *real valued* $f(x)$ there are suitable x_1, x_2 on (a, b) with

$$f'(x_1) \leqq \frac{f(b) - f(a)}{b - a} \quad \text{and} \quad f'(x_2) \geqq \frac{f(b) - f(a)}{b - a}$$

respectively (actually, the real-variable proof of the theorem of the mean shows that there is an ξ in (a, b) for which *both* of these inequalities hold simultaneously).

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, State University of New York

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Harpur College, Endicott, New York. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1101. *Proposed by M. S. Webster, Purdue University*

The inequality of Bernoulli is often stated as follows: If $h > -1$ and $h \neq 0$, then $(1+h)^n > 1+nh$, where n is an integer greater than unity. What is the generalization if n is an arbitrary real number?

E 1102. *Proposed by Harley Flanders, University of California*

Let C be an $n \times n$ matrix such that whenever $C=AB$ then $C=BA$. What is C?

E 1103. *Proposed by Paul Monsky, Brooklyn, N. Y.*

Find the locus of the vertex of a tri-rectangular trihedral angle which moves

so that its edges intersect a given circle.

E 1104. *Proposed by J. R. Hatcher, Fisk University*

Let $g(x)$ be a given continuous function satisfying $\int_\alpha^\beta g(x)dx=0$. Find $f(x)$ such that $\int_\alpha^\beta f^2(x)dx$ is a minimum subject to the conditions $\int_\alpha^\beta f(x)dx=\gamma$ and $\int_\alpha^\beta f(x)g(x)dx=\delta$, where $\alpha, \beta, \gamma, \delta$ are constants.

E 1105. *Proposed by R. M. Mason, Washington, D. C.*

"He had a line or two rigged up to help him cross the widest spaces—Long John's earrings, they were called: and he would hand himself from one place to another, now using the crutch, now trailing it alongside by the lanyard, as quickly as another man could walk."—Robert Louis Stevenson, *Treasure Island.*

Find the probability that on the nth step Silver is not supported by his crutch.

SOLUTIONS

Value of a Derivative

E 1071 [1953, 417]. *Proposed by Albert Wilansky, Lehigh University*

If $f(x)=\int_0^x \cos(1/t)dt$, show that $f'(0)=0$.

Solution by George Piranian, University of Michigan. Since $f(0)=0$, $f'(0) =\lim_{x\to 0} x^{-1}f(x)$. Also, since

$$f(x) = -x^2 \sin(1/x) + \int_0^x 2t \sin(1/t)dt,$$

$|f(x)| \leqq 2x^2$, and the result follows.

Also solved by P. M. Anselone, Richard Courter, N. J. Fine, Donald Greenspan, M. S. Klamkin, S. Leja, A. E. Livingston, C. S. Ogilvy, S. Parameswaran, G. B. Parrish, L. L. Pennisi, M. Perisastri, L. A. Ringenberg, O. E. Stanaitis, and the proposer.

A Four-Digit Number

E 1072 [1953, 417]. *Proposed by S. J. Jasper, East Tennessee State College*

Find a four-digit number in base ten which, when the order of the digits is reversed, becomes an equivalent number in base seven.

Solution by Julian Braun, Washington, D. C. Let the number in the base ten appear as $ABCD$. Then

$$1000A + 100B + 10C + D = 343D + 49C + 7B + A,$$

whence

$$38D - 111A = (31B - 13C)/3.$$

Because B, $C \leqq 6$, the values of (A, D) are limited to (1, 2), (1, 3), (2, 5), and (2, 6), which give values for $38D-111A$ of -35, 3, -32, and 6, respectively. There are only seventeen pairs (B, C) which yield values of $(31B-13C)/3$ which are integers. Only in the case $(B, C)=(1, 1)$ do we obtain a value of $(31B-13C)/3$ matching a value of $38D-111A$ given above, namely the case for $(A, D)=(2, 6)$. Thus the *unique* four-digit number in the base ten satisfying the given requirement is 2116.

Also solved by A. N. Aheart, Norman Anning, Leon Bankoff, Charles Berndtson and W. S. Wilf (jointly), W. E. Briggs, W. E. Buker, P. L. Chessin, J. E. D'Atri, Monte Dernham, Fred Discepoli, Mildred Going, H. W. Gould, Vern Hoggatt, Douglas Holdridge, A. R. Hyde, John Jones, Jr., Allen Kirchberg, M. Kitamura, M. S. Klamkin, Sidney Kravitz, D. D. McCracken, Sylvia Manheim, Fred Marer, D. C. B. Marsh, George Millman, Sharon Murnick, E. J. Musch, J. B. Muskat, E. A. Nordhaus, C. S. Ogilvy, Margaret Olmsted, S. Parameswaran, G. B. Parrish, M. J. Pascual, L. A. Ringenberg, H. A. Robinson, R. Shafer, L. J. Shiflett, J. D. Thomas, C. W. Trigg, W. R. Van Voorhis, R. Z. Vause, J. W. Young, and the proposer.

Trigg remarked that the above is the only solution for bases x and y where $10 \geqq x > y$. Dernham pointed out the following curiosity: The corresponding problem for two-digit numbers yields $23_{10}=32_7$. Doubling each side we get $46_{10}=64_7$. There are no other two-digit solutions, and there are no three-digit solutions. Squaring each side of the last equality we find $2116_{10}=6112_7$.

Moment of Inertia of a System

E 1074 [1953, 417]. *Proposed by Vern Hoggatt, Oregon State College*

A regular n-gon with the center O has a particle of mass m at each vertex. Let PO be a segment perpendicular to the plane of the n-gon and let l be a line through P parallel to the plane of the n-gon. Show that the sum of the moments of inertia of the n masses about the line l is independent of the (restricted) orientation of l.

Solution by J. F. Burke, Lehigh University. Let I be the moment of inertia of the system with respect to l, and let r be the radius of the n-gon. Through O draw a line l_x parallel to l and assume that this line makes an angle ϕ with some radius of the n-gon. Then

$$\begin{aligned} I &= \sum_{k=0}^{n-1} mr^2 \sin^2 (\phi + 2k\pi/n) + mn\overline{OP}^2 \\ &= mr^2(n/2) + mn\overline{OP}^2. \end{aligned}$$

Also solved by A. G. Grace and W. J. Klimczak (jointly), Douglas Holdridge, A. R. Hyde, M. S. Klamkin, A. E. Livingston, Beckham Martin, S. Parameswaran, G. B. Parrish, M. Perisastri, L. A. Ringenberg, O. E. Stanaitis, C. A. Swanson, J. V. Whittaker, and the proposer.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4573. *Proposed by M. R. Spiegel, Rensselaer Polytechnic Institute, Troy, N. Y.*

Show that, if $p=\log(\sqrt{2}+1)$,

$$\int_0^p \frac{x \sinh x}{\sqrt{1-\sinh^2 x}}\,dx = \frac{\pi}{4}\log 2.$$

4574. *Proposed by H. S. Shapiro, Bell Telephone Laboratories, Murray Hill, N. J.*

Let $f(z)=\sum_{n=0}^{\infty} a_n z^n$ be the power series of a rational function, and let the a_n be rational integers. If $a_n=0$ for $n=(k!+k)$, $k=1, 2, \cdots$, then f is a polynomial.

4575. *Proposed by the late Joseph Rosenbaum, Hartford, Connecticut*

Find the complete solution in positive integers of the Diophantine system,

$$2uv - xy = 16, \qquad xv - uy = 12.$$

4576. *Proposed by G. A. Garreau, Northampton Polytechnic College, London, England*

Construct an infinite matrix A such that A^{m-1} exists, whereas A^m does not.

4577. *Proposed by Albert Wilansky, Lehigh University*

Given a point x in a Hausdorff space, does there exist a set of neighborhoods of x whose intersection contains only x, and such that of any two neighborhoods in the set one includes the other (*i.e.* the neighborhoods are "nested")?

SOLUTIONS

Deficient Numbers

4507 [1952, 555]. *Proposed by Leo Moser, University of Alberta, Canada.*

In L. E. Dickson, *History of the Theory of Numbers*, we find the statement: Charles de Neuvegelise proved that the products $3\cdot 4, \cdots, 8\cdot 9$ of two consecutive numbers are abundant.

Prove that there are infinitely many numbers of the form $a(a+1)$ which are deficient.

Solution by Bryant Tuckerman, Oberlin College and Institute for Advanced Study. Let p_i be the consecutive primes ($p_2=3$); let $b_n=4\prod_{i=2}^{i=n} p_i$, ($n\geqq 2$); and let $c_n=(b_n-2)(b_n-1)$. Then c_n has the form $a(a+1)$, and will be shown to be deficient for all sufficiently large n.

None of $p_2, \cdots, p_n$ divides b_n-2 or b_n-1, and b_n-2 is not a multiple of 4. Hence $c_n=2\prod_{j=1}^{j=k} q_j^{d_j}$ where $k>0$, $d_j>0$, and the q_j are certain distinct primes each greater than p_n. Furthermore, $k<2n+1$ (for otherwise, $c_n>2p_n^k\geqq 2p_n^{2n+1}>2^4p_n^{2n-2}\geqq b_n^2>c_n$, a contradiction).

The ratio of the sum of the divisors of c_n to c_n is

$$\frac{\sigma(c_n)}{c_n}=\frac{3}{2}\prod_{j=1}^{k}\frac{q_j^{d_j+1}-1}{(q_j-1)q_j^{d_j}}<\frac{3}{2}\prod_{j=1}^{k}\frac{q_j}{q_j-1}<\frac{3}{2}\left(\frac{p_n}{p_n-1}\right)^k<\frac{3}{2}\left(\frac{p_n}{p_n-1}\right)^{2n+1}$$
$$=\frac{3}{2}\left[\left(1+\frac{1}{p_n-1}\right)^{p_n-1}\right]^{(2n+1)/(p_n-1)},$$

which approaches $3/2$, since $p_n\to\infty$ and $n/p_n\to 0$ as $n\to\infty$. Hence for some N, and for all $n>N$, $\sigma(c_n)/c_n<2$, so that c_n is deficient.

Also solved by Fritz Herzog and the Proposer.

Editorial Note. Herzog also remarks that it can be shown similarly that for fixed k, infinitely many of the binomial coefficients $\binom{b}{k}$ are deficient.

A Polynomial Assuming Prescribed Values

4508 [1952, 640]. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn, N. Y.*

Find a polynomial $F(x)$ of lowest degree such that $F(x)+\omega^s$ is divisible by $(x-\omega^s)^r$ for $s=0, 1, 2, \cdots, p-1$, where ω is a primitive pth root of unity and r is a given positive integer. (This generalizes a problem in Goursat-Hedrick, *A Course in Mathematical Analysis*, vol. 1, p. 32.)

Solution by Chih-yi Wang, University of Minnesota. Since $F'(x)$ is divisible by $(x^p-1)^{r-1}$, and the binomials $x-\omega^s$ are relatively prime to one another, it is evident that the degree of the polynomial $F'(x)$ cannot be lower than $p(r-1)$ and hence that of $F(x)$ cannot be lower than $p(r-1)+1$. Let

$$F'(x)=a(x^p-1)^{r-1},$$

where $a(\neq 0)$ is some constant which has to be determined. By integrating from 0 to x we get

$$F(x)-F(0)=a\int_0^x(t^p-1)^{r-1}dt.$$

But $F(\omega^s)+\omega^s=0$ for $s=0, 1, \cdots, p-1$, so that

$$
\begin{aligned}
-\omega^s - F(0) &= \int_0^{\omega^s} a(t^p - 1)^{r-1}dt \\
&= \omega^s\left\{a\sum_{k=0}^{r-1}(-1)^k\binom{r-1}{k}\frac{1}{p[(r-1)-k]+1}\right\} \\
&= \omega^s\left\{a\int_0^1 (t^p-1)^{r-1}dt\right\}, \qquad s = 0, 1, \cdots, p-1.
\end{aligned}
$$

These p equations are satisfied if and only if

$$
F(0) = 0, \qquad a = -1\Big/\int_0^1 (t^p - 1)^{r-1}dt.
$$

Hence the required polynomial is

$$
F(x) = -\int_0^x (t^p - 1)^{r-1}dt\Big/\int_0^1 (t^p - 1)^{r-1}dt.
$$

Also solved by C. N. Campopiano, Harley Flanders, M. S. Klamkin, Norman Miller, and the Proposer.

A Multiplicative Function

4509 [1952, 640]. *Proposed by D. J. Newman, Republic Aviation Corporation, Farmingdale, N. Y.*

Prove that the most general multiplicative function $f(n)$ such that $f(n+1) \sim f(n)$ is $f(n) = n^a$ for some constant a.

Solution by A. E. Livingston, University of Washington. Since $f(x)$ is multiplicative, we have $f(xy) = f(x)f(y)$ for all real x, y. We note that $[f(0)]^2 = f(0)$ and, hence, that $f(0) = 0$ or $f(0) = 1$. If $f(0) = 1$, then we have $1 = f(0) = f(0)f(x) = f(x)$ for all x. In what follows, then, we take $f(0) = 0$. Furthermore it is clear that $f(1) = 1$. Hence $[f(-1)]^2 = f(1) = 1$, so that $f(-1)$ is either 1 or -1. There are thus two solutions when $f(0) = 0$: either $f(x)$ is an odd function of x or $f(x)$ is even. Consequently, we need only consider the problem for $x > 0$.

We have

$$
f(x+1) = f(x)f\left(1 + \frac{1}{x}\right), \qquad f(x) = f(x+1)f\left(1 - \frac{1}{x+1}\right).
$$

Since $f(1) = 1$ and $f(x+1)/f(x) \to 1$ as $x \to \infty$, we have $f(1^-) = f(1^+) = f(1)$, from which we deduce in the usual way that $f(x)$ is continuous for all x.

Since $f(x)$ is continuous, we have $f(x^b) = [f(x)]^b$ for every real b. In particular, if $b = \log c$, $c > 0$ and $c \neq 1$, then

$$
[f(x)]^{\log c} = f(x^{\log c}) \equiv f(c^{\log x}) = [f(c)]^{\log x}.
$$

Therefore

$$f(x) = \left\{ [f(c)]^{1/\log c} \right\}^{\log x} = x^{\log f(c)/\log c}$$

which equals x^a, say.

Also solved by Harry Furstenberg, M. S. Klamkin, Albert Wilansky, and the Proposer.

A Monotone Non-increasing Sequence

4510 [1952, 640]. *Proposed by H. P. Thielman, Iowa State College*

If $a_1, a_2, a_3, \cdots$ is a sequence of real numbers such that $a_{n+1} = 1/n^p a_n$, where $p > 0$ and $n = 1, 2, 3, \cdots$, prove that the sequence $\{a_n\}$ is monotone non-increasing if and only if $a_1 = (\pi/2)^{p/2}$.

Solution by R. H. Boyer, Carnegie Institute of Technology, Pittsburgh. Using the recurrence relation we find

$$a_{2n+1} = \left(\frac{1\cdot 3 \cdots (2n-1)}{2\cdot 4 \cdots (2n)}\right)^p a_1, \qquad \frac{1}{a_{2n}} = \left(\frac{3\cdot 5 \cdots (2n-1)}{2\cdot 4 \cdots (2n-2)}\right)^p a_1.$$

$\{a_n\}$ will be monotone non-increasing if and only if $a_{2n+1} \leqq a_{2n} \leqq a_{2n-1}$ for all n. Thus

$$\left(\frac{1\cdot 3 \cdots (2n-1)}{2\cdot 4 \cdots (2n)}\right)^p a_1 \leqq \left(\frac{2\cdot 4 \cdots (2n-2)}{3\cdot 5 \cdots (2n-1)}\right)^p \frac{1}{a_1} \leqq \left(\frac{1\cdot 3 \cdots (2n-3)}{2\cdot 4 \cdots (2n-2)}\right)^p a_1,$$

or

$$\left(\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5} \cdots \frac{2n-2}{2n-1}\cdot\frac{2n}{2n-1}\right) \geqq a_1^{2/p} \geqq \left(\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5} \cdots \frac{2n-2}{2n-3}\cdot\frac{2n-2}{2n-1}\right).$$

But we know from Wallis' formula that, for all n,

$$\left(\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5} \cdots \frac{2n-2}{2n-1}\cdot\frac{2n}{2n-1}\right) \geqq \frac{\pi}{2} \geqq \left(\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5} \cdots \frac{2n-2}{2n-3}\cdot\frac{2n-2}{2n-1}\right).$$

It is therefore necessary and sufficient that $a_1 = (\pi/2)^{p/2}$.

Also solved by Harley Flanders, Harry Furstenberg, C. V. Gregg, Emil Grosswald, P. G. Kirmser, Norman Miller, S. Parameswaran, R. F. Reeves, A. J. Rodriguez, O. E. Stanaitis, Chih-yi Wang, J. V. Whittaker, J. E. Wilkins, Jr., and the Proposer.

A Diophantine Equation

4511 [1952, 640]. *Proposed by W. V. Parker, Alabama Polytechnic Institute*

Solve the Diophantine equation

$$x^2 + y^2 + (xy)^2 = z^2.$$

Solution by C. V. Gregg, Huddersfield Technical College, England.

I. It is easily verified (and was noted by Diophantus) that solutions are given by the following for any positive integer x:

$$y = 0, \quad z = x; \qquad y = x \pm 1, \quad z = x^2 \pm x + 1.$$

II. If x is held fixed the given equation becomes a Pell-Fermat equation in y and z. Hence if (x, y_0, z_0) is a solution and $(y_0, z_0), (y_1, z_1), \cdots$, is a sequence of number pairs such that

$$(1) \qquad y_{n+1} = (2x^2 + 1)y_n + 2xz_n, \qquad z_{n+1} = 2x(x^2 + 1)y_n + (2x^2 + 1)z_n,$$

then (x, y_n, z_n) is a solution for all n. The sequence can be extended backward by

$$(2) \qquad y_{n-1} = (2x^2 + 1)y_n - 2xz_n, \qquad z_{n-1} = - 2x(x^2 + 1)y_n + (2x^2 + 1)z_n.$$

For each arbitrary x there are thus three infinite sequences of solutions obtained by (1) from **I.** However, the last two may be shown to be the same sequence in reverse order with the sign of y_n changed. Furthermore, each solution found by developing these sequences can, by interchanging the values of x and y, initiate a new sequence, and so on.

III. It will be shown that, by using the procedures of **II**, all possible solutions of the given equation may be obtained from **I**, in fact, may be obtained from solutions $(0, N, N)$ or $(N, 0, N)$ where N is an integer. Let (x_0, y_0, z_0) be any solution in positive integers in which $y_0 > x_0$. Then

$$z_0^2 = x_0^2 + y_0^2 + (x_0y_0)^2 > (x_0y_0)^2$$

so that $z_0 > x_0y_0$. Also $z_0 < y_0(x_0 + 1/x_0)$ because

$$z_0^2 < (x_0y_0)^2 + 2y_0^2 = y_0^2(x_0^2 + 2) < y_0^2(x_0^2 + 2 + 1/x_0^2).$$

Hence

$$(3) \qquad \begin{aligned} (2x_0^2 + 1)y_0 - 2x_0z_0 &< (2x_0^2 + 1)y_0 - 2x_0^2y_0 = y_0, \\ (2x_0^2 + 1)y_0 - 2x_0z_0 &> (2x_0^2 + 1)y_0 - 2x_0y_0(x_0 + 1/x_0) = - y_0. \end{aligned}$$

Let X, Y denote the transformations

$$X[x, y, z] \rightarrow [(2y^2 + 1)x + 2yz, y, 2y(y^2 + 1)x + (2y^2 + 1)z],$$

$$Y[x, y, z] \rightarrow [x, (2x^2 + 1)y + 2xz, 2x(x^2 + 1)y + (2x^2 + 1)z].$$

Now if $y_0 > x_0$, there is by (3) a finite integer p such that

$$Y^{-p}(x_0, y_0, z_0) \to (x_0, y_1, z_1)$$

where $|y_1| < x_0$. If $y_1 \neq 0$ there exists a further finite integer q such that

$$X^{-q}(x_0, |y_1|, z_0) \to (x_2, |y_1|, z_2)$$

where $|x_2| < |y_1|$. Therefore by a finite number of transformations any solution is reducible to a solution of one of the forms $(0, N, N)$, $(N, 0, N)$. Since every step in the process is reversible, the conclusion follows.

The result of the repeated transformation $Y^n(x, y_0, z_0) \to (x, y_n, z_n)$ may, by a familiar device, be expressed in the form

$$y_n = \frac{y_0}{2}(v_1^{2n} + v_2^{2n}) + \frac{z_0}{2\sqrt{x^2+1}}(v_1^{2n} - v_2^{2n}),$$

$$z_n = \frac{z_0}{2}(v_1^{2n} + v_2^{2n}) + \frac{y_0\sqrt{x^2+1}}{2}(v_1^{2n} - v_2^{2n}),$$

where $v_1 = x + (x^2+1)^{1/2}$ and $v_2 = x - (x^2+1)^{1/2}$. As the transformations X, Y are not commutative, no simple expression for the result of a random sequence of transformations is possible.

Also solved partially by Norman Anning, Leon Bankoff, Daniel Finkel, Arthur Gregory, M. R. Kenner, R. C. Lyness, Joseph Rosenbaum, K. Subbarao, R. Venkatachalam Iyer, and the Proposer.

Editorial Note. The Diophantine equation treated above can be written in the form $(x^2+1)(y^2+1) = z^2+1$. This suggests the equation

$$(x^2 + a)(y^2 + b) = z^2 + c,$$

where no two of a, b, c, vanish. There need not be any solution, *e.g.*, $(x^2+1)(y^2-1) = z^2+1$; but if any solution, not $(0, 0, 0)$ exists, then, by reasoning similar to II above, there is an infinite number of solutions with x fixed, and an infinite number with y fixed.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, Oberlin College, Oberlin, Ohio, and not to any of the other editors or officers of the Association.

Elementary Analytic and Projective Geometry. By D. J. Struik. Cambridge, Addison-Wesley Press, Inc., 1953. 9+291 pages. \$6.50.

Projective geometry may be developed either projectively or metrically. In the former case, no metric idea is used so that a projective mentality is estab-

lished without Euclidean prejudices. In the latter case, the concepts are approached from the metric side so that the treatment can be based on students' previous training in metric geometry. This book belongs to the second category.

The avowed purpose of the book is "to find the legitimate place for this field inside our present mathematical curriculum and to stress those fundamentals which are most vital for the understanding of our science as a whole." It is to be covered in one semester and only enough knowledge of elementary analytic geometry to recognize conics and quadrics is presupposed. Algebraic methods are occasionally exchanged for synthetic ones. The book is built around the Erlanger Program and contains well selected materials. Of the well known texts on this subject perhaps it most nearly parallels Graustein's Higher Geometry in subject matter and in method. But it enlarges the horizon by the introduction of historical remarks and connections with other fields while being only a little over half as long—an advantage for those colleges whose curriculum is so crowded that only a one-term course can be allotted to analytic geometry beyond the traditional course. It is certainly a very suitable and long-awaited text.

There are eleven chapters in the book. The first seven chapters cover the projective geometry of the plane and the last four chapters cover that of space. The titles of the chapters are: (1) Point Sets on a Line; (2) Line Pencils; (3) Line Coordinates. Homogeneous Coordinates; (4) Transformations of the Plane; (5) Projective Theory of Conics; (6) Affine and Euclidean Theory of Conics; (7) Projective Metric; (8) Points, Lines, and Planes; (9) Projective Theory of Quadrics; (10) Affine and Euclidean Theory of Quadrics; (11) Transformations of Space. Cross ratio is defined in terms of division ratios which are expressed in terms of Cartesian coordinates of points. Projectivity is defined as one-one algebraic correspondence between division ratios. Projective coordinates of a point are defined in terms of its trilinear coordinates and those of a line by means of its equation in projective point coordinates, which are shown to be expressible as cross ratios. That the projective plane is nonorientable and closed is illustrated with great clearness by its topologically equivalent figures. Einstein's summation convention is used to simplify the notation for projective transformation. Transformations of projective coordinates are called projective transformations when they are passive or collineations when they are active. Elementary divisors are used for the classification of collineation. A conic is defined as the locus of intersections of corresponding lines in two projective pencils. The concepts of point, line and plane in space are developed without metric ideas. Several sections are devoted to non-Euclidean geometries, vectors, force systems, central projection, perspective, parallel projection, oblique projection, and orthogonal projection.

The book is provided with collateral reading, with answers to exercises and with an index. The exercises are well selected, and the special exercises at the end of the book are most interesting.

There are several inconsistencies with regard to symbols and terminology

used in the book. On page 3, it is unfortunate but convenient to assign "the values $\lambda = \pm \infty$" to a point in order to establish correspondence between division ratio and point on a line. On page 110, one finds the statement "the value $\pm \infty$." On pp. 9, 11, 15, 23, 24, 55, the symbol ∞ is used instead of $\pm \infty$. Similarly, for a slope, $\pm \infty$ is used on p. 27 and ∞ is used on p. 136. With respect to Ex. 1 on p. 6, cross-ratio of p. 8, and the statement about the bilinear relation (5-1) on p. 10, the symbol $\pm \infty$ should be assumed as one value and hence ∞ seems a better symbol for the division ratio of that certain point B on page 3. Operations with this symbol should be specifically indicated, since (5-1) is taken as the definition of projectivity and plays an important role in its following development.

On page 3 the author decides to use "ideal versus ordinary." But on pp. 12, 59, 138, 142, 185, 210, the terms "at infinity," "improper," and "special" are adopted instead of "ideal." This use of different terms for the same kind of object on different pages might cause confusion to the beginner. Similar cases happen on p. 14 and p. 15 concerning the terms "standard form" and "canonical form" and on p. 224 with respect to "quartic" and "biquadratic."

On page 11, after proving the theorem that the cross ratio of four points on a line is invariant under a (non-singular) projectivity, the author states that "we return to this theorem in Chapter 2," which is found on p. 42 and reads as "the cross ratio is invariant under projection." The term "projection" is not defined. From the figures on the same page, it seems to have the general geometrical meaning as described on p. 232. Although the equivalence between projection and projectivity, which is defined algebraically, follows immediately from a theorem on p. 24, it seems to the reviewer an explanation might be helpful to convey to the reader a clear geometrical concept of projectivity.

The equation of a plane in geometry of space is usually understood to be a shorthand statement for "the equation connecting the coordinates of the points of a plane." Such a term as "the equation of a plane in plane coordinates" is misleading and it is unfortunately used for coordinates of a plane on p. 178 and p. 198.

There are many typographical errors in the book. Though most of the errors are obvious, yet they do cause a lot of trouble to the reader, especially to the beginner. It is the reviewer's hope that a list of errata be immediately set up and sold with the book.

T. K. Pan
University of Oklahoma

Computing Manual. Third Edition. By Fred Gruenberger, Madison, University of Wisconsin Press, 1952. 123 pages.

The title of this small paper-bound manual is misleading. Opposed to the broadness the title suggests, it is devoted exclusively to technical problems

arising in the use of standard I.B.M. punched card equipment for computing, and was intended as an aid in training personnel for such work.

Most of the chapters of the manual are devoted to the handling of some familiar computing problem. Digiting and differencing with a tabulator, and chi-squared analysis on a calculating punch, 602A, are some of the topics discussed. There is also a short section on the layout of questionnaires, keeping in mind that I.B.M. punched cards are going to be used in the analysis. In keeping with its practical intent, throughout the book, emphasis is placed on ease of handling, expense and error detection. The last quarter of the book has various laboratory problems and exercises.

Although there is a cursory discussion of the card programmed electronic Calculator, C.P.C. model one, there is no mention of the more recent I.B.M. machines such as the 101, 604, 701, *etc.*

Because the author assumes a working knowledge of the machines discussed and of the associated I.B.M. terminology, one unfamiliar with these would find little to interest him. On the other hand, an experienced operator would most likely be familiar with the contents. Although the manual was used as a text in a course on computing machines at Wisconsin, I feel that it is too technical to interest those students not going directly into I.B.M. computing. As was said, the manual was designed to aid the instruction of personnel in I.B.M. computing and, as such, should prove very useful.

GLENN LEWIS
Computer Project
Institute for Advanced Study

Real Functions. By Casper Goffman. New York, Rinehart and Company, Inc. 1953. 12+263 pages. $6.00.

This is a text-book based on a course in functions of a real variable given by the author to first-year graduate students at the University of Oklahoma. It is intended for a year course meeting three times per week. The material covered is indicated by the following Chapter headings: 1. Sets and Operations. 2. Equivalence of Sets. 3. The Real Numbers. 4. Limit Theorems. 5. Simple Properties of Sets. 6. The Cantor Ternary Set. 7. Functions. 8. Sequences of Functions. 9. The Derivative of a Function. 10. Order Types and Ordinals. 11. Borel Sets and Baire Functions. 12. Applications of Well-Ordering. 13. Measure and Measurable Sets. 14. Metric Properties of Sets. 15. Measurable Functions. 16. Approximation of Measurable Functions. 17. The Lebesgue Integral and the Riemann Integral. 18. The Lebesgue Integral as a Set Function. 19. The Fundamental Theorem of the Calculus. 20. Planar Measure and Double Integration.

The book is carefully and clearly written, being directed primarily to the student who is meeting the subject for the first time rather than to the expert. Especially in the earlier chapters, the author patiently leads the reader over

those pitfalls which are so obvious in hindsight but give students so much trouble when first encountered. There are numerous examples and exercises. The exercises vary greatly in difficulty from rather trivial remarks on the one hand to some really difficult problems on the other which will tax the most mature students. In the reviewer's opinion, the exercises would be considerably more useful if there were some indications of difficulty with more hints and cross-references to the text, other exercises or outside reading.

In Chapter 3 the real numbers are constructed using the rationals (defined as an ordered field) as a starting point. Both the Cantor and Dedekind methods for completing the rationals are outlined. However it should be pointed out that the author gives an inadequate proof of completeness for the real numbers so obtained. What is missing is a proof that his definitions of fundamental sequence and limit of a sequence of real numbers are equivalent to the usual definitions in terms of absolute values.

Slightly less than half the book is devoted to measure theory and Lebesgue integration. The theory is developed for sets on the unit intervals (except for Chapter 20). Unbounded sets are avoided as a rule except in a number of exercises where the student is asked to carry out a discussion for the general case. The last chapter contains a proof of the Fubini theorem.

This book should prove to be very useful either as a text-book or as a reference book for a first course in real variables. The average beginning graduate student with a reasonably good background should be able to read the book on his own with very little trouble except possibly with the more difficult exercises.

C. E. RICKART
Yale University

NEW BOOKS RECEIVED

Mechanics. By K. R. Symon. Cambridge, Mass., Addison-Wesley Press, Inc., 1953. 13+358 pages. $7.50.

Sample Survey Methods and Theory. Volume I. By M. H. Hansen, W. N. Hurwitz, and W. G. Madow. New York, John Wiley & Sons, Inc., 1953. 22+638 pages. $8.00.

An Introduction to the History of Mathematics. By Howard Eves. New York, Rinehart & Company, Inc., 1953. 15+422 pages. $6.00.

Methods of Mathematical Physics, Volume I. By R. Courant and D. Hilbert. New York, Interscience Publishers, 1953. 15+571 pages. $9.50.

Mathematical Research in India. By A. C. Banerji. Lahore, India. The Indian Science Congress Association and the Council of Scientific & Industrial Research, 1953. 43 pages.

Graphs of the Compton Energy-Angle Relationship and the Klein-Nishina Formula from 10 Kev to 500 Mev. By Ann T. Nelms. National Bureau of Standards Circular 542. 89 pages. $0.55.

Original Investigation or How to Attack an Exercise in Geometry. By E. S. Loomis. Columbus, Bonded Scale and Machine Co., 1953. No charge.

Introduction to Dynamics. By L. A. Pars. New York, Cambridge University Press, 1953. 22+501 pages. $6.00.

An Introduction to Homotopy Theory. By P. J. Hilton. New York, Cambridge University Press, 1953. 8+142 pages. $3.00.

An Introduction to Relaxation Methods. By F. S. Shaw. New York, Dover Publishing Company, 1953, 396 pages. $5.50.

Partial Differential Equations in Engineering Problems. By K. S. Miller. New York, Prentice-Hall, Inc., 1953. 8+254 pages. $4.75.

Plane Trigonometry. Second Edition. By A. W. Weeks and H. G. Funkhouser. New York, D. Van Nostrand Company, 1953. 8+197+37 pages. $2.68 (without tables), $2.88 (with tables).

A Mathematician's Miscellany. By J. E. Littlewood. London, Methuen and Company, 1953. 7+136 pages. 15s.

Plane Trigonometry. By P. R. Rider. New York, The Macmillan Company, 1953. 8+180 pages. $3.00.

Differential Line Geometry. By Vaclav Hlavaty. Translation based on text by H. Levy. Groningen, Holland, P. Noordhoff Ltd., 1953. 10+495 pages. f 25.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

NATIONAL REGISTER OF SCIENTIFIC PERSONNEL

The National Science Foundation is organizing a National Register of Scientific and Technical Personnel. Unlike the Roster compiled during World War II and the Register of 1950–51, this one will be a continuing affair: information will be kept up to date.

The Register is organized in sections. The American Mathematical Society has agreed to compile and maintain the section of the Register covering the mathematical sciences.

There are two major uses to which this Register can be put. In case of a national emergency the Register will quickly provide lists of specialists to whom essential work might be referred by panels of experts chosen by the appropriate scientific organizations.

The second use of the Register is in connection with statistical studies of scientific manpower. There is no doubt that the country's demand for scientific manpower far exceeds the supply. Effective policies for increasing this supply must be based on accurate information about scientists. Such information is not now available but can be furnished to a considerable extent by this Register.

Both these uses of this Register are highly important, and so you are urged to complete and return the questionnaire which you will receive soon.

SYMPOSIUM ON MONTE CARLO METHODS

Preliminary Announcement: A Symposium on Monte Carlo Methods, sponsored by the Aeronautical Research Laboratory, Wright Air Development Center, will be conducted by the Statistical Laboratory, University of Florida, at Gainesville on March 16 and 17, 1954. Registration will be on Monday, March 15, for those who arrive early. An invitation is issued to those interested in the field to attend. Further information may be obtained by writing Professor H. A. Meyer, Building OE, University of Florida, Gainesville.

Following the Symposium, an Eastern Regional meeting of the Institute of Mathematical Statistics is being planned for Thursday, March 18, 1954, at Gainesville. The Biometric Society, ENAR is meeting on March 18, 19 and 20.

PERSONAL ITEMS

Dr. Morris Ostrofsky of the Westinghouse Research Laboratories, Pittsburgh, represented the Association at the Diamond Jubilee Ceremonies celebrating the founding of Duquesne University on November 11–12, 1953.

Dean H. R. Kingston of the University of Western Ontario received an honorary degree of Doctor of Laws from Queen's University in June, 1953.

Dr. C. V. Newsom, Associate Commissioner of Higher Education in the State of New York, was granted an honorary degree of Doctor of Humane Letters by Hofstra College in June, 1953.

Professor G. deB. Robinson of the University of Toronto has been elected President of the Canadian Mathematical Congress for 1953–57.

Boston University announces the following: Dr. D. W. Blackett, formerly research associate at Princeton University, has been appointed to an assistant professorship; Dr. F. E. Browder is now a Guggenheim Fellow and is at the Institute for Advanced Study; Dr. V. R. Staknis has been appointed to an assistant professorship at Northeastern University.

Brooklyn College reports the following: Professor Walter Prenowitz is the recipient of a Ford Foundation Fellowship for the year 1953–54 and is at the Institute for Advanced Study; Dr. William Forman has been appointed to an instructorship; Mr. Leon Herbach has resigned and is with the Army Chemical Corps at New York University.

At California Institute of Technology: Dr. A. G. Mackie has been appointed Bateman Fellow; Dr. Frank Spitzer has been appointed to an instructorship.

Emory University announces the following: Dr. Morris Friedman, previously acting assistant professor at Tulane University, has been appointed to an assistant professorship; Associate Professor C. E. Clark has resigned.

Hampton Institute reports the following: Mr. S. R. Beyma, previously with the United States Army, has been appointed to an instructorship; Associate Professor C. Y. Wang is now an instructor at the University of Minnesota.

Illinois Institute of Technology announces: Mr. George Crane, Mr. Thomas Erber, and Mr. Bernard Galler have been appointed to part-time instructorships; Associate Professor Michael Sadowsky has been appointed Professor of Mechanics at Rensselaer Polytechnic Institute.

Iowa State College makes the following announcements: Dr. G. W. Peglar has been promoted to an assistant professorship; Dr. W. D. Lindstrom, previously at the State University of Iowa, Dr. D. E. Sanderson, formerly at the University of Wisconsin, and Dr. F. M. Wright of Northwestern University have been appointed to instructorships; Dr. H. J. Weiss, formerly located at Brown University, has been appointed to an assistant professorship.

Marquette University reports the following: Dr. C. B. Hanneken, formerly an assistant at the University of Illinois, and Mr. William Golomski of St. Louis University have been appointed to instructorships; Associate Professor B. F. Dostal, recently retired from the University of Florida, has been appointed a part-time lecturer; Associate Professor L. V. Toralballa is now at New York University.

Ohio State University announces: Dr. Erwin Kleinfeld, previously an instructor at the University of Chicago, has been appointed to an assistant professorship; Dr. D. W. Dubois of the University of Oklahoma has been appointed to an instructorship.

Oregon State College announces the following: Mr. R. L. Brock, who is on leave from Boeing Airplane Company, Seattle, Washington, has been appointed Acting Instructor; Assistant Professor W. M. Stone, who has returned after a two-year leave of absence spent at Boeing Airplane Company has been promoted to an associate professorship.

Rensselaer Polytechnic Institute makes the following announcements: Dr. Kurt Bing of the University of California has been appointed to an assistant professorship; Associate Professor Horace Komm of the University of the South and Instructor Valdemars Punga of the University of Massachusetts have also been appointed to assistant professorships; Mr. W. R. Beck has resigned and is at White Plains High School; Professor Dis Maly has been elected a member of the Council of the Association of Mathematics Teachers of New York State.

At St. Louis University: Reverend J. F. Daly of Rockhurst College and Mr. W. E. Perrault of Fairfield University have been appointed to instructorships; Instructor Paolo Lanzano has been promoted to an assistant professorship;

Assistant Professor A. J. Lorenz has resigned from his teaching duties and is doing graduate work at the University.

Syracuse University reports the following: Dr. K. L. Chung, previously visiting associate professor at Cornell University, has been appointed to an associate professorship; Associate Professor R. D. Whitney has retired; Assistant Professor Erik Hemmingsen has received an award from the Danish National Science Foundation.

State College of Washington announces: Professor S. G. Hacker has been appointed Professor of Mathematics and Director of the J. R. Jewett Observatory; Assistant Professor J. L. Brenner has been promoted to an associate professorship; Dr. P. A. Clement has been promoted to an assistant professorship; Dr. W. J. Firey of Stanford University, Dr. W. M. Gilbert of the Analytical Research Group of Princeton University, and Dr. M. T. Wechsler of Princeton University have been appointed to instructorships.

Texas Southern University reports the following: Professor J. A. Pierce, head of the Department of Mathematics and chairman of the Division of Natural Sciences and Mathematics, has been appointed Dean of the Graduate School; Miss Thyrsa A. Frazier, previously an analytical statistician at the Wright-Patterson Air Force Base, has been appointed to an instructorship; Mr. John Westberry and Mr. A. H. Wardlow have received renewals of their General Education Board Fellowships for study at the University of Michigan.

At the University of Alabama: Professor M. O. Gonzalez of the University of Havana has been appointed to a professorship; Dr. H. C. Filgo, previously a graduate assistant at Rice Institute, Instructor C. C. Buck of Wayne University, Mr. J. W. Jewett, formerly a teaching fellow at the University of Michigan, and Mr. B. M. Seelbinder, previously a part-time instructor at the University of North Carolina, have been appointed to assistant professorships; Mrs. Margaret B. Seelbinder, formerly a part-time instructor at the University of North Carolina, has been appointed to an instructorship.

University of California at Berkeley announces: Professors D. H. Lehmer and Hans Lewy have returned to their positions in the Department; Associate Professor J. L. Kelley, who has been promoted to a professorship, is on sabbatical leave for the year 1953–54 and holds a National Science Foundation grant; Professor Harald Cramer, who is also Rector of the University of Stockholm, served as Visiting Professor during the fall term; Professor Jan van der Corput is Visiting Professor during the year; Associate Professor Leon Henkin of the University of Southern California has been appointed to a professorship; Professor Herman Meyer of the College of the University of Chicago is Visiting Associate Professor for the year; Dr. Louise C. Lim has been awarded a Ford Foundation Fellowship and was at the University during the fall term; Associate Professor A. Seidenberg is on sabbatical leave for the year 1953–54 and is at Cambridge, Massachusetts; Professor L. H. Seinford is on leave of absence because of illness; Dr. Anne C. Davis, Dr. M. P. Epstein, Dr. D. C. Kleinecke, Dr. J. P. Roth, and Dr. R. A. Wijsman have been appointed to instructorships; Dr.

Joseph Putter has been appointed Lecturer in the Statistical Laboratory.

University of California at Los Angeles reports the following: Dr. J. R. Jackson, previously a research assistant at the University, has been appointed to an instructorship; Dr. Maurice Sion, formerly an instructor at the University of California at Berkeley, has been appointed to an instructorship; Assistant Professor P. G. Hodge, Jr., has been promoted to an associate professorship; during the year 1953–54, Professor Hodge is on leave and holds the position of Associate Professor of Applied Mathematics at Polytechnic Institute of Brooklyn; Associate Professor Richard Arens is on sabbatical leave and is at the Institute for Advanced Study; Professor P. G. Hoel is also on sabbatical leave and is in Norway on a Fulbright Fellowship; Assistant Professor L. J. Paige is on sabbatical leave and is at the Institute for Advanced Study.

University of Colorado announces: Dr. E. B. McLeod, previously a research assistant at Stanford University, and Dr. A. Zirakzadeh, who has been an instructor at Oklahoma Agricultural and Mechanical College, have been appointed to instructorships; Dr. Robert Osserman has resigned.

University of Connecticut announces the following appointments to instructorships: Dr. Oscar Litoff, Mr. A. E. Nussbaum, Dr. E. E. Osborne, and Mr. John Rausen.

University of Delaware reports: Assistant Professor Russell Remage, Jr., has been promoted to an associate professorship; Dr. E. J. Pellicciaro, previously a graduate fellow at the University of North Carolina, has been appointed to an instructorship; Assistant Professor G. O. Peters has resigned.

At the University of Florida: Associate Professor E. H. Hadlock has been promoted to a professorship; Mr. R. W. Bagley, previously a graduate assistant at the University, Miss Elaine Hundertmark, formerly a teaching assistant at the University of Illinois, and Mr. J. D. Neff, member of the Technical Staff, Bell Telephone Laboratories, New York City, have been appointed to instructorships; Associate Professor B. F. Dostal has retired; Mr. L. W. Blanton is on leave of absence to continue his graduate study at Columbia University.

University of Houston makes the following announcements: Assistant Professors Blanche Grover, Ruth Kissel, and C. A. Rogers have been promoted to associate professorships; Dr. J. C. Douglas, Jr., Mrs. Thelma I. Hammerling, Miss Eleanor S. Mohr, Mr. W. R. Strickler, and Mr. R. O. Young have been appointed part-time instructorships; Mr. W. S. Rees has returned to the University as an assistant professor; Mr. D. O. Gray has returned from military leave.

University of Idaho announces: Assistant Professor A. E. Halteman has been promoted to an associate professorship; Mr. G. E. Witter has been promoted to an instructorship; Mr. P. E. Livermore and Mr. R. G. Schrandt, who have been discharged from military service recently, have been appointed to instructorships; Mr. R. J. Rohlfs, previously a graduate student at the University of South Dakota, has been appointed to an instructorship.

University of Kentucky announces the following: Professor M. C. Brown

has been appointed Acting Head of the Department; Mr. Howard Burnette, Miss Sara Ripy, Mr. Richard Sprague, Mr. W. C. Swift, Mr. Sherman Vanaman, Mr. J. B. Wells, and Mr. Wilson Zaring have been promoted to full-time instructorships; Mr. Cephas Bevins and Mr. Thomas Rowland have been appointed to part-time instructorships; Dr. Cordell Moore has resigned and has accepted a position with Convair Aircraft Corporation, Fort Worth, Texas; Mrs. Virginia B. Leach has resigned.

The University of Maine announces the appointments of Mr. J. S. Dinsmore, previously a graduate student at the University of North Carolina, and Assistant Professor D. A. Kearns of Merrimack College, Andover, Massachusetts, to instructorships.

At the University of Massachusetts: Mr. August Newlander, who has been a staff research mathematician at the Denver Research Institute, and Mr. C. W. Naylor, formerly master at the Darrow School, New Lebanon, New York, have been appointed to instructorships; Miss Lorraine D. Lavallee, previously a student at Mount Holyoke College, Mr. E. I. Pina and Mrs. Louise E. Rice, formerly students at the University, have been appointed to part-time instructorships; Mr. R. C. Scott has been appointed to a teaching fellowship; Mr. A. A. Kheiralla has resigned.

The University of Miami announces the following appointments: Dr. Charles Capel of Tulane University and Dr. R. A. Roberts of the University of West Virginia to assistant professorships; Professor E. J. Moulton of Northwestern University to a visiting professorship.

University of Mississippi reports the appointments of Mr. Russell Stokes and Mrs. Fred White to instructorships and the retirement of Assistant Professor A. H. Samuels.

University of New Mexico makes the following announcements: Dr. J. V. Lewis, previously at the Ballistic Research Laboratory, Aberdeen Proving Ground, has been appointed to an associate professorship; Dr. Oswald Wyler, formerly a lecturer at Northwestern University, has been appointed to an assistant professorship; Mr. Milton Hoehn of the University of Idaho and Mr. Arthur Steger, previously a graduate assistant at the University of California, have been appointed to instructorships; Professor M. S. Hendrickson has been elected Chairman of the Southwestern section of the Association for the year 1953–54.

University of North Dakota announces the following: Instructor W. H. McBride has been promoted to an assistant professorship; Associate Professor W. J. Lyche of Augustana College and Mr. K. H. Olson, previously a teaching assistant at the University, have been appointed to instructorships.

University of Oklahoma reports: Assistant Professor Earl LaFon has been promoted to an associate professorship; Dr. T. K. Pan, formerly visiting assistant professor at the University of California, has been appointed Visiting Associate Professor; Dr. J. B. Giever of the Instrumentation Laboratory, Massachusetts Institute of Technology, has been appointed to an assistant

professorship; Associate Professor Andrew Sobczyk has resigned and has accepted a position as a mathematician at Los Alamos Scientific Laboratory.

University of Oregon announces: Dr. S. G. Ghurye, previously a research associate at the University of North Carolina, and Assistant Professor Bertram Yood of Cornell University, have been appointed to assistant professorships; Dr. R. L. San Soucie, formerly a National Science Foundation Fellow at the University of Wisconsin, has been appointed to an instructorship; Dr. H. J. Reiter has returned to Austria; Assistant Professor F. J. Massey has been promoted to an associate professorship; Professor Massey is on leave and is a Ford Foundation Fellow at Harvard University.

University of Rochester reports the following: Assistant Professor H. P. Adkins has been promoted to an associate professorship and also holds the position of Assistant to the Dean of Men; Mrs. Mary Estill Rudin, of Duke University has been appointed to a part-time assistant professorship; Assistant Professor D. S. Miller has resigned and is with the Eastman Kodak Company.

At the University of Tennessee: Dr. Hanan Rubin, who is on leave from the Institute of Mathematical Sciences, New York University, has been appointed Visiting Assistant Professor; Dr. H. C. Griffith has been promoted to an instructorship; Professor Wallace Givens is on leave and is at the Computing Center of New York University.

University of Toronto announces the following: Professor L. J. Mordell, who has retired from the Sadlerian Professorship of Pure Mathematics in the University of Cambridge, is Visiting Professor for the Session 1953–54; Dr. P. C. Gilmore, who was awarded one of the Postdoctorate Fellowships of the National Research Council of Canada, is in residence as Research Associate; Assistant Professors Cecilia C. Krieger and Dr. D. A. S. Fraser have been promoted to associate professorships; Mr. W. O. J. Moser has been appointed to an instructorship; Dr. T. J. Jenkins, Mr. R. F. Johnston, Mr. E. E. Noonan, Mr. W. A. Skirrow, and Mr. A. W. Walker have been appointed to assistantships; Mr. D. W. Allan, Mr. L. L. Campbell, Mr. I. Guttman, Mr. A. H. Lightstone, Mr. K. Okashimo, Mr. F. G. Robinson, Mr. D. A. Sprott, and Mr. G. Zyskind have been appointed to teaching fellowships; Dr. G. A. Dirac has been appointed Senior Lecturer at King's College, London.

University of Tulsa reports the following: Associate Professor E. A. Howard has been promoted to a professorship; Assistant Professor Sarah M. Burkhart has been promoted to an associate professorship; Dr. Kenneth Walters, previously a graduate assistant at the University of Florida, has been appointed to an instructorship.

University of Utah reports the following: Assistant Professors R. E. Chamberlain and J. H. Wolfe have been promoted to associate professorships; Dr. C. E. Burgess has been promoted to an assistant professorship; Dr. Miriam L. Dickman, Mr. Louis Barrett, previously graduate assistants at the University, and Mr. C. W. Thompson, formerly part-time instructor at the University of Colorado, have been appointed to instructorships; Associate Professor R. N.

Thomas has resigned and is at the Harvard College Observatory; Dr. E. N. Parker is now Assistant Professor of Physics at the University; Dr. Henry Hiz has resigned to accept an appointment at the University of Pennsylvania; Mr. LaMar Deverall is now engaged as a mathematician at the Dugway Proving Grounds; Mr. Marcus Peterson has accepted a position with Consolidated Aircraft Company, Fort Worth, Texas; Dr. Elmo Stewart is with the Bendix Corporation.

At the University of Western Ontario: Mr. Arthur Woods has been appointed to a professorship; Professor R. H. Cole has been appointed to the Council of the Canadian Mathematical Congress.

Washington University announces the following: Dr. K. S. Shih, previously a graduate student at the University of Illinois, has been appointed Visiting Assistant Professor; Dr. Anne E. Scheerer, formerly a graduate student at the University of Pennsylvania, has been appointed to an instructorship; Professor Walter Leighton is on partial leave of absence from the University and is serving temporarily as Chief of the Mathematics Division of the Office of Scientific Research, United States Air Force; Professor T. L. Downs has been appointed Vice-Chairman for the current academic year; Dr. H. M. Schaerf has been awarded a Ford Foundation Fellowship for the year 1953–54.

Wayne University announces: Dr. Fred Meyer, formerly a research chemist with Ethyl Corporation, Detroit, has been appointed to an instructorship; Professor H. D. Huskey is now Assistant Director of the Institute for Numerical Analysis, National Bureau of Standards, Los Angeles.

Associate Professor Floyd Bowling of Lincoln Memorial University has been promoted to a professorship.

Dr. R. D. Branstetter of Iowa State College is affiliated now with the Operations Analysis Research Group, Colorado Springs, Colorado.

Mr. M. R. Bryson of the University of Idaho has been appointed to an instructorship at Drake University.

Mr. L. C. Damsgard of Pasadena City College has retired.

Mr. A. G. Davis of the University of Massachusetts has been appointed to an instructorship at Clarkson College of Technology.

Assistant Professor E. A. Davis is on leave and is studying at the University of Chicago under a Ford Foundation grant.

Assistant Professor Mary P. Dolciani of Vassar College has been awarded a Ford Foundation Fellowship for the year 1953–54 and is spending the year as an Honorary Research Assistant at University College, University of London.

Mr. A. B. Finkelstein of Long Island University has been promoted to an assistant professorship.

Miss Gloria C. Ford of Virginia State College is teaching now at Morgan College, Baltimore, Maryland.

Dr. M. L. Juncosa of the Ballistic Research Laboratories, Aberdeen Proving Ground, Maryland, has accepted a position as mathematician with the Rand Corporation, Santa Monica, California.

Mr. R. G. Kuller has been appointed to an instructorship at Dartmouth College.

Associate Professor Marguerite Lehr of Bryn Mawr College gave a television course entitled "Invitation to Mathematics" during the first semester of 1952–53.

Dr. R. D. Luce of Massachusetts Institute of Technology has been appointed Managing Director of the Behavioral Models Project, Bureau of Applied Social Research, Columbia University.

Dr. Emanuel Parzen of the University of California has a position as a research scientist at the Hudson Laboratories, Columbia University.

Dr. R. L. Plunkett of the University of Virginia has been appointed to an assistant professorship at Vanderbilt University.

Dr. Gustave Rabson of Purdue University has been appointed to an assistant professorship at Antioch College.

Dr. C. R. Rao of Presidency College, Calcutta, India, has been appointed Visiting Professor of Mathematical Statistics at the University of Illinois.

Miss Margaret Roston has been appointed to an assistant professorship at Hood College.

Miss Ruth R. Royer of Iowa State College has been appointed to an instructorship at Chico State Teachers College, California.

Professor J. P. Scholz of Lebanon Valley College has been appointed to a professorship at Western College.

Associate Professor S. R. Smith of the University of Wyoming has been promoted to a professorship.

Professor G. W. Spencely of Miami University has retired with the title of Professor Emeritus.

Professor P. W. Stoner, previously chairman of the Department of Mathematics and Astronomy at Pasadena City College, has been appointed to a professorship at Westmount College.

Professor Emeritus J. I. Tracey of Yale University has been appointed Distinguished Professor of Mathematics at Texas Christian University for the year 1953–54.

Dr. A. B. Willcox has been appointed to an instructorship at Amherst College.

Reverend R. B. Eiten, who was Associate Professor of Mathematics at the University of Detroit, died on October 6, 1953.

Associate Professor O. W. Irwin, formerly of the faculty of Brooklyn College, died on August 22, 1953.

Professor Emeritus J. H. McDonald of the University of California died on July 4, 1953.

Mr. R. H. Mason of the University of Florida died on July 24, 1953.

Dr. Joseph Rosenbaum of the Milford School, Connecticut, died on November 10, 1953; he was a charter member of the Association.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 45 persons have been elected to membership by the Board of Governors on applications duly certified.

E. N. BRANDT, JR., Student, University of Oklahoma.

P. N. W. BURGOYNE, Student, McGill University.

L. N. CAPLAN, Student, Carnegie Institute of Technology.

G. E. CARUSO, Student, St. John's University.

D. H. CLANTON, M.A. (Baylor) Instr., University of South Carolina.

G. M. CLOUGH, B.S. (New Mexico Inst.) Mathematician, Battelle Memorial Institute, Columbus, Ohio.

P. E. DALBEC, Student, Boston College.

S. H. DALRYMPLE, Student, University of Texas.

A. E. DEAN, B.S. (U.S. Naval Post.) Grad. Student, University of Mississippi.

S. H. EISMAN, M.A. (Pennsylvania) Optometrist, 6751 Souder Street, Philadelphia 49, Pa.

J. F. ELLIOTT, M.S. (Tennessee) Teacher, Dobyns-Bennett High School, Kingsport, Tenn.

ARTHUR EVANS, JR., Student, Carnegie Institute of Technology.

A. B. FINKELSTEIN, Ph.D. (N.Y.U.) Asst. Professor, Long Island University.

C. E. FLANAGAN, Ph.M. (Wisconsin) Director, Academic Education Department, Wisconsin State College, Whitewater, Wis.

EDWARD HALAS, Student, University of Detroit.

C. B. HANNEKEN, Ph.D. (Illinois) Instr., Marquette University.

ROBERT KALIN, M.A. (Harvard) Test-Specialist, Educational Testing Service, Princeton, N. J.

H. E. KANTER, Student, Carnegie Institute of Technology.

LOLA F. KISER, B.S. (Memphis S.C.) Teaching Assistant, University of Georgia.

HORACE KOMM, Ph.D. (Michigan) Asst. Professor, Rensselaer Polytechnic Institute.

STEPHEN KULIK, Dr.Math. (Kiev) Asso. Professor, University of South Carolina.

R. G. KULLER, M.S. (Michigan) Instr., Dartmouth College.

R. F. LISKOVEC, B.S. (Kent State) 4126 East 131st Street, Cleveland 5, Ohio.

R. D. LUCE, Ph.D. (M.I.T.) Managing Director, Behavorial Models Project, Bureau of Applied Social Research, Columbia University.

MRS. HELEN M. MARSTON, M.A. (Duke) Head, Mathematics Section, Educational Testing Service, Princeton, N. J.

G. E. MEIKE, B.S. (Aquinas) Teaching Fellow, University of Detroit.

M. W. MILLIGAN, M.A. (Illinois) Asst. Professor, Adams State College, Alamosa, Colo.

M. A. OBERDICK, Student, Seton Hill University.

T. K. PAN, Ph.D. (California) Visiting Asso. Professor, University of Oklahoma.

R. N. PENDERGRASS, M.A. (Missouri) Asso. Professor, Radford College, Virginia.

J. H. POWELL, M.A. (Michigan S.C.) Grad. Research Assistant, Michigan State College.

VALDEMARS PUNGA, M.Math.Sci. (Latvia) Asst. Professor, Rensselaer Polytechnic Institute.

REV. J. H. RAYMOND, Ph.D. (U. of Washington) Chairman, St. Martin's College, Olympia, Wash.

MARGARET A. REIFF, M.S. (Oklahoma A. & M.) Temporary Instr., Oklahoma Agricultural and Mechanical College.

J. D. RICE, M.A. (Rice) Asst. Professor, Lamar State College of Technology, Beaumont, Tex.

D. A. ROBINSON, B.A. (New York S.T.C., Albany) Grad. Student, Rensselaer Polytechnic Institute.
P. G. ROONEY, Ph.D. (C.I.T.) Lecturer, University of Alberta.
R. S. ROTH, B.A. (Kenyon) Grad. Student, Carnegie Institute of Technology.
MIRIAM J. RUSSELL, M.A. (George Peabody) Instr., University of Arizona.
J. P. SCHOLZ, Ph.D. (Vienna) Acting Head, Department of Physics; Professor-elect of Mathematics, Western College.
MRS. OLIVIA H. SHANKS, M.A. (Vanderbilt) Chairman, Belmont College.
SISTER ANTONIETTA FITZPATRICK, M.A. (Texas) Instr., Incarnate Word College.
M. W. STONE, M.A. (George Peabody) Mathematician, Rohm and Haas Company, Redstone Arsenal, Huntsville, Ala.
D. J. VAN VRANKEN, B.S. (Union C.) Instr.; Grad. Student, Rensselaer Polytechnic Institute.
KENNETH WOLSSON, Student, Brooklyn College.

THE OCTOBER MEETING OF THE MINNESOTA SECTION

The October meeting of the Minnesota Section of the Mathematical Association of America was held at the Bemidji State Teachers College in Bemidji, Minnesota, on October 10, 1953. Sessions were held in the forenoon, at luncheon and in the afternoon. Professors H. D. Colson, R. C. Staley and A. G. Hill, Chairman of the Section, presided at the respective sessions.

Twenty-nine persons attended the meeting including the following twenty members of the Association:

F. J. Arena, J. M. Calloway, E. J. Camp, H. D. Colson, Ruby M. Grimes, F. C. Hatfield, A. G. Hill, Karlis Kaufmanis, W. H. McBride, Margaret Owchar, J. C. Peterson, P. A. Rognlie, L. W. Sheridan, F. C. Smith, R. C. Staley, O. E. Stanaitis, A. G. Swanson, Matilda B. Thompson, K. W. Wegner, F. L. Wolf.

By invitation of the Executive Committee, Professor O. E. Stanaitis of St. Olaf College delivered an address at the morning session entitled "Remarks on Abel's Partial Summation." Abstract of this address follows:

Abel's partial summation and the following more special criteria of convergence of series were reviewed: the test of Abel, Dirichlet, du Bois-Reymond and Dedekind, J. Hadamard, E. B. Eliot, and G. H. Hardy. It was shown that Euler's summation formula and a test superior to Hardy's test follow almost immediately from Abel's partial summation. Though the result is in any case a far-reaching one, the convergence of somewhat more general series cannot be proved, and additional refinement of the formulas obtained are necessary. The statements were illustrated by examples.

The following short papers were presented:

1. *The size of the visible universe*, by Professor Karlis Kaufmanis, Gustavus Adolphus College.

The author discussed the period-luminosity curve and pointed out the reasons that have caused the astronomers to multiply all distances determined by means of the classical cepheids by the factor of two. The maximum range of the 200-inch telescope on Mt. Palomar is $2\cdot 10^9$ light years as estimated by Baade.

2. *P-adic representations of p-primary groups without elements of finite order*, by Professor G. C. Preston, Macalester College, introduced by Professor G. K. Kalisch.

Let G be a locally compact p-primary group* and $p^\infty = Q_p - z_p$ where Q_p represents the p-adic numbers and Z_p the p-adic integers. (G, p^∞) will represent the group of all continuous homomorphisms of G into p^∞ provided with the following topology: a neighborhood of zero will consist of all homomorphisms of (G, p^∞) which map some fixed compact set of G onto the zero element of p^∞. Then $(G, p^\infty) = \widehat{G}$ where $\widehat{G}$ is the ordinary Pontrjagin character group of G. Therefore, $((G, p^\infty), p^\infty) = \widehat{\widehat{G}} = G$. From this fact it is also clear that G has "sufficiently many" continuous homomorphisms into p^∞; *i.e.*, given $a \neq b$ in G, there is a continuous homomorphism α of G into p^∞ such that $\alpha(a) \neq \alpha(b)$.

Now if G' is a linear topological space over Q_p, the set of continuous homomorphisms of G' into Q_p (1-dimensional representations in Q_p) is algebraically isomorphic to (G, p^∞). Hence G' has sufficiently many 1-dimensional representations in Q_p. If G is a locally compact p-primary group without elements of finite order, there exists a linear topological space G' over Q_p such that $G \subset G'$ (Braconnier). Hence G has sufficiently many 1-dimensional representations in Q_p.

3. *Some remarks concerning 110 formulas for the area of a plane triangle*, by Professor F. J. Arena, North Dakota Agricultural College.

In this paper the writer discusses the various types of formulas for the area of a plane triangle published by Marcus Baker in the Annals of Mathematics, vol. 1, 1884, pp. 134–138 and vol. 2, 1885, pp. 11–18.

4. *Vector methods in analytical geometry*, by Professor W. L. Woodley, North Dakota Agricultural College, introduced by Professor A. G. Hill.

In recent years vector methods have been applied to analytics with considerable success.

A line segment is written in matrix form:

$$P_1P_2 = (x_2 - x_1 \quad y_2 - y_1 \quad z_2 - z_1) \quad \text{or} \quad l_1(\cos\alpha \quad \cos\beta \quad \cos\gamma).$$

In plane analytics, any two lines may be written:

$$L_1 = (a \quad b) = l_1(\cos\alpha_1 \quad \sin\alpha_1); \qquad L_2 = (c \quad d) = l_2(\cos\alpha_2 \quad \sin\alpha_2)$$

with the scalar product

$$L_1L_2 = ac + bd = l_1l_2(\cos\alpha_1\cos\alpha_2 + \sin\alpha_1\sin\alpha_2) = l_1l_2\cos\theta,$$

or

$$\cos\theta = \frac{L_1L_2}{l_1l_2}.$$

These formulas are readily extended to three or more dimensions. The perpendicularity condition is $L_1L_2 = 0$. The area of the triangle determined by L_1 and L_2 is $\pm\frac{1}{2}L_1'L_2$ where $L_1' = (b \quad -a)$ or $(-b \quad a)$ is perpendicular to L_1.

If four components are used, a world line may be written with time as a fourth component as follows:

$$(x - x_1 \quad y - y_1 \quad z - z_1 \quad t - t_1) = \phi(x_2 - x_1 \quad y_2 - y_1 \quad z_2 - z_1 \quad t_2 - t_1)$$

where ϕ is a parameter. This is a component equation. By equating components and eliminating ϕ,

* See J. Braconnier, Sur les groupes topologiques localement compacts, J. Math. Pures Appl., vol. 27, pp. 1–85, 1948; see also I. Kaplansky, Dual modules over a valuation rank, Proc. Amer. Math. Soc., vol. 4, pp. 213–219, 1953.

we may find the coordinate equations of the line. If ϕ is replaced by $r_1/(r_1+r_2)$, the segment P_1P_2 is divided in the ratio $r_1:r_2$.

A tangent to a curve may be written $(cdt\ idx\ idy\ idz)$ and has length $k_1ds=\sqrt{c^2dt^2-dx^2-dy^2-dz^2}=\sqrt{c^2-v^2}\,dt$. The solution of this equation has some interesting properties.

5. *Report on the summer conference in collegiate mathematics*, by Professor J. M. Calloway, Carleton College.

The purpose of the conference and the nature of the lectures were indicated as well as an evaluation of the successes and failures of the conference.

F. C. SMITH, *Secretary*

THE OCTOBER MEETING OF THE OKLAHOMA SECTION

The Oklahoma Section of the Mathematical Association of America was the guest of Oklahoma City University, Oklahoma City, Oklahoma, on October 30, 1953. Professor W. N. Huff, Chairman of the Section, presided at the meeting. A total of eighty-four persons attended the meeting, including the following forty-two members of the Association:

R. V. Andree, Arthur Bernhart, P. M. Berry, J. C. Bradford, J. C. Brixey, Sarah M. Burkhart, R. B. Deal, R. C. Dragoo, J. R. Foote, I. E. Glover, E. V. Greer, L. D. Gregory, L. A. Guest, O. H. Hamilton, Claire A. Harrison, J. O. Hassler, E. F. Heimann, J. E. Hoffman, W. N. Huff, P. W. M. John, L. W. Johnson, Douglas Jones, J. E. LaFon, Gene Levy, B. L. Mackin, Dora McFarland, G. E. Meador, Dorothea Meagher, R. R. Murphy, F. J. Palas, D. L. Patten, G. M. Petersen, C. M. Pirrong, E. C. Rice, J. W. Sehestedt, M. G. Shults, O. S. Spears, C. E. Springer, Vivian Spurgeon, J. D. Thomas, G. R. Vick, J. H. Zant.

At the business meeting the following officers were elected: Chairman, Professor R. B. Deal, Oklahoma Agricultural and Mechanical College; Vice-Chairman, Professor C. M. Pirrong, Oklahoma City University; Secretary-Treasurer, Professor R. V. Andree, University of Oklahoma.

Mathematical literature, including the *O.U. Mathematics Letter*, was distributed after the business meeting.

The following papers were presented:

1. *Summation methods and continuity*, by Dr. G. M. Petersen, University of Oklahoma.

The usefulness of various methods of evaluating divergent series with relation to boundedness, circle of convergence of the related power series, *etc.*, was discussed. Examples of methods which evaluate rapidly diverging series, but which fail to sum bounded series, were given.

2. *Need for a dynamic theory of games*, by Professor Arthur Bernhart, University of Oklahoma.

The static theory of Von Neumann is not *relevant* to actual parlor game strategy. Several examples of *dynamic* situations where static theory needs modification were given.

3. *The Fregier point in euclidean n-space*, by Professor R. B. Deal, Oklahoma Agricultural and Mechanical College.

Orthogonal pairs of lines through a fixed point on a conic meet the conic in pairs of points of an orthogonal involution whose center lies on the lines joining corresponding points. This center is called the Fregier point for the given point and the conic. The locus of the Fregier point for a given conic is called the Fregier conic. The purpose of this investigation is to show that the concept generalized readily for orthogonal n-types of lines through a given point on a hyperquadric.

4. *The functional equation* $f^2(x)-f^2(y)=f(x+y)f(x-y)$, by Mr. J. E. Hoffman, University of Oklahoma, and Professor R. B. Deal, Oklahoma Agricultural and Mechanical College.

The only real continuous solutions of the functional equation $f^2(x)-f^2(y)=f(x+y)f(x-y)$ are found to be $f(x)=cx$, $f(x)=a \sinh bx$, and $f(x)=a \sin bx$. The methods, particularly those used in proving differentiability from continuity and the functional equation, are shown to be applicable to a number of other functional equations. Discontinuous solutions are shown to exist.

5. *Two typical elementary problems in artillery research*, by Major O. S. Spears, Fort Sill, Oklahoma.

In armed forces research, there are many important, interesting problems in elementary mathematics. The following are typical of certain problems which frequently arise in artillery research:

Problem 1: *The effect of a multiple-fuzing system on the time at which a projectile or warhead is expected to explode.* Ordinarily, the explosion of a projectile occurs when a pre-set timing mechanism (fuze) functions. If a normal distribution, $f(t)$, is assumed for a given type of fuze, then an interesting statistical problem arises when several identical fuzes are attached to the same projectile in such a manner that the projectile explodes when at least n of the fuzes function. (The variable t refers to the time at which a fuze functions.)

Problem 2: *Probability of artillery hits on aircraft flying along a battle front.* It is sometimes necessary for aircraft to fly parallel to a battle front in support of infantry, while friendly artillery is firing in a direction perpendicular to the same battle front, and in support of the same infantry. There exists, therefore, a certain probability that a given aircraft flying under stipulated conditions will be accidentally hit by an artillery shell. The magnitude of such probabilities can materially affect tactical doctrine. An approach to the problem is outlined.

6. *Hyperbolic non-euclidean geometry*, by Professor J. O. Hassler, University of Oklahoma. (By invitation.)

Professor Hassler substituted the Gauss-Bolyai-Lobachewski definition of parallel lines for the euclidean and the postulate that there are two parallels ("right-hand" and "left-hand") to a given line through an outside point for the Playfair form of the euclidean postulate of parallels, but included the other postulates and definitions of euclidean geometry in developing the usual sequence of theorems leading to the proof that the sum of the angles of a triangle is less than one straight angle.

The afternoon was devoted to a series of papers and panel discussions in conjunction with the Mathematics Section of the Oklahoma Education Association.

R. V. ANDREE, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirty-fifth Summer Meeting, University of Wyoming, Laramie, Wyoming, August 30–31, 1954.

Thirty-eighth Annual Meeting, University of Pittsburgh, Pittsburgh, Pennsylvania, December 30, 1954.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Marshall College, Huntington, West Virginia, May 1, 1954.

ILLINOIS, Knox College, Galesburg, May 14–15, 1954.

INDIANA, Rose Polytechnic Institute, Terre Haute, May, 1954.

IOWA, Iowa State College, Ames, April, 1954.

KANSAS, Baker University, Baldwin City, March 27, 1954.

KENTUCKY, April 24, 1954.

LOUISIANA-MISSISSIPPI, Tulane University, New Orleans, February 19–20, 1954.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA

METROPOLITAN NEW YORK, St. John's University, Brooklyn, March 27, 1953.

MICHIGAN, University of Michigan, Ann Arbor, March 27, 1954.

MINNESOTA, Hamline University, St. Paul, May 8, 1954.

MISSOURI, University of Missouri, Columbia, May 7, 1954.

NEBRASKA, Omaha, April 24, 1954.

NORTHERN CALIFORNIA

OHIO, Ohio State University, Columbus, April 17, 1954.

OKLAHOMA, Oklahoma City University, October, 1954.

PACIFIC NORTHWEST, Reed College, Portland, Oregon, June 18, 1954.

PHILADELPHIA

ROCKY MOUNTAIN, Colorado Agricultural and Mechanical College, Fort Collins, April, 1954.

SOUTHEASTERN, University of South Carolina, Columbia, March 19–20, 1954.

SOUTHERN CALIFORNIA, George Pepperdine College, Los Angeles, March 13, 1954.

SOUTHWESTERN, Arizona State College, Tempe, April 16–17, 1954.

TEXAS, Texas Technological College, Lubbock, April 23–24, 1954.

UPPER NEW YORK STATE, College for Teachers at Albany, May 1, 1954.

WISCONSIN, State Teachers College, Eau Claire, May, 1954.

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 61 NUMBER 3

CONTENTS

MARCH 1954

The AMERICAN MATHEMATICAL MONTHLY

(Founded in 1894 by Benjamin F. Finkel)

EDITORIAL CORRESPONDENCE should be addressed to the Editor, C. B. Allendoerfer, Department of Mathematics, University of Washington, Seattle 5, Washington.

ADVERTISING CORRESPONDENCE should be addressed to J. F. Randolph, Morey Hall, University of Rochester, Rochester 3, N. Y.

NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the Monthly should be sent to the Secretary-Treasurer, H. M. Gehman, University of Buffalo, Buffalo 14, N. Y.

THIS IS THE OFFICIAL JOURNAL OF THE

MATHEMATICAL ASSOCIATION OF AMERICA, INC.

(Devoted to the Interests of Collegiate Mathematics)

Entered as second class matter at the post office at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

Subscription Price: To Members, $4 a Year, To Others, $5 a Year.

Published by the Association at Menasha, Wisconsin, and Seattle, Wash. during the months of January, February, March, April, May, June–July, August–September, October, November, December.

PRINTED IN THE UNITED STATES OF AMERICA

OF COURSE AND COURSES*

SAUNDERS MACLANE, The University of Chicago

Mr. President, members of the Assocation, and guests. I am here to retire, but I do not wish to be retiring. Both ends can be accomplished if I address myself to the principal objective of the Mathematical Association of America: that of the steady and imaginative improvement of collegiate education in Mathematics.

Collegiate education in Mathematics must not be construed too narrowly. On the one hand, we cannot ignore high school mathematics; this is the source of our entering students with their manifold difficulties in Mathematics. On the other hand, we cannot neglect graduate instruction in Mathematics; this is the source of our new teachers and the goal of some of our ablest students.

Collegiate education in Mathematics needs the most imaginative and vigorous reform, for it is now beset by numerous troubles and inadequacies.

These are internal troubles. Many of our courses cleave valiantly to a weak and obsolete tradition. Calculus has a perspicuous and beautiful intellectual structure, but its usual presentation is distorted by the unhappy fact that each new "standard" calculus text must copy the weakness of a long line of equally imitative predecessors. Trigonometry is in worse state; the publishers and authors of trigonometry texts conspire to demonstrate in exhaustive detail the combinatorial fact that an infinity of different texts is possible. The demonstration can be given more briefly: just combine all the alternative orderings of the following: define first the trignometric functions of an acute angle, or first those of a general angle; first triangles by logarithms or triangles by natural functions; identities first or equations first, *etc.*, *etc.*, *etc.* Among the standard courses, College Algebra is in perhaps the worst shape. Years ago, when it was first established†, it perhaps made sense, but under the pressure from masses of weaker and weaker entering students, the course has been analytically continued by continued dilutions. Today "College Algebra" stands for a subject which ought to be taught in the high school and which has nothing to do with algebra.

These are only some of the internal troubles of collegiate mathematics. There are also numerous external troubles. On the one hand, the social scientists have discovered that Mathematics can be of use. They properly complain that calculus and linear algebra are currently taught with a view only to the engineering and physical applications. They urge needed reform, and they are under a great temptation to urge too much in the way of special courses addressed primarily to the social scientists. They may fall thereby into the basic error of assuming that one can prepare students to make the necessary new applications

* Retiring Presidential Address before the Mathematical Association of America, December 31, 1953.

† The title "College Algebra" is an old one; there are American books under this title by Thomson and Quimby (1880), Bowser (1888) and J. M. Taylor (1889)—and perhaps others earlier.

of novel mathematical ideas by training them primarily in the old applications of older ideas. It cannot be too often reiterated that the aim of collegiate mathematics is the understanding of mathematical ideas *per se*. The applications support the understanding, and not vice versa.

Mathematics, like the rest of the academic community, is also bedevilled by the current fashion of general education. In its basic assumption this fashion was necessary. The emphasis on the great books was a needed corrective to our sorry adherence to text books that were far from great. The emphasis on an education that is broad and common to all students was a necessary answer to the American problem of giving a mass education to masses of students barely removed from illiteracy. However, the general educators cannot avoid one basic observation about the intellectual status of our age: there in fact is no common and accepted conceptual organization of present day knowledge, hence there is no possibility of constructing courses to convey this organization. For this reason and others, the fashion of general education has now been over-extended. As evidence, I cite you the fact that every University president treats general education as the Russians treat science: each president claims that his university discovered it first, or at any rate discovered its only true realization. These enthusiastic presidents miss the point that general education was probably started without benefit of presidents, as at Columbia, which in effect had no president, and as at Chicago, which had a Chancellor and not a president.

In the face of these fashions and troubles, we must keep our minds fixed on the real objectives of collegiate education in Mathematics. We must contrive ever anew to expose our students—be they general students or specialized students—to the beauty and excitement and relevance of mathematical ideas. We must set forth the extraordinary way in which mathematics, springing from the soil of basic human experience with numbers and data and space and motion, builds up a far-flung architectural structure composed of theorems which reveal insights into the reasons behind appearances and of concepts which relate totally disparate concrete ideas.

The program of our sessions today will amply cover the various problems which here arise for the more elementary collegiate courses. I therefore propose to turn my attention to the problems which arise in advanced undergraduate and beginning graduate courses. It is here that the sweeping reforms are necessary; the curriculum in advanced mathematics must be so overhauled that it can set forth the real structure of Mathematics as it is today. This structure can no longer be presented by piecemeal courses, for it is simply no longer true that advanced mathematics can be split neatly into compartments labelled "algebra," "analysis," "geometry," and "applied mathematics."

The fact that these subdivisions have ceased to be relevant may be seen most strikingly by observing how many of the most significant current discoveries in Mathematics refuse to be classified in the old compartments. Basic problems in functions of several complex variables have recently been solved by

using the notion of a *faisceau* or "sheaf"—a notion coming straight from algebraic topology. Hilbert's fifth problem has been solved—using techniques derived from topology, from Hilbert-space theory, and from Lie groups. In which compartment does this lie? In algebra, some of the most fruitful advances arise because of the demands for new algebraic tools for use in topology or in algebraic geometry. One is forced to the realization that algebra is not, and indeed never has been, an independent discipline. Modern analysis is replete with distributions, with rings of operators and with representations, so much so that it is difficult to say whether it is analysis, or a wholly new subject.

You may protest that I am talking about research and not about education, and that this should be the business of the Society, not the Association. I answer, first: The character and direction of current research is the best indication of the ideas which we ought to be teaching. I answer, second: One of our main responsibilities is that of training the research mathematicians of the future. American Mathematics has made tremendous strides forward in the last two decades; an essential ingredient in this advance has been the infusion of European mathematical talent. In the decades to come, we must produce a similar infusion on our own and from our own students.

In training research mathematicians the old ideals are not sufficient. When research mathematics was first developed in this country, it was necessary that the emphasis be just on the idea of research: something new, something unpublished, with no especial attention to the significance of the results or their place in Mathematics. Now it is necessary that the research be done with full appreciation of the significance of the various parts of our science, and with full availability of the techniques from the various disciplines which may become necessary even for the apparently most specialized problem.

My topic is "Of Course and Courses;" I mean "of the courses which may be constructed to produce an integrated course in mathematics." As an existence proof for such a course, I offer you the *Elements de Mathematique* by that dipsomaniac, Nicholas Bourbaki. He has achieved a Gallic version of a conceptual integration of mathematics; you may point out that he has paid a high price by way of abstraction and length, but I submit you that the goal is worth the price.

To be more specific, I would like to describe the mathematics curriculum which has been developed by the Department of Mathematics at the University of Chicago. This curriculum is intended to carry serious students of Mathematics from the beginning of the Junior year of college through a Master's degree in Mathematics. The objective is the ambitious one of providing a complete introduction to all those ideas which are basic to Mathematics in the sense that they occur in their several aspects in more than one part of Mathematics.

I must confess at once that our curriculum is still divided for administrative convenience into algebra, geometry, and analysis, but our emphasis is and should be on the use of each of these techniques in the other fields. Some subdivision is

necessary because our courses must appear in units of one quarter each, but we view these individual courses—and we hope that the students so view them—as part of a larger pattern.

In algebra we present a sequence of five quarter courses. The first of these is an introduction to abstract algebra by way of number theory, group theory, and vector spaces. That abstract notions require introduction by way of examples is clear; this is done. But abstract notions are not "hard." The young students like to think in these terms. It is a happy omen for the progress of civilization that here (and elsewhere in the curriculum) the beginning student takes more easily to abstractions and generalities than does his professor, who had to understand them the hard way.

The beginning algebra course also offers an ideal place for the introduction of the notion of a mathematical proof. This must be done, even for students who don't aim to be mathematicians. Algebra provides a better locus for this introduction than does the traditional plan of introducing the student to rigor and complex variables at the same time—for this type of introduction led to the assumption that rigor was the same thing as epsilontics. Bringing in rigor with algebra brings it in sooner and in simpler form.

The second and third algebra courses deal completely with vector spaces and linear transformations with especial attention to such topics as the invariant description of linear transformations (elementary divisors and the Jordan and rational canonical forms), the properties of quadratic forms (in particular, the principal axis theorem and its geometrical meaning), and finally the various relevant general notions, such as invariants, equivalence, and dual vector spaces. These topics belong early in the training of a mathematician. The proper treatment of calculus for functions of several variables requires vector ideas; the budding statistician and the coming physicist need them; modern analysis is unthinkable without the notion of linear dependence and all that flows from it. Throughout these courses the infusion of a geometrical point of view is of paramount importance. A vector is geometrical; it is an element of a vector space, defined by suitable axioms—whether the scalars be real numbers or elements of a general field. A vector is not an n-tuple of numbers until a coordinate system has been chosen. Any teacher and any text book which starts with the idea that vectors are n-tuples is committing a crime for which the proper punishment is ridicule. The n-tuple idea is not "easier," it is harder; it is not clearer, it is more misleading. By the same token, linear transformations are basic and matrices are their representations.

The fourth algebra course reverts to abstract algebra proper, with a treatment of rings, homomorphisms, ideals, groups, normal subgroups, the Jordan-Hölder theorem, and the Sylow theorems. The important notion of tensor product of groups and of spaces is often included and probably should always be there, for it is needed to understand modern geometry and modern algebraic topology. It is proved that every finitely generated abelian group is a direct sum of cyclic groups. Indeed, this theorem is an almost ideal example of an algebraic

"structure theorem." I wish that I could report that this theorem is always established as a special case of the corresponding theorem about modules over a principal ideal ring, but I must confess that the latter notion goes down hard—even though it is necessary as a connection between the groups treated in this course and the canonical forms for linear transformations from the previous course. In any event, abstract algebra is done with emphasis on the basic idea of homomorphism—the object of algebra is not just the study of a mathematical system *per se*, but of mappings of one such system into another. This idea is not restricted to algebra, and thus underlines again the conceptual unity of Mathematics.

The fifth course of the algebra sequence deals with the Galois theory—one of the most beautiful examples of a self-contained mathematical discipline, and one of the most convincing demonstrations of the power of the notions "homomorphism" and "automorphism." The course terminates with the basic structure theorems for linear algebra, which are the models for current developments in the structure of rings and in the study of rings of operators on a Hilbert space.

The geometry sequence consists of three courses. The first of these takes up analytic geometry, already treated to some extent in the calculus course, and carries it further. At the same time, analytic projective geometry appears, not in the flowery decadence which this subject reached in its American heyday, but as a necessary introduction of geometrical ideas of duality and of locus, and as a first demonstration that geometry starts with the space of ordinary experience but has the fertility to conceive new spaces representing and extending that experience.

The second geometry course deals with the foundations of geometry. The axiomatics of projective geometry, with the introduction of coordinates on the basis of these axioms, is one of the most beautiful instances of the power of axiomatic method, and at the same time emphasizes how geometry leads to algebra and how abstract notions like those of endomorphism have concrete meanings. The course continues with n-dimensional projective geometry—collineations, correlations, and the classification of hyperplanes. Finally, it turns to non-euclidean and inversive geometry, where these geometries are given in terms of subgroups of the projective group, thus illustrating again the relevance of group theory.

The third geometry course treats differential geometry. Here again we see the inadequacies of the "standard" course in this subject in comparison with the actual state of Mathematics. It is no longer sufficient to consider curves, surfaces, curvature, torsion, and first and second fundamental forms, all as an elegant application of the calculus. One must pay attention to differential geometry as it now is: with this in view, the course omits some of the more extended and uninteresting parts of the classical doctrine and instead provides an introduction to ideas of differential geometry in the large (the four vertex theorem, the theorem on turning tangents, *etc.*) and to the modern ideas of differentiable manifolds. The geometry on a surface is just not adequate if it is

done only "locally"—the consideration of changes of coordinate systems must build up to the notion of a differentiable manifold, and the study of the first and second differential form must lead to the notion of exterior differential forms on a manifold. The business of the young mathematicians is with ideas, and these are the ones he must meet in this field, the sooner, the better.

Finally I turn to analysis. The ideas of calculus are presented to freshmen and sophomores, as usual. We attempt to treat calculus with proper attention to rigor and more rapidly than is the custom, this by trimming some of the barnacles which have accumulated through the years. One just doesn't need an infinity of different applications of the definite integral! The calculus for several variables is a hard nut; for example, the proper treatment of Stokes' and Green's theorem really requires the notion of exterior forms; I must confess that we have not yet found a good way to introduce these ideas where they belong in calculus.

The final course in our calculus sequence covers various topics in advanced calculus. It starts with the idea of uniform convergence for series; this is then applied to establish the standard results for Fourier series. The idea of successive approximation is then introduced and used to provide existence and continuity proofs for differential and integral equations. The course terminates with a survey of the methods of complex variable theory up through contour integration, leaving the more sophisticated treatment of these topics for a later course on the subject.

The further reaches of analysis can no longer be treated in isolation from other topics in Mathematics; properly construed, they rest essentially upon algebra and topology and in turn fructify these subjects. Hence the student next takes a sequence of two courses in topology. The first of these, on point sets and metric spaces, starts with the basic algebra of sets, including cartesian products, and develops the cardinal and ordinal numbers and the technique of using Zorn's lemma in its various forms. Then comes a study of metric spaces, including completeness and compactness; the power of the notion of a metric is illustrated by showing how uniform convergence can be realized as convergence in the metric of suitable function space. This treatment of metric spaces presents the ϵ-δ technique of analysis in its proper setting and motivates the more general notions of topology.

The latter ideas are covered in the second course, starting with the definition of a topological space and continuing through the study of the separation axioms, connectivity, and compact spaces. The connection with metric spaces is re-established by means of metrization theorems. The possibility of various topologies on function spaces is used to illustrate the breadth and power of the general notion of a topological space, while the compactification theorems illuminate the processes of constructing new spaces from old ones. The course ends with the Weierstrass-Stone approximation theorem and, when possible, with a brief treatment of fundamental groups and of covering spaces. The last topic in particular has great merit, with application on the one hand to topo-

logical groups, and on the other to algebraic topology, where it serves as an introduction to the more general notions of fibre bundles.

On this basic array of topological instruments the student is then well prepared to take up the further essential topics in analysis; a systematic study of complex variable theory (one quarter) and an examination of measure theory and Lebesgue and Stieltjes integrals (one quarter). The traditional material of a course in real variables is thus subdivided and dispersed—as it should be. It appears in part in the more general notions of metric spaces, and in part in more advanced specialized courses.

This then is the curriculum which my colleagues at the University of Chicago have laid out to cover mathematical ideas on a broad front. For the student who aims to teach, it provides a sound knowledge of what mathematics is about. For the student going further in Mathematics, it provides a broad base and technical equipment for the attack on further ideas—functional analysis, topological groups, algebraic topology, differentiable manifolds, algebraic geometry, the structure of rings, Lie groups, and the rest. The basic requirement on a sound curriculum is precisely that it give the necessary background for these and other studies, and thereby exhibit the unity of Mathematics.

In so outlining the curriculum which has been set up at Chicago, I do not wish to claim that it is perfect or unique. It has some gaps (for example, an introduction to partial differential equations). Other quite different organizations of material could be made; indeed some of these organizations could better stress the relation between various ideas. I claim then no perfection, I submit rather that this is but one first approximation to the pressing need for basic mathematical training. One must design modern and coherent curricula, cleared of traditional impedimenta and providing rapid access to new and general ideas, so that they can then be applied in special domains. I submit that this objective is a vital one and urge that you go and do better.

MOTIVES AND TRENDS IN MATHEMATICS*

H. K. FULMER, Georgia Institute of Technology

1. The main idea of this paper. The purpose of this paper is to set forth the thesis that in basic discoveries in science generally, and in mathematics particularly, the ordinary motives of men do not seem to operate. For example, if we list as ordinary motives (1) the desire for personal prestige, (2) the wish to create something useful—that is, the utility motive—and (3) (strongest of all)

* Condensed from a paper presented at the annual meeting of the Southeastern Section of the Mathematical Association of America at Alabama Polytechnic Institute, March 13–14, 1953.

the desire for personal profit, we hope to show that these have not usually been associated with basic discoveries. Indeed, and this is really the main idea, we hope to show that there seems to exist an opposition, a sort of settled aversion, a repulsion between these ordinary motives and the will to recognize and create new ideas. The notion is rather closely associated with the unwritten law that no really great teacher is in the profession purely for the money. And in similar vein, others have speculated on how Newton or Euler would regard the forty-hour week with ideas given a project number and reduced to small sub-sections each in charge of a director.

The study is pertinent at the present time since most of our current research (aside from the defense effort) is sponsored with the sole motive: Will it pay? If in the past this motive has not produced, there is reason to doubt that it will produce in the future and that our elaborate set-ups for study will yield a return.

2. An example from history. To cite in the history of mathematics a simple illustration of the fact that utility and the creative urge have not as a rule been found together, consider the history of plane geometry in Egypt and Greece. Legend claims that the early development of geometry occurred in Egypt, the reason being that the priestly class of that country had the leisure to study it. In this environment it might have taken root and developed into a permanent science. But in response to the needs of government the study of geometry was diverted toward practical ends. Granted that our historical accounts of how land must be measured to apportion taxes and how washed-out boundaries must be replaced may be in considerable part fiction, nevertheless geometry came to have the utilitarian motive. It became, under the Egyptians, a geometry of rules without reasons. It was applied geometry—no theorems, no coherence, no plan. What was the result? We are familiar with it: the Egyptians, always with the motive of utility, produced no permanent geometry. They failed utterly to produce any semblance of a logical system in this field.

Now consider geometry several hundred years later in Greece. The Greek mind, particularly Euclid's, entertained no notion of profit or application. The Greeks were not concerned with usefulness. But they did possess an instinctive urge toward learning for its own sake, a craving to discover the reasons for things. And so, with no thought of utility, but with every thought of reason and logical structure, the Greeks produced a geometry that still endures, a work of surpassing strength and beauty.

3. Another historical example. Consider the mathematics of the Roman Empire. The record here is one of peculiar sterility, a desert, a stretch of hundreds of years without a single noteworthy advance in our field. The Romans added nothing to their scientific inheritance. No Roman mathematician has a place in history. Even the few historical attempts at kindly references are open to question, since the historians of that period were impressed with the necessity of pleasing their rulers. Over this long period of hibernation of mathe-

matical reasoning, what characteristic was dominant in Roman thought? The answer, as we all know, is that the Romans were a practical people, that their primary thought was utility. Thus they sought a practical mathematics, a geometry of immediate usefulness, an arithmetic that would solve the problems of the day. The Roman mind had no time for definitions, for logical structure, for theorems and proofs. Though these people had inherited riches in abundance, they had not inherited the inquiring spirit, the native curiosity that had distinguished the Greeks. They had inherited rather the practical viewpoint, the notion that there was no place except for the useful. And so they produced nothing, added nothing, and the Roman Empire affords another historical example of the barren results that seem inevitably to attend the utility motive. Here again we could point to an apparent inborn opposition between the spirit to create and the desire to use.

4. Motives as expressed by great mathematicians. It is useful, in our study, to examine the motives of history's great mathematicians as expressed by themselves. What were their motives? Did they consider whether their studies would be useful or profitable? We quote the well-known remark of Archimedes,

> "Every kind of art which is connected with daily needs is ignoble and vulgar."

Galileo said:

> "I am filled with infinite astonishment and also infinite gratitude to God that it has pleased Him to make me alone the first observer of such wonderful things, which have been hidden in all past centuries."

And the remark of Newton is well known:

> "I do not know what I may appear to the world; but to myself I seem to have been only like a boy playing on the seashore, and diverting myself in now and then finding a smoother pebble or a prettier shell than ordinary, whilst the great ocean of truth lay all undiscovered before me."

And still considering motives, recall that Descartes, while yet a soldier, occupied all his leisure time examining new ideas, testing out his first faint notions of analytic geometry. Newton spent his declining years editing and improving his notes even though he had no intention of publishing them. These glimpses into the lives and thoughts of Archimedes, Galileo, Descartes, and Newton illustrate again the main thesis of this paper—the idea of conflict between profit and utility as against the patient speculative curiosity that marks the true pioneer of new ideas.

5. Mathematicians in government and industry. We conclude this paper with a forward look. A trend has set in that is difficult to assess, a trend that is new and growing. We refer to the increasing drift of men highly trained in mathematics into the realm of government and of business. Thirty years ago a man or woman specializing in mathematics had only one field open—the teach-

ing profession. With the exception of a few men employed in statistics by insurance companies practically no mathematicians found a place in industry. This meant that the colleges were able year after year to recruit the most talented and accomplished mathematicians from the graduate schools. How rapidly this is changing we all know full well. No longer do the colleges have a monopoly on the best brains; the colleges are fighting a losing battle against better paying government and still better paying industry.

A recent study of the membership of this Association shows that at least 10% are engaged in work other than teaching. In round numbers this group of non-teaching members totals 450 and roughly equals in number the Association's entire membership in our Southeastern States. Many of these mathematicians work in various divisions of the Federal government. Many others are in research bureaus of vast industrial organization.

Are their surroundings and general motivation of the sort which in the past have stimulated learning? Consider that in the main their studies are directed. Few are free to follow their own inclinations. The mathematics is definitely a means toward a technical end. Groups are highly organized with chiefs, assistant chiefs, directors and project managers. Perhaps in industry there is less organization than in the Federal government, but in industry the profit motive is frank and open. The job in industry, with rare exceptions, is not the advancement of natural science at all. The job is to develop, produce, and sell goods at a profit. Stated simply, the job in industry is to outwit one's competitors, to produce a better article at a lower price.

6. Contrast of motives and conclusion. Contrast the motives here with the motives of those who in history advanced the frontiers of knowledge. How remote from organizational matters, from chiefs, directors, and advisory councils, from projects and reports is the patient, inquiring, doubting, questioning attitude of a Galileo or a Newton. Could it be that there exists, as history seems to show, an implacable conflict between the selfish motive and the will to create? Could it be that in the long years of the future only meager returns will attend our vast and expensive research organizations, that man's progress still must wait on those who hear and heed the ancient call

> "Something lost beyond the ranges,
> Something hidden, go and find it."

A CHAIN OF CIRCLES ASSOCIATED WITH THE 5-LINE

J. W. CLAWSON, Ursinus College

1. Introduction. In the second half of the nineteenth century there was considerable activity in the field of elementary geometry. Many new points, straight lines and circles associated with the triangle were discovered. Some of the most interesting are found in current textbooks on Modern Geometry. In the same period investigations into the geometry of the complete quadrilateral or 4-line uncovered many interesting relations. The writer collected many of these in an article published in 1919. [1]. The complete pentagon or 5-line has not been so extensively explored. The important theorem that the mid-diagonal lines (Newtonians) of the five 4-lines obtained by omitting in turn each side of the 5-line concur at the center of the conic inscribed in the 5-line is implicit in a lemma of Newton [2]. The point of concurrency might well be termed the Newtonian point of the 5-line. Apart from this, few theorems are well-known except those connected with the Clifford and Morley chains. These are the basis for this article and are stated in section 2. In sections 3–5 we apply these theorems to the 5-line and add some connective material. In sections 6–8 we use the method of inversion to add some new points and circles to the geometry of the pentagon. The methods of pure Euclidean geometry are used throughout.

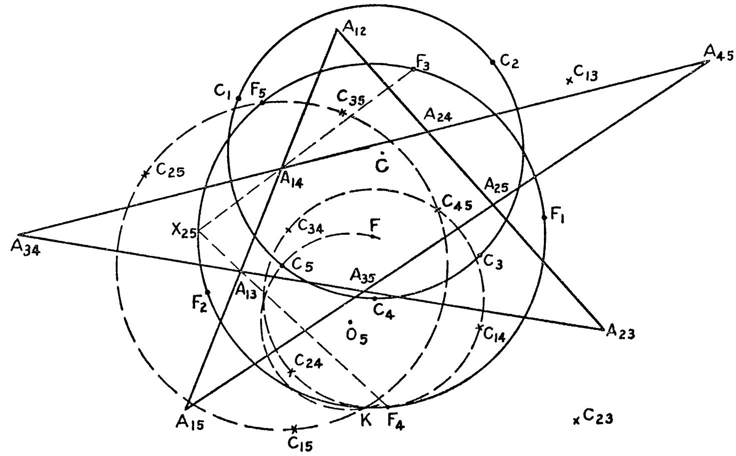
2. Clifford and Morley chains. In 1871, W. K. Clifford announced [3] a series of theorems: Two straight lines determine a point. Three straight lines determine a circle. In a 4-line, the circles circumscribing the four triangles obtained by omitting in turn each of the four sides, pass through a point—the Wallace [4] or focal point of the 4-line. In a 5-line, the five Wallace points of the 4-lines obtained by omitting in turn each of the five sides, lie on a circle, [5] which we shall call the Clifford circle of the 5-line. In a 6-line, the six Clifford circles of the 5-lines obtained by omitting in turn each of the six sides, pass through a point, which we shall call the Clifford point of the 6-line. And so on, so that in a $2n$-line, the $2n$ Clifford circles of the $2n-1$-lines obtained by omitting in turn each of the $2n$ sides, pass through the Clifford point of the $2n$-line; while in a $2n+1$-line, the $2n+1$ Clifford points of the $2n$-lines obtained by omitting in turn each of the $2n+1$ sides, lie on the Clifford circle of the $2n+1$-line.

In 1877, G. de Longchamps [6] stated the following series: In a 4-line, the centers of the circles circumscribing the four triangles, obtained by omitting in turn each of the four sides, lie on a circle which we shall call the Morley (circumcentric) circle of the 4-line; this circle also passes through the focal (Wallace) point. In a 5-line, the five centers of the Morley circles of the 4-lines, obtained by omitting in turn each of the five sides, lie on a circle which we shall call the Morley circle of the 5-line; while these five circles themselves pass through a point which we shall call the de Longchamps point of the 5-line. And so on, so

that in an n-line, the n centers of the Morley circles of the $n-1$-lines, obtained by omitting in turn each of the n sides, lie on a circle which we shall call the Morley circle of the n-line; while these n-circles themselves pass through a point which we shall call the de Longchamps point of the n-line.

Proofs of these theorems may be found in the original papers, more systematically in such books as Coolidge, "Treatise on the Circle and the Sphere" [**7**] and, by less elementary methods, in Morley and Morley, "Inversive Geometry" [**8**].

3. Clifford circle, Morley circle, de Longchamps point of the 5-line. We now re-state these theorems insofar as they apply to the 5-line, introducing the notation that will be used in this paper, and adding a theorem, (f) which was announced by F. Bath in 1940 [**9**].



Let l_1, l_2, l_3, l_4, l_5 be five straight lines in a plane, no two parallel and no three concurrent. Let A_{12} be the intersection of l_1, l_2; and so on. Let the circumcircle of $A_{12}A_{23}A_{13}$ be $\mathfrak{C}_{45}$, its center C_{45}; and so on. Then

(a) *$\mathfrak{C}_{12}$, $\mathfrak{C}_{13}$, $\mathfrak{C}_{14}$, $\mathfrak{C}_{15}$ have a common point, F_1.*

(b) *C_{12}, C_{13}, C_{14}, C_{15}, F_1 lie on a circle, $\mathfrak{C}_1$, its center C_1. There are five points like F_1 and five circles like $\mathfrak{C}_1$. F_p and F_q both lie on $\mathfrak{C}_{pq}$.*

(c) *F_1, F_2, F_3, F_4, F_5 lie on a circle, $\mathcal{J}$, its center F. This is the Clifford circle of the 5-line.*

(d) *$\mathfrak{C}_1$, $\mathfrak{C}_2$, $\mathfrak{C}_3$, $\mathfrak{C}_4$, $\mathfrak{C}_5$ have a common point, K. This is the de Longchamps point of the 5-line.*

(e) *C_1, C_2, C_3, C_4, C_5 lie on a circle, $\mathfrak{C}$, its center C. This is the Morley circle of the 5-line.*

(f) *$\mathcal{J}$ also passes through K.*

Since Bath's proof of (f) is difficult, it may be well to give a simple proof here. We use R. A. Johnson's notation [**10**]; PQR is the angle through which the

line PQ taken as a whole must be rotated counterclockwise in order for it to coincide with QR taken as a whole.

Taking K to be with C_{45} an intersection of $\mathfrak{C}_4$ and $\mathfrak{C}_5$, $F_4KF_5 = F_4KC_{45} + C_{45}KF_5 = F_4C_{34}C_{45} + C_{45}C_{35}F_5$.

But, since $\mathfrak{C}_{34}$ and $\mathfrak{C}_{45}$ intersect in A_{12} and F_4, $F_4C_{34}C_{45}$ is half the central angle $F_4C_{34}A_{12}$ of $\mathfrak{C}_{34}$ and hence equal to $F_4F_3A_{12}$. Similarly $C_{45}C_{35}F_5 = A_{12}F_3F_5$. So, adding, $F_4KF_5 = F_4F_3F_5$. Hence circle $\mathcal{J}$ passes through K.

4. Ten points on the Clifford circle. We add the theorem: *F_pA_{qr}, F_qA_{pr}, F_rA_{pq} concur at a point on $\mathcal{J}$. Here p may be any of the digits* 1, 2, 3, 4, 5; *q any different digit; r any third digit. There are thus ten possible combinations.*

To prove this theorem we shall need to use Clifford's proof of theorem (c) above.

Let F_3A_{14} and F_4A_{13} intersect at X_{25}. Consider the triangle $A_{13}A_{14}X_{25}$ with the points F_3 on $X_{25}A_{14}$, F_4 on $X_{25}A_{13}$ and first the point A_{12}, second the point A_{15} on $A_{13}A_{14}$. Then since, in the first case, circles $A_{12}A_{13}F_4$ or $\mathfrak{C}_{45}$ and $A_{12}A_{14}F_3$ or $\mathfrak{C}_{35}$ intersect in F_5, in the second case, $A_{13}A_{15}F_4$ or $\mathfrak{C}_{24}$ and $A_{14}A_{15}F_3$ or $\mathfrak{C}_{23}$ intersect in F_2, it follows from a converse of Miquel's theorem that the circle $F_3X_{25}F_4$ passes through both F_5 and F_2. This proves that F_2, F_3, F_4 and F_5 are concyclic.

Since the intersection of F_1A_{34} and F_3A_{14} may be proved in the same way to lie on circle $\mathcal{J}$, it follows that F_1A_{34}, F_3A_{14} and F_4A_{13} concur at a point on $\mathcal{J}$. This does not seem to have been noted before.

5. A connective relation. We add another theorem which does not seem to have been stated before:

(g) *The de Longchamps point and the center of the Clifford circle are inverse points with respect to the Morley circle.*

We shall need the following lemma: $F_pF_qF_r - C_pC_qC_r = F_pA_{st}F_r$.

Proof. $F_4F_5F_3 = F_4X_{25}F_3 = A_{34}A_{14}X_{25} + X_{25}A_{13}A_{34} + A_{13}A_{34}A_{14}$. Now $A_{13}A_{34}A_{14} = A_{13}F_5A_{14} = C_{45}C_{25}C_{35}$, (since $\mathfrak{C}_{25}$, $\mathfrak{C}_{45}$ intersect in A_{13}, F_5 and $\mathfrak{C}_{25}$, $\mathfrak{C}_{35}$ in A_{14}, F_5) $= C_{45}KC_{35} = C_4C_5C_3$, (since $\mathfrak{C}_4$, $\mathfrak{C}_5$ intersect in C_{45}, K and $\mathfrak{C}_3$, $\mathfrak{C}_5$ in C_{35}, K).

Hence $F_4F_5F_3 = A_{24}A_{14}F_3 + F_4A_{13}A_{23} + C_4C_5C_3$.

Then $F_4F_5F_3 - C_4C_5C_3 = A_{24}A_{12}F_3 + F_4A_{12}A_{23} = F_4A_{12}F_3$. This is generalized above.

We shall now prove that the circle FC_5K is orthogonal to $\mathfrak{C}$. Let O_5 be the center of this circle.

Now $O_5C_5C = O_5C_5F + FC_5C$. But $O_5C_5F = \frac{1}{2}\pi - FKC_5 = \frac{1}{2}\pi - (F_5C_{45}K - F_5F_3K)$ since C_5, F are centers of circles passing through F_5, K. But $F_5F_3K = F_5F_3F_4 - KF_3F_4 = F_5F_3F_4 - KF_5F_4$. Hence $O_5C_5F = \frac{1}{2}\pi - F_5C_{45}K + F_5F_3F_4 - KF_5F_4$.

Also $FC_5C = C_4C_5C - C_4C_5F = \frac{1}{2}\pi - C_5C_3C_4 - C_{45}KF_5$, since C_4C_5 is perpendicular to KC_{45} and FC_5 to KF_5.

Thus $O_5C_5C = \pi + F_5F_3F_4 - C_5C_3C_4 - F_5C_{45}K - C_{45}KF_5 - KF_5F_4$. However, $\pi - F_5C_{45}K - C_{45}KF_5 = KF_5C_{45}$ and $KF_5C_{45} - KF_5F_4 = F_4F_5C_{45}$. Also, by the lemma, $F_5F_3F_4 - C_5C_3C_4 = F_5A_{12}F_4$. Hence, $O_5C_5C = F_5A_{12}F_4 + F_4F_5C_{45}$.

But $F_4F_5C_{45} = \frac{1}{2}\pi - F_5A_{12}F_4$, considering $\mathfrak{C}_{45}$. Thus $O_5C_5C = \frac{1}{2}\pi$. Since FC_4K may similarly be proved orthogonal to $\mathfrak{C}$, theorem (g) follows.

6. New systems of circles and points. Now let us use a circle of inversion $\mathfrak{K}$ with K as center and with any definite radius. The five circles $\mathfrak{C}_1$, $\mathfrak{C}_2$, $\mathfrak{C}_3$, $\mathfrak{C}_4$, $\mathfrak{C}_5$, will invert into a 5-line c_1, c_2, c_3, c_4, c_5 with c_1, c_2 intersecting in c_{12}, and so on. We then make the same constructions as in the original 5-line. The circumcircle of $c_{12}c_{23}c_{13}$ shall be $\mathfrak{d}_{45}$, its center d_{45}. Then

(a′) $\mathfrak{d}_{12}$, $\mathfrak{d}_{13}$, $\mathfrak{d}_{14}$, $\mathfrak{d}_{15}$ have a common point, g_1.

(b′) d_{12}, d_{13}, d_{14}, d_{15}, g_1 lie on a circle, $\mathfrak{d}_1$, its center d_1. There are five points like g_1 and five circles like $\mathfrak{d}_1$.

(c′) g_1, g_2, g_3, g_4, g_5 lie on a circle $\mathfrak{g}$, its center g.

(d′) $\mathfrak{d}_1$, $\mathfrak{d}_2$, $\mathfrak{d}_3$, $\mathfrak{d}_4$, $\mathfrak{d}_5$ have a common point, l.

(e′) d_1, d_2, d_3, d_4, d_5 lie on a circle $\mathfrak{d}$, its center d.

(f′) $\mathfrak{g}$ also passes through l.

(g′) g and l are inverse points with respect to $\mathfrak{d}$.

We now invert this system of circles with respect to the same circle $\mathfrak{K}$, so that c_{12} goes back into C_{12}, and so on. We should recall that of a pair of inverse circles, the inverse of the center of one circle is the inverse of the center of inversion in the other circle [**11**]. We obtain at once the following theorems which apply to the original 5-line and its circles.

(A) *Circle $C_{34}C_{35}C_{45}$, which we shall call $\mathfrak{D}_{12}$, and three other such circles, $\mathfrak{D}_{13}$, $\mathfrak{D}_{14}$, $\mathfrak{D}_{15}$ have a common point, G_1.*

(B) *If the inverse of K with respect to $\mathfrak{D}_{12}$ is D_{12}, then D_{12}, D_{13}, D_{14}, D_{15} and also G_1 lie on a circle, $\mathfrak{D}_1$.*

(C) *G_1, G_2, G_3, G_4, G_5 lie on a circle, $\mathfrak{G}$.*

(D) *$\mathfrak{D}_1$, $\mathfrak{D}_2$, $\mathfrak{D}_3$, $\mathfrak{D}_4$, $\mathfrak{D}_5$ have a common point, L.*

(E) *If the inverse of K with respect to $\mathfrak{D}_1$ is D_1, then D_1, D_2, D_3, D_4, D_5 lie on a circle, $\mathfrak{D}$.*

(F) *$\mathfrak{G}$ also passes through L.*

(G) *If the inverse of K with respect to $\mathfrak{G}$ is G, then G and L are inverse points with respect to $\mathfrak{D}$.*

But the circle $\mathfrak{F}$ also passes through K, by (f). If the inverse of $\mathfrak{F}$ in $\mathfrak{K}$ is f, then for example the five circles $\mathfrak{C}_2$, $\mathfrak{C}_3$, $\mathfrak{C}_4$, $\mathfrak{C}_5$, $\mathfrak{F}$ invert into a 5-line c_2, c_2, c_4, c_5, f. Let the intersection of c_2 and f be f_2, and so on. Repeating the same constructions as above with this 5-line and denoting the circle circumscribing $f_3f_4c_{34}$ by $\mathfrak{f}_{34}$, its center f_{34}, and so on, we have

(a₁) $\mathfrak{f}_{34}$, $\mathfrak{f}_{35}$, $\mathfrak{f}_{45}$, $\mathfrak{d}_{12}$ have a common point, g_{12}.

(b₁) f_{34}, f_{35}, f_{45}, d_{12}, g_{12} lie on a circle $\mathfrak{d}'_{12}$, its center d'_{12}.

(c_1) g_{12}, g_{13}, g_{14}, g_{15}, g_1 lie on a circle $\mathfrak{g}_1'$, its center g_1'.
(d_1) $\mathfrak{d}_{12}'$, $\mathfrak{d}_{13}'$, $\mathfrak{d}_{14}'$, $\mathfrak{d}_{15}'$, $\mathfrak{d}_1$ have a common point, l_1.
(e_1) d_{12}', d_{13}', d_{14}', d_{15}', d_1 lie on a circle $\mathfrak{d}_1'$, its center d_1'.
(f_1) $\mathfrak{g}_1'$ also passes through l_1.
(g_1) g_1' and l_1 are inverse points with respect to $\mathfrak{d}_1'$.

If we now invert this system with respect to the same circle $\mathfrak{K}$, so that f_2 goes back into F_2 and so on, we obtain at once the following theorems which apply to the original 5-line and its circles.

(A_1) *Circles $F_3F_4C_{34}$, which we shall call $\mathfrak{F}_{34}$ and two other such circles $\mathfrak{F}_{35}$, $\mathfrak{F}_{45}$, together with $\mathfrak{D}_{12}$ have a common point, G_{12}.*

(B_1) *If the inverse of K with respect to $\mathfrak{F}_{34}$ is F_{34}, then F_{34}, F_{35}, F_{45}, D_{12}, G_{12} lie on a circle $\mathfrak{D}_{12}'$.*

(C_1) *G_{12}, G_{13}, G_{14}, G_{15}, G_1 lie on a circle, $\mathfrak{G}_1'$.*

(D_1) *$\mathfrak{D}_{12}'$, $\mathfrak{D}_{13}'$, $\mathfrak{D}_{14}'$, $\mathfrak{D}_{15}'$, $\mathfrak{D}_1$ have a common point, L_1.*

(E_1) *If the inverse of K with respect to $\mathfrak{D}_{12}'$ is D_{12}', then D_{12}', D_{13}', D_{14}', D_{15}', D_1 lie on a circle, $\mathfrak{D}_1'$.*

(F_1) *$\mathfrak{G}_1'$ also passes through L_1.*

(G_1) *If the inverse of K with respect to $\mathfrak{G}_1'$ is G_1', then G_1' and L_1 are inverse points with respect to $\mathfrak{D}_1'$.*

There are four other systems of this type.

7. Connective relations. Now c_1, c_2, c_3, c_4, c_5, f form a 6-line. Hence, taking the next step in the Clifford and Morley chains, the circles, $\mathfrak{d}$, $\mathfrak{d}_1'$, $\mathfrak{d}_2'$, $\mathfrak{d}_3'$, $\mathfrak{d}_4'$, $\mathfrak{d}_5'$ pass through a point, p, the Clifford point of the 6-line.

Also, circles $\mathfrak{g}$, $\mathfrak{g}_1'$, $\mathfrak{g}_2'$, $\mathfrak{g}_3'$, $\mathfrak{g}_4'$, $\mathfrak{g}_5'$ pass through a point, q, the de Longchamps point of the 6-line.

And the centers of the circles $\mathfrak{d}$, $\mathfrak{d}_1'$, $\mathfrak{d}_2'$, $\mathfrak{d}_3'$, $\mathfrak{d}_4'$, $\mathfrak{d}_5'$ lie on a circle, the Morley circle of the 6-line.

It follows at once that:

(H) *The circles $\mathfrak{D}$, $\mathfrak{D}_1'$, $\mathfrak{D}_2'$, $\mathfrak{D}_3'$, $\mathfrak{D}_4'$, $\mathfrak{D}_5'$ pass through a point P.*

(I) *The circles $\mathfrak{G}$, $\mathfrak{G}_1'$, $\mathfrak{G}_2'$, $\mathfrak{G}_3'$, $\mathfrak{G}_4'$, $\mathfrak{G}_5'$ pass through a point, Q.*

(J) *The inverse points of K with respect to the circles $\mathfrak{D}$, $\mathfrak{D}_1'$, $\mathfrak{D}_2'$, $\mathfrak{D}_3'$, $\mathfrak{D}_4'$, $\mathfrak{D}_5'$ lie on a circle.*

8. The chain. In each of the six systems of circles described in section 6, there are six circles passing respectively through the points L, L_1, L_2, L_3, L_4, L_5. Thus, inverting and re-inverting with respect to circles with each of these points as center and with definite radii, we may obtain thirty-six new systems in each of which theorems like (A) to (G) apply, linked by six systems to which theorems like (H) to (J) apply. This process may be continued indefinitely, with 6^p new

systems obtained at the pth repetition of the process linked by 6^{p-1} of the other systems.

References

1. Annals of Mathematics, vol. 20, page 232. Also, vol. 23, page 40.
2. Principia, 1684, vol. 1, sect. 5, lemma 25, cor. 3.
3. Messenger of Mathematics, vol. 5.
4. Leybourn's Mathematical Repository, 1804, p. 170.
5. Miquel, Journ. de Math., vol. 3, 1838, p. 485.
6. Nouvelle Correspondance Mathématique, vol. 3, p. 306.
7. Oxford, 1916, p. 90 and p. 92.
8. Boston, 1933, p. 268 and p. 160.
9. Mathematical Gazette, p. 518.
10. This MONTHLY, vol. 24, p. 101.
11. *E.g.*, R. A. Johnson, "Modern Geometry," Boston, 1929, p. 56.

ROTORS WITHIN ROTORS

MICHAEL GOLDBERG, Bureau of Ordnance, U. S. Navy

"As if a wheel had been in the midst of a wheel."—Ezekiel x.10

1. Introduction. This note describes a relationshp between two curves each of which is known for its own remarkable properties. One curve is an involute of the deltoid. The deltoid is known also as the tricuspid, or the hypocycloid of three cusps. The other curve is known as the curve of Ribaucour. It is one of the parallels to the astroid of the hypocycloid of four cusps. Hence, the curve of Ribacour is called a para-astroid [1, 2].

The following theorem will be demonstrated.

THEOREM. *The curve of Ribaucour can be inscribed in the deltoid involute and may be turned continously in a constrained manner through all orientations.*

2. The deltoid involute rotor in the square. The deltoid is traced by the ends of a line segment of length $4b$ whose midpoint moves along a fixed circle of radius b through the angle 2ϕ while the line segment rotates in the opposite direction about this moving midpoint through the angle ϕ. The well-known fact that the line segment is always tangent to the curve will be shown. The tangential polar equation of the deltoid is

$$p = b \sin 3\phi \tag{1}$$

where p is the distance from the origin to the tangent whose inclination with the x-axis is $90° - \phi$. The right member is called the supporting function (translating Minkowski's term *Stützfunktion*).

In the rectangular coordinate system, if (x_E, y_E) is the contact point, s_E is the element of arc length traced by E, and $p' = dp/d\phi$, we have in general

$$(2) \qquad \begin{aligned} x_E &= p\cos\phi - p'\sin\phi, & dx_E &= -\sin\phi\, ds_E \\ y_E &= -p\sin\phi - p'\cos\phi, & dy_E &= -\cos\phi\, ds_E, \end{aligned}$$

and the radius of curvature $\rho = ds/d\phi$ and the intercept n of the normal are given by the formulas

$$(3) \qquad \begin{gathered} \rho = s'_E = p + p'' \qquad s_E = p' + \int p\,d\phi \\ n = -y\csc\phi = p + p'\cot\phi. \end{gathered}$$

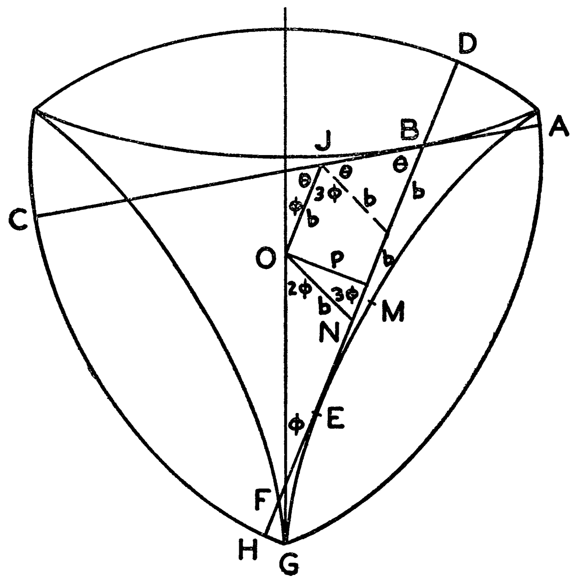


FIG. 1

From Figure 1, the familiar parametric coordinates of the variable point B on the curve are as follows:

$$(4) \qquad \begin{aligned} x_B &= 2b\sin\phi + b\sin 2\phi, & x'_B &= 4b\cos(3\phi/2)\cos(\phi/2) \\ y_B &= 2b\cos\phi - b\cos 2\phi, & y'_B &= 4b\cos(3\phi/2)\sin(\phi/2). \end{aligned}$$

We find that the arc length s_B, measured from the midpoint of an arch for a change of $\phi/2$ in the inclination of the tangent, is given by

$$s'_B = 4b\cos(3\phi/2), \qquad s_B = (8b/3)\sin(3\phi/2).$$

Also, since the slope at B is $y'/x' = \tan(\phi/2)$, the angle between the two tangents at B in Fig. 1 is $\theta = 90° - \phi - \phi/2$.

In equations (4), if ϕ is replaced by $180° - 2\phi$, the coordinates take the values in equations (2) showing that the contact point E is on the locus of point B.

If a straight line of length $16b/3$ (the length of one of the three arches of the deltoid) is placed so that it is tangent to a cusp, one end coinciding with a vertex, and then rolled over the deltoid without slipping, its ends will describe an involute of the deltoid which passes through all the vertices of the deltoid. Since this rolling line of fixed length is normal to the involute at its ends, the involute is a curve of constant width. If a pair of parallel tangents is held fixed, the involute may be turned between them, remaining tangent to both lines. Another pair of parallel tangents may be added without restraining the rotation. Therefore, the deltoid involute is said to be a rotor in a square (or any rhombus).

Since

$$DE = 16b/3 - GE = 8b/3 + EM = 8b/3 + p' - \int p\,d\phi$$
$$= (8b/3)(1 + \cos 3\phi)$$

and

$$BD = DE - p' + b\sin\phi - 2b = (2b/3)(1 + \cos 3\phi), \tag{5}$$

the parametric coordinates of the deltoid involute are given by

$$\begin{aligned} x_D &= x_B + BD\sin\phi = (b/3)(8\sin\phi + 2\sin 2\phi + \sin 4\phi) \\ y_D &= y_B + BD\cos\phi = (b/3)(8\cos\phi - 2\cos 2\phi + \cos 4\phi). \end{aligned} \tag{6}$$

3. The Ribaucour rotor in the triangle. The curve of Ribaucour is described by the tangential polar equation

$$p = a(3 + \cos 2\theta). \tag{7}$$

In parametric form, the coordinates of this curve are

$$\begin{aligned} x &= (a/2)(9\cos\theta - \cos 3\theta) = 2a(3\cos\theta - \cos^3\theta) \\ y &= (a/2)(3\sin\theta - \sin 3\theta) = 2a\sin^3\theta. \end{aligned} \tag{8}$$

The Cartesian form is

$$x^2 + y^2 + 3(2ay^2)^{2/3} = 16a^2. \tag{9}$$

This curve, which resembles an ellipse, has a major axis of twice the length of the minor axis. If an equilateral triangle is circumscribed about the curve, the sum of the distances of its sides from the origin is constant, as is shown in the following equation:

$$
(10)\qquad
\begin{aligned}
&p(\theta) + p(\theta + 120°) + p(\theta - 120°) \\
&\qquad = a(3 + \cos 2\theta) + a(3 + \cos 2\theta \cos 120° - \sin 2\theta \sin 120°) \\
&\qquad\quad + a(3 + \cos 2\theta \cos 120° + \sin 2\theta \sin 120°) = 9a.
\end{aligned}
$$

Since the altitude of the triangle is equal to the constant sum of these distances, the triangle has the same size for all orientations. Therefore, the curve of Ribaucour is a rotor in an equilateral triangle.

4. The normal of the curve of Ribaucour. The distance between the point of contact and the foot of the perpendicular from the origin to the tangent is given by p'. If the intercept of the normal between the curve and the major axis is denoted by n, its value is derivable as follows:

$$(11)\qquad n = p + p' \cot \theta = a(3 + \cos 2\theta - 2 \sin 2\theta \cot \theta) = a(1 - \cos 2\theta).$$

The radius of curvature ρ is given by the following equation:

$$(12)\qquad \rho = p + p'' = a(3 + \cos 2\theta - 4 \cos 2\theta) = a(3 - 3 \cos 2\theta) = 3n.$$

This relation (the radius of curvature is three times the normal intercept) is sometimes used as the definition of the curve of Ribaucour.

5. Demonstration of the theorem. If the rolling line of Section 2 carries a plane with it, the successive positions of the fixed deltoid in the moving plane will have a curve as an envelope. This curve is said to be molded in the moving plane by the fixed deltoid involute. In Figure 1, construct the line AC tangent to an arch of the deltoid at B. Through B pass another line DH tangent at E to another arch of the deltoid, cutting the deltoid again at F. Denote the midpoint of BF by N. Let the angle GON be 2ϕ, and the angle between BF and OG be ϕ, as described in Section 2. Similarly, the line OJ has rotated through the angle ϕ while the line AC has rotated through $\phi/2$ in the opposite direction. We have

$$2\theta - 3\phi = 180°$$

from which the angle θ between the two tangents through B is given by

$$(13)\qquad \theta = 90° - 3\phi/2.$$

This confirms a result derived in Section 2.

The point B is the instantaneous center of rotation of the plane which carries AC. Therefore, the curve molded in this plane must touch the deltoid involute at D which is the foot of the perpendicular from B. From equations (5) and (13) we have:

$$(14)\qquad BD = (2b/3)(1 + \cos 3\phi) = (2b/3)(1 - \cos 2\theta).$$

Comparison of this value of the normal intercept with equation (11) shows that

the molded curve is the curve of Ribaucour with $a=2b/3$.

In Figure 2, the curve of Ribaucour lies inside the deltoid involute. An equilateral triangle is circumscribed about the curve of Ribaucour and a square is circumscribed about the deltoid involute. The figure illustrates one of the ∞^3 configurations of the rotors and their polygons.

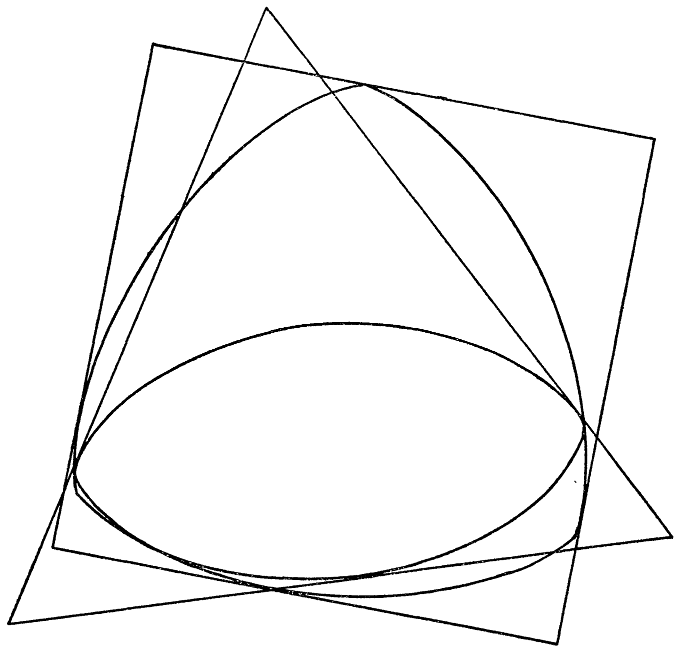


FIG. 2

6. Parallel rotors. The theorem can be generalized to parallel rotors. If the deltoid involute is replaced by a parallel curve at any arbitrary distance, the curve of Ribaucour can be replaced by a parallel curve separated by the same distance. These curves also are rotors in the square and the triangle respectively. As the arbitrary distance is increased, the curves approach circular shapes. Internal parallel curves cannot be used since the radii of curvature at the vertices of the original curves are zero; internal parallel curves would no longer be convex.

7. Similar theorems. The theorem derived here is similar to a chain of known theorems. Every hypocycloid of n cusps is a rotor in another hypocycloid of $n+1$ cusps. Also, every epicycloid of n cusps is a rotor in an epicycloid of $n+1$ cusps. Their parallel curves have been applied in the design of rotary pump elements [3].

The theorem described here is not derivable as a special case of the foregoing theorems. Similarly, the possible generalization to involutes of higher

order hypocycloids is not an immediate consequence.

References

1. Gino Loria, Curve piane speciali algebriche e trascendenti, vol. 2, Curve trascendenti, Milan, 1930, pp. 157–167, 328.

2. H. Wieleitner, Spezielle ebene Kurven, Leipzig, 1908, pp. 299–302.

3. M. F. Hill, Kinematics of Gerotors, Philadelphia. 1927.

LINEAR RECURRENCE RELATIONS*

J. L. BRENNER, State College of Washington

1. Introduction. The relation

$$u_{n+1} = u_n + u_{n-1},$$

which is satisfied by the Fibonacci sequence $u_{-1}=0$, $u_0=1$, $u_1=1$, $u_2=2$, $u_3=3$, $u_4=5$, $\cdots$, can be generalized. For example, the sequence of Lucas [5] can be defined similarly from the relation

$$u_{n+1} = Pu_n - Qu_{n-1}, \tag{1}$$

where u_{-1}, u_0 have the same values as before, and u_1, u_2, $\cdots$ are calculated in succession from (1). Another linear recurrence relation is discussed in [6] in connection with a divisibility question. The most general linear recurrence relation (over integers) can be written in the form

$$u_{n+1} = \alpha_0 u_n + \alpha_1 u_{n-1} + \cdots + \alpha_{k-1} u_{n-k+1} + a, \tag{2}$$

where k is positive, a and α_i are integers ($i=0, \cdots, k-1$), and α_{k-1} is not 0. The integer k is called the degree of the relation, and is fixed henceforth. If (initial) integral values u_{-1}, u_0, $\cdots$, u_{k-2} are prescribed, the values u_{k-1}, u_k, $\cdots$ can be computed in succession from (2). If α_{k-1} happens to be 1 or -1, the sequence u_k can be computed for negative values of k also, by solving (2) for u_{n-k+1}, as in [6]. Linear recurrence relations and the sequences associated with them are useful in solving certain problems concerning primality or divisibility; Lehmer [4] gave a test for the primality of Mersenne numbers 2^p-1 (p, a prime) which has been used to discover a prime with several hundred digits; this test was based on properties of the Lucas sequence. The relation $u_n | u_{nr}$ holds generally for Lucas sequences. An illuminating account of linear recurring sequences is given in [3].

Two interesting (known) properties of recurring sequences are studied in this article by the use of matrix methods. Firstly, the residues mod p of the u_n must be periodic; so that secondly, if u_{-1} is 0, then some u_n must be divisible by

* The writing of this article was sponsored by the Office of Ordnance Research, U. S. Army, under contract DA-04-200-ORD-113. This is technical report 7 under that contract.

p, if p is a prime not dividing α_{k-1}. General theorems about matrices lead quickly to fairly precise results concerning the size of this period. On the other hand, it would be possible to establish theorems concerning the period of a matrix belonging to a certain class of matrices by starting with results concerning recurrence relations which have already been established in other ways.

2. The matrix formula. Let A stand for the $(k+1)\times(k+1)$ matrix

$$\begin{pmatrix} 1, & 0, & 0, & 0, & \cdots & 0, & a \\ 0, & 0, & 0, & 0, & \cdots, & 0, & \alpha_{k-1} \\ 0, & 1, & 0, & 0, & \cdots, & 0, & \alpha_{k-2} \\ 0, & 0, & 1, & 0, & \cdots, & 0, & \alpha_{k-3} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0, & 0, & 0, & 0, & \cdots, & 1, & \alpha_0 \end{pmatrix}.$$

In case k is 1, A is the matrix $\binom{1,\ a}{0,\ \alpha_0}$. Let B stand for the $(k+2)\times(k+2)$ matrix

$$\begin{pmatrix} 0, & 1, \ u_{-1}, \ \cdots, \ u_{k-2} \\ 0 & \\ 0 & A \\ \cdot & \\ \cdot & \\ 0 & \end{pmatrix}.$$

From (2), the formula

$$B^{n+1} = \begin{pmatrix} 0, & 1, \ u_{n-1}, \ \cdots, \ u_{n+k-2} \\ 0 & \\ 0 & A^{n+1} \\ \cdot & \\ \cdot & \\ 0 & \end{pmatrix} \qquad (n > -1) \tag{3}$$

is easily proved by induction.

3. Periodicity. Apparition of primes. Since the vector $(u_{n-1}, u_n, \cdots, u_{n+k-2})$ can take at most p^k values which are distinct modulo the prime p, the following theorem is true.

THEOREM 1. (Lagrange) *Modulo an arbitrary prime, the values u_n given by the linear recurrence relation* (2) *are periodic with period not exceeding p^k.*

Not all periods less than $1+p^k$ are actually possible. To show this, it is convenient to use some information from the theory of automorphisms of abelian groups.

Let G be an elementary abelian group of order p^{k+1}, that is, an abelian group

generated by $k+1$ independent generators, each of order p. [G is isomorphic to the additive group of $(k+1)$-vectors $(\beta_0, \beta_1, \cdots, \beta_k)$, the components of the vectors being integers mod p.] An automorphism of G can be described by mapping the first generator $(1, 0, \cdots, 0)$ onto any of the $p^{k+1}-1$ elements which have order p; by mapping the second generator $(0, 1, \cdots, 0)$ onto any of the $p^{k+1}-p$ elements independent of this element; *etc.* Thus the order of the group of automorphisms of G is $(p^{k+1}-1)(p^{k+1}-p) \cdots (p^{k+1}-p^k)$, which is later denoted by $\phi(p, k)$.

COROLLARY. *The order of any automorphism of G is a divisor of $\phi(p, k)$.*

Actually, more than this is known [1].

THEOREM. (Burnside) *The group of automorphisms of G is isomorphic to the group of $(k+1)\times(k+1)$ nonsingular matrices* mod p [2, p. 242].

COROLLARY. *The order of any such matrix is a divisor of $\phi(p, k)$.*

Thus if A mod p is non-singular, its order is a divisor of this number. The condition for this is that α_{k-1} shall not be divisible by p. Here, the order of A mod p means the smallest positive exponent r such that the relation $A^r \equiv I$ (mod p) holds. When A mod p is non-singular, such an exponent must exist.

THEOREM. *Let r be the order of A* mod p. *Then the relation $B^{r+1} \equiv B$* (mod p) *holds.*

This theorem follows from the relation $B^{r+1} = BB^r$ together with the displayed matrix equation (3).

COROLLARY. *Modulo an arbitrary prime p which does not divide α_{k-1}, the values u_n given by the linear recurrence relation* (2) *are periodic with period dividing $(p^{k+1}-1)(p^{k+1}-p) \cdots (p^{k+1}-p^k)$; indeed the period divides the order r of A* mod p.

COROLLARY. *If any one of $u_{-1}, u_0, \cdots, u_{k-2}$ is* 0, *then every prime p, not a divisor of α_{k-1}, has a finite rank of apparition; that is, an infinite number of terms u_n are divisible by the fixed prime p.*

The methods of this article can be generalized in various directions, which will be noted by the sophisticated reader.

References

1. Brenner, J. The linear homogeneous group. Annals of Math. (2) 39, 1938, pp. 472–493.
2. Burnside, W. The Theory of Groups of Finite Order. Cambridge, 1897.
3. Dickson, L. E. History of the Theory of Numbers, I, ch. xvii.
4. Lehmer, D. H. An extended theory of Lucas' functions. Annals of Math. (2) 31, 1930, pp. 419–448.
5. Lucas, E. Amer. J. Math. 1, 1878, pp. 184–220.
6. Mills, W. H. A system of quadratic diophantine equations. Pacific J. Math. 3, 1953, pp. 209–220.

MATHEMATICAL NOTES

EDITED BY F. A. FICKEN, University of Tennessee

Material for this department should be sent to F. A. Ficken, University of Tennessee, Knoxville 16, Tenn.

A NOTE ON WOLSTENHOLME'S THEOREM

L. CARLITZ, Duke University

1. A well-known theorem of Wolstenholme asserts that if p is a prime greater than 3, then

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{p-1}\equiv 0 \pmod{p^2}. \tag{1.1}$$

For a proof, see for example [2, p. 87]. It is perhaps of interest to examine the sum

$$S_{k,p}=\frac{1}{kp+1}+\frac{1}{kp+2}+\cdots+\frac{1}{kp+p-1}, \tag{1.2}$$

where k is an arbitrary integer.

Put

$$\prod_{r=1}^{p-1}(x-kp-r)=x^{p-1}-A_1^{(k)}x^{p-2}+\cdots+A_{p-1}^{(k)}. \tag{1.3}$$

If we take $x=(2k+1)p$ in (1.3), it is clear that

$$A_{p-2}^{(k)}-(2k+1)pA_{p-3}^{(k)}+(2k+1)^2p^2A_{p-4}^{(k)}-\cdots=0. \tag{1.4}$$

Since by (1.3)

$$A_r^{(k)}\equiv A_r^{(1)}\equiv 0 \pmod{p}, \qquad (1\leqq r\leqq p-2),$$

it is clear that $A_{p-2}^{k}\equiv 0 \pmod{p^2}$ for all k (provided $p>3$). Moreover if $p^e \mid (2k+1)$ then

$$A_{p-2}^{(k)}\equiv 0 \pmod{p^{e+2}}.$$

Also (1.4) implies the congruence

$$A_{p-2}^{(k)}\equiv (2k+1)pA_{p-3}^{(k)} \pmod{(2k+1)^2p^3}; \tag{1.5}$$

indeed (1.5) holds (mod $(2k+1)^2p^4$) for $p>5$.

Since

$$S_{k,p}=A_{p-2}^{(k)}/[(kp+1)(kp+2)\cdots(kp+p-1)],$$

it is clear in particular that

$$S_{k,p} \equiv 0 \pmod{p^{e+2}}$$

provided $p^e | (2k+1)$, $p>3$. As for $p=3$, we have

$$S_{k,3} = \frac{1}{3k+1} + \frac{1}{3k+2} = \frac{3(2k+1)}{(3k+1)(3k+2)},$$

so that $S_{k,3} \equiv 0 \pmod{p^{e+1}}$. It therefore follows that for properly chosen k, $S_{k,p}$ is divisible by arbitrarily high powers of p.

2. Wolstenholme's theorem has been generalized by Leudesdorf. In the simplest case we have

$$S_m = \sum_{r=1,\,(r,m)=1}^{m} \frac{1}{r} \equiv 0 \pmod{m^2}$$

for $(m, 6)=1$; for a proof of this result see [2, p. 100], also [1]. Extending (1.2) we define

$$S_{k,m} = \sum_{r=1}^{m}{}' \frac{1}{km+r}, \tag{2.1}$$

where the prime denotes that the summation is restricted to r prime to m, and k is an arbitrary integer.

Exactly as in [1], we have

$$\begin{aligned} 2S_{k,m} &= \sum_{r=1}^{m}{}' \left(\frac{1}{km+r} + \frac{1}{km+m-r}\right) = (2k+1)m \sum_{r=1}^{m}{}' \frac{1}{(km+r)(km+m-r)} \\ &= -(2k+1)m^2 \sum_{r=1}^{m}{}' \frac{1}{(km+r)^2(km+m-r)} + (2k+1)m \sum_{r=1}^{m}{}' \frac{1}{(km+r)^2}. \end{aligned}$$

Now

$$\sum_{r=1}^{m}{}' \frac{1}{(km+r)^2} \equiv \sum_{r=1}^{m}{}' \frac{1}{r^2} \equiv 0 \pmod{m}. \tag{2.2}$$

Chowla derives the latter congruence from

$$\sum{}' \frac{1}{r^2} \equiv \sum{}' \frac{1}{(ar)^2} \pmod{m},$$

where a is chosen so that $a^2 \not\equiv 1 \pmod{p}$ for any prime p dividing m; since m is prime to 6 it is possible to find such a. Alternatively the result can be proved by means of the formula

$$\sum_{r=1}^{m}{}' \frac{1}{r^2} \equiv \phi\left(\frac{m}{p^\alpha}\right) \sum_{r=1,\,p\nmid r}^{p^\alpha} \frac{1}{r^2} \pmod{p^\alpha},$$

where $p^\alpha | m$, $p^{\alpha+1} \nmid m$.

Combining (2.1) and (2.2) we see that

$$S_{k,m} \equiv 0 \pmod{(2k+1)m^2} \qquad ((m, 6) = 1). \tag{2.3}$$

In particular for properly chosen k, it is clear that $S_{k,m}$ is divisible by arbitrarily high powers of m; this last is true for all m.

References

1. S. Chowla, Leudesdorf's generalization of Wolstenholme's theorem, J. London Math. Soc., vol. 9, 1934, p. 246.
2. G. H. Hardy and E. M. Wright, An Introduction to the Theory of Numbers, Oxford, 1938.

ON RULED SURFACES

S. BEATTY, University of Toronto

1. Introduction. Treatments of skew curves invariably deal with the theory of the developable surfaces in space generated by the planes of the moving trihedral. The following represents an attempt to discuss the ruled surface in space generated by the axis of rotation of the rigid body supporting the trihedral throughout its progress and moving with it. For the sake of simplicity, we shall identify s, the distance covered by the moving point as it progresses along the curve, with t, the time required.

2. The ruled surface. Let (x, y, z) be the coordinates, relative to fixed axes in space, of the moving point on the curve associated with the value s, and let the direction cosines of the tangent, principal normal, binormal, be

$$(l_1, m_1, n_1), \qquad (l_2, m_2, n_2), \qquad (l_3, m_3, n_3) \tag{1}$$

respectively. Let (u, v, w) be the coordinates, relative to the trihedral at (x, y, z) of a given point in the rigid body, and let (X, Y, Z) be the coordinates of the same point, relative to the fixed axes in space. We have, therefore,

$$X = x + l_1u + l_2v + l_3w, \tag{2}$$

with corresponding expressions for Y and Z.

Employing κ and τ to denote the curvature and torsion of the curve at (x, y, z) and making use of the Frenet-Serret formulae, we have

$$\dot{X} = l_1(1 - v\kappa) + l_2(u\kappa + w\tau) - l_3(v\tau), \tag{3}$$

with corresponding expressions for $\dot{Y}$ and $\dot{Z}$, where the dot notation indicates derivation with respect to s at (x, y, z). The square of the velocity of the moving point (X, Y, Z) is, therefore,

$$(1 - v\kappa)^2 + (u\kappa + w\tau)^2 + (v\tau)^2, \tag{4}$$

which is at its least on the axis of rotation

$$u\kappa + w\tau = 0, \qquad v = \frac{\kappa}{R^2} \tag{5}$$

in the rigid body, where R^2 stands for $\kappa^2+\tau^2$. This line is the s-generator of the ruled surface in space, and its equations become

$$\begin{gathered}(X - x)(l_1\kappa + l_3\tau) + (Y - y)(m_1\kappa + m_3\tau) + (Z - z)(n_1\kappa + n_3\tau) = 0,\\ (X - x)l_2 + (Y - y)m_2 + (Z - z)n_2 = \frac{\kappa}{R^2},\end{gathered} \tag{6}$$

when made to refer to the fixed axes in space.

The square of the velocity, as given in (4), may be written in the form

$$T^2 + \Delta^2 R^2, \tag{7}$$

where T denotes τ/R, the velocity in the direction of the s-generator, while Δ^2 denotes

$$\left(\frac{u\kappa + w\tau}{R}\right)^2 + \left(v - \frac{\kappa}{R^2}\right)^2,$$

the square of the distance of the moving point from the axis of rotation, so that ΔR represents the velocity of rotation and R the angular velocity of rotation about this line. In particular, for the point (x, y, z) itself, expression (7) reduces to unity, as required. If s is allowed to vary, equations (6) supply the ruled surface in space.

3. The curve of striction. The s-generator, as given in (6), has direction cosines

$$\left(\frac{l_1\tau - l_3\kappa}{R}, \frac{m_1\tau - m_3\kappa}{R}, \frac{n_1\tau - n_3\kappa}{R}\right). \tag{8}$$

To these we must add

$$\frac{\kappa\dot{\tau} - \tau\dot{\kappa}}{R^2}\, ds \text{ times} \left(\frac{l_1\kappa + l_3\tau}{R}, \frac{m_1\kappa + m_3\tau}{R}, \frac{n_1\kappa + n_3\tau}{R}\right)$$

respectively, to obtain the direction cosines of the $(s+ds)$-generator. It follows that the common perpendicular to these two generators has direction cosines

$$(l_2, m_2, n_2), \tag{9}$$

which means that it is parallel to the principal normal at (x, y, z).

But the principal normal at (x, y, z) intersects the s-generator in the point (X, Y, Z) given by

$$\frac{X - x}{l_2} = \frac{Y - y}{m_2} = \frac{Z - z}{n_2} = \frac{\kappa}{R^2}. \tag{10}$$

It is easy to see that (X, Y, Z) is the point of striction on the s-generator. Indeed,

$$dX = \left(\frac{l_1\tau - l_3\kappa}{R}\right)Tds + l_2 d\frac{\kappa}{R^2}, \tag{11}$$

with corresponding expressions for dY and dZ. In other words,

$$X + l_2 d\frac{\kappa}{R^2} = X + dX - \left(\frac{l_1\tau - l_3\kappa}{R}\right)Tds,$$

with corresponding expressions for $Y+m_2d(\kappa/R^2)$ and $Z+n_2d(\kappa/R^2)$. That is, to proceed from (X, Y, Z) a distance of $d(\kappa/R^2)$ in the direction indicated in (9) lands us finally at the same point as to proceed from $(X+dX, Y+dY, Z+dZ)$ a distance of $T\,ds$ in the direction opposite to that indicated in (8), and this means that (X, Y, Z) is the point of striction on the s-generator. Strictly speaking, we should have used the direction of the $(s+ds)$-generator rather than that of the s-generator, but the conclusion would still have been the same, since none but the coefficients of higher powers of ds than the first could have been affected. If s is allowed to vary, equations (10) supply the curve of striction.

4. **Properties.** The curve of striction in space corresponds to one or more segments of the principal normal in the rigid body, with the possibility of such segment or segments being swept out again and again. The distance between the s-generator and the $(s+ds)$-generator in space is $d(\kappa/R^2)$, which is also the distance between them in the rigid body. The points of striction on these two generators in space are advanced, one relative to the other, a distance of Tds in their virtually common direction as given in (8), whereas in the rigid body there is no such advance whatever of one relative to the other, since both points lie on the principal normal.

The tangent plane to the ruled surface at the point of striction on the s-generator passes through

(i) the s-generator, with direction cosines as given in (8),

(ii) the principal normal orthogonal to (i), with direction cosines as given in (9),

(iii) the tangent line to the curve of striction at the point, with direction ratios (dX, dY, dZ) as begun in (11) and completed immediately after.

Where θ and ϕ denote the angles which (iii) makes with (i) and (ii) respectively, we easily find that

$$\frac{\cos\theta}{Tds} = \frac{\cos\phi}{d\dfrac{\kappa}{R^2}} = \frac{1}{\left\{(Tds)^2 + \left(d\dfrac{\kappa}{R^2}\right)^2\right\}^{1/2}}.$$

The s-generator is tangent to the curve of striction at (X, Y, Z) as given in (10) when and only when κ/R^2 is constant. In that case, and provided the orig-

inal curve is not a cylindrical helix, the ruled surface is the tangent surface of the curve of striction and is developed by its osculating planes, so that the binormal of the curve of striction is identical with the principal normal of the original curve. The proof of this is routine. If however κ/R^2 is constant, with the original curve being a cylindrical helix, it is clear that κ and τ are both constant, which means that the osculating planes of the curve of striction prove to be indeterminate. Also, the tangent plane to the ruled surface remains the same for all the points of contact on a single generator, provided κ/R^2 is constant, but rotates about the generator as the point of contact recedes along it, provided κ/R^2 is not constant.

In particular, if the original curve is not skew, the ruled surface is a cylinder, with generators normal to the plane of the original curve. In that case, the curve of striction is not unique but may be taken as the intersection of the cylinder with the plane of the original curve and so is the evolute of the original curve. In the special case where the original curve is a circle, the ruled surface is a single line normal to its plane and passing through the centre of the circle.

THE NON-EXISTENCE OF RATIONAL SOLUTIONS FOR CERTAIN DIFFERENCE EQUATIONS

F. R. OLSON, Duke University

1. Introduction. In this paper we prove:

THEOREM I. *The difference equation*

$$\frac{1}{x^k} = F(x+1) - F(x), \qquad k = 1, 2, \cdots, \tag{1}$$

has no rational solution with coefficients in the complex field.

THEOREM II. *The difference equation*

$$\frac{1}{x^k} = F(x+1) + F(x), \qquad k = 1, 2, \cdots, \tag{2}$$

has no rational solution with coefficients in the complex field.

We present two methods of proof.

2. Theorem I. $k=1$. Suppose that $F(x)$ is a rational function of x. Then we can write

$$F(x) = \frac{R(x)}{T(x)} \tag{3}$$

where $R(x)$ and $T(x)$ are polynomials in x having no factor in common and of degree r and t respectively. Hence

$$\frac{1}{x} = \frac{R(x+1)}{T(x+1)} - \frac{R(x)}{T(x)}. \tag{4}$$

Let x take successively the values $1, 2, \cdots, n$. (We observe that $T(x)$ vanishes for none of these values of x for were it to vanish for one such value it would vanish for infinitely many.) Adding the resulting equations we have

$$1 + \frac{1}{2} + \cdots + \frac{1}{n} = \frac{R(n+1)}{T(n+1)} - \frac{R(1)}{T(1)}. \tag{5}$$

Letting $n \to \infty$ we consider two cases.

Case 1. $r \leqq t$. The left side of (5) grows without bound while the right side approaches a constant.

Case 2. $r > t$. Both sides of (5) grow large but with different orders of magnitude, the left side on the order of $\log n$, the right side as a power of n.

Thus the assumption that $F(x)$ is rational leads to a contradiction.

3. **Theorem I.** $k > 1$. The above procedure lends itself to the more general case. Assuming first that k is an even integer, say $k = 2p$, $p \geqq 1$, and that $F(x)$ is rational, we have

$$\frac{1}{x^{2p}} = \frac{R(x+1)}{T(x+1)} - \frac{R(x)}{T(x)}. \tag{6}$$

Letting x take on the values $1, 2, \cdots, n$ and adding, there results

$$1 + \frac{1}{2^{2p}} + \cdots + \frac{1}{n^{2p}} = \frac{R(n+1)}{T(n+1)} - \frac{R(1)}{T(1)}. \tag{7}$$

As $n \to \infty$, the left side of (7) has as its limiting value the irrational number [1]

$$\frac{(2\pi)^{2p}}{2(2p)!} B_p \tag{8}$$

where B_p is a Bernoulli number. If the right side of (7) is to converge it has for its limit either zero or a number dependent upon the coefficients of $R(x)$ and $T(x)$. That this number is rational follows from (7) and from the fact that if a rational function has rational values for all positive integral values of x then its coefficients are rational numbers [2]. Therefore we are led once again to a contradiction.

There remains to consider the case where k is odd, say $k = 2p - 1$, $p \geqq 1$. Writing

$$\frac{1}{x^{2p-1}} = F(x+1) - F(x), \tag{9}$$

and taking derivatives we have

$$\frac{1}{x^{2p}} = \frac{F'(x+1) - F'(x)}{-2p+1}. \tag{10}$$

But the rationality of $F(x)$ implies the rationality of $F'(x)$ and (10) is of the same form as (6). Thus we have an immediate contradiction.

4. Theorem II. In a similar fashion we examine equation (2). Again assuming $F(x)$ a rational function and considering first $k=2p$, $p\geqq 1$, we employ the method of §3, this time alternately adding and subtracting the equations that result from letting x take successively the values $1, 2, \cdots, n$. We obtain

$$1 - \frac{1}{2^{2p}} + \cdots + \frac{(-1)^{n+1}}{n^{2p}} = (-1)^{n+1}F(n+1) + F(1). \tag{11}$$

The limit of the left side as $n\to\infty$ is the irrational number [1]

$$\frac{(2^{2p-1}-1)\pi^{2p}}{(2p)!} B_p. \tag{12}$$

Since the right side of (11) must have, employing the argument of §3, a rational number as its limiting value, it follows that the assumption that $F(x)$ is a rational function is not true.

For the case $k=2p-1$, $p\geqq 1$, we need only to employ differentiation as in §3 to reach a contradiction.

5. Alternative proof. Another method of proof for which I am indebted to L. Carlitz is as follows. As before, with respect to (1), let

$$\frac{1}{x^k} = \frac{R(x+1)}{T(x+1)} - \frac{R(x)}{T(x)}. \tag{13}$$

Clearing of fractions we have

$$T(x)T(x+1) = x^kR(x+1)T(x) - x^kR(x)T(x+1). \tag{14}$$

Since $R(x)$ and $T(x)$ have no factor in common, (14) implies on the one hand that $T(x)$ divides $x^kT(x+1)$, on the other that $T(x+1)$ divides $x^kT(x)$. Thus we have

$$x^kT(x+1) = T(x)M(x), \qquad x^kT(x) = T(x+1)N(x), \tag{15}$$

where $M(x)$ and $N(x)$ are polynomials. Hence $x^{2k}=M(x)N(x)$ and so $M(x) = N(x) = \pm x^k$ so that $T(x+1)=\pm T(x)$ which is evidently impossible.

For equation (2) the proof is exactly the same with a change of sign in (13) and (14).

References

1. L. B. W. Jolley, Summation of Series, London, 1925, pp. 46–47.
2. G. Pólya and G. Szegö, Aufgaben und Lehrsätze aus der Analysis, II, Berlin, 1925, p. 133.

MATRICES AND POLYNOMIALS

W. V. PARKER, Alabama Polytechnic Institute

1. Introduction. In 1932, Pierce [1] gave a method for computing the resultant of two polynomials by applying a theorem of Frobenius to the companion matrices of the polynomials. The writer [2] showed how the degree of the greatest common divisor is determined from this form of the resultant. In the present note it is shown how the greatest common divisor may be determined from this form of the resultant.

2. The greatest common divisor of two polynomials. Let $c=(c_1, c_2, \cdots, c_n)$ be a vector in n-space and denote by $c(x)$ the polynomial $c_1+c_2x+ \cdots +c_nx^{n-1}$. This correspondence between vectors of n-space and polynomials of virtual degree $n-1$ will be assumed throughout this paper. Basic vectors e_i are defined by $e_i(x)=x^{i-1}$ so that $c=\sum_{i=1}^n c_ie_i$. Let C be the n-rowed matrix whose rows are $e_2, e_3, \cdots, e_n, c$. Then $e_iC=e_{i+1}$ $(i=1, 2, \cdots, n-1)$ and $e_nC=c$. If C^0 is defined to be the identity matrix, it follows that the rows of C^{i-1} $(i=1, 2, \cdots, n)$ are $e_i, e_iC, e_iC^2, \cdots, e_iC^{n-1}$ and the rows of C^n are $c, cC, cC^2, \cdots, cC^{n-1}$; hence $C^n=c(C)$. Also, if k is any other vector of n-space and $k(x)$ is the corresponding polynomial, the matrix $k(C)$ has the rows $k, kC, kC^2, \cdots, kC^{n-1}$.

THEOREM 1. *If k is a vector of n-space, $k(C)=0$ if, and only if, $k=0$.*

COROLLARY. *The minimum function of the matrix C is $f(x)=x^n-c(x)$.*

The matrix C is usually referred to as the companion matrix of the polynomial $f(x)$. If $h(x)=q(x)f(x)+r(x)$ then $h(C)=r(C)$ and the greatest common divisor of $h(x)$ and $f(x)$ is the greatest common divisor of $r(x)$ and $f(x)$. Hence, it is not necessary to consider polynomials of degree greater than $n-1$.

If $h(x)$ and $k(x)$ are two polynomials of degrees less than n, the first row of the matrix $h(C)k(C)$ is $hk(C)$ and the first row of $k(C)h(C)$ is $kh(C)$. But $h(C)k(C)=k(C)h(C)$; hence

THEOREM 2. *If h and k are two vectors of n-space, the vectors $hk(C)$ and $kh(C)$ are identical.*

If the vectors $k, kC, \cdots, kC^{r-1}$ are linearly independent and $kC^r=k(h_1I+h_2C+ \cdots +h_rC^{r-1})$ then $kC^s(s>r)$ is also a linear combination of the first r rows of $k(C)$. That is, the first r rows of $k(C)$ form a basis for the row space of $k(C)$ and consequently $k(C)$ is of rank r. Thus we have

THEOREM 3. *If the matrix $k(C)$ is of rank r the first r rows of $k(C)$ are linearly independent.*

We note that $kC^r=kh(C)$ and indicate by $g(x)$ the polynomial $x^r-h(x)$ where $h(x)$ is of degree $r-1$. Then $kg(C)=0$ and $g(x)$ is the minimum function of C relative to the vector k. But $kg(C)=0$ if, and only if, $k(C)g(C)=0$ and the latter

is true if, and only if, $f(x)$ is a factor of $k(x)g(x)$. Since $g(x)$ is the monic polynomial of lowest degree such that $k(x)g(x)$ is divisible by $f(x)$ it follows that $g(x) = f(x)/d(x)$ where $d(x)$ is the greatest common divisor of $f(x)$ and $k(x)$. This establishes

THEOREM 4. *The degree of the greatest common divisor of $f(x)$ and $k(x)$ is the nullity of $k(C)$* [2].

Also from the above we have

THEOREM 5. *Let $k(x)$ be any polynomial and let $d(x)$ be the greatest common divisor of $k(x)$ and $f(x)$. If $k(C)$ is nonsingular, then $d(x)=1$. If $k(C)$ is of rank $r<n$ and $g(x)$ is the monic polynomial of degree r such that $gk(C)=0$ then $d(x) = f(x)/g(x)$.*

3. The matrix equation $AX=XB$. Consider now the matrix equation $CX=XD$ where D has m rows. The rows of CX are e_2X, e_2XD, $e_2XD^2, \cdots, e_2XD^{n-1}$; hence $e_2X=u_1D$ where u_1 is the first row of X and consequently the rows of X are $u_1, u_1D, \cdots, u_1D^{n-1}$. Also the last row of CX is $u_1c(D)$ and the last row of XD is u_1D^n so that $u_1f(D)=0$. In particular if $m=n$ and D is the companion matrix of $k(x)$, then X is a polynomial in D. If P is a nonsingular matrix and $A=PCP^I$ and $B=PDP^I$, the equation $AX=XB$ has the same solution as $CP^IXP=P^IXPD$. Hence, we have

THEOREM 6. *If A and B are nonderogatory matrices which can be transformed into rational canonical form by the same nonsingular matrix P the only solutions of $AX=XB$ are polynomials in B.*

If $f(x)$ and $k(x)$ are monic polynomials of degree n and their respective companion matrices are C and D, then $CX=XD$ if and only if $X=h(D)$ where $h(x)$ is a polynomial of degree less than n such that $hf(D)=0$. We have seen above that $hf(D)=0$ if and only if $h(D)f(D)=0$ and the latter is true if and only if $k(x)$ divides $h(x)f(x)$. We may then say that all solutions of $CX=XD$ are given by $X=h(D)$ where $h(x)$ is any polynomial of degree less than n having $k(x)/d(x)$ as a factor.

If A and B are as defined in Theorem 6, then $AX=XB$ is equivalent to $CP^IXP=P^IXPD$. Hence $P^IXP=h(D)$ and $X=Ph(D)P^I=h(B)$, where $h(x)$ is as described above. We also have the well known

COROLLARY. *If A is nonderogatory, X is commutative with A if, and only if, X is a polynomial in A.* [3]

References

1. T. A. Pierce, The practical evaluation of resultants, this MONTHLY, vol. 39, 1932, p. 161.

2. W. V. Parker, The degree of the highest common factor of two polynomials, this MONTHLY, vol. 42, 1935, p. 164. See also MacDuffee, this MONTHLY, vol. 57, 1950, p. 156 and [3] p. 45.

3. W. V. Parker, The matrix equation $AX=XB$, Duke Math. Journal, vol. 17, 1950, pp. 43–51.

NOTE ON POSITIVE POLYNOMIALS

S. S. ABHYANKAR, Harvard University

Introduction.* Increased attention has been given, during recent years, to the synthesis of electric networks. By synthesis is meant the design of an electric network which has certain properties. The impedance function of a network may be represented by a polynomial $P(X)$ whose coefficients are sums of products of the parameters of the network and whose independent variable X is the frequency.

If $P(X)$ is the impedance function of the network and $P_1(X), P_2(X), \cdots, P_m(X)$ are impedance functions of the various branches of the network then $P(X)=P_1(X)+P_2(X)+\cdots+P_k(X)$.

Obviously the smaller the number of branches the less costly the network. Hence it is the object of the engineer to synthesize a network with as few branches as possible. Certain physical requirements further demand that the roots of $P_i(X)$ be negative. Hence the problem is to determine the minimum number of polynomials whose coefficients are positive, whose roots are negative, and whose sum is equal to a given polynomial with positive coefficients. In this paper we shall determine this minimum number and also give a construction to compute these polynomials. It being equally simple, we shall deal with polynomials in any arbitrary ordered field. By a positive polynomial we shall mean a polynomial all of whose coefficients are positive numbers in the given ordered field. Now we state the

THEOREM. *Let $m(n)$ be the least integer such that any nth degree positive polynomial could be expressed as a linear combination with positive coefficients of not more than $m(n)$ positive polynomials each of which has all its roots negative.*

Then $m(n)$ = the least integer greater than $n/2$.

Proof. Without loss of generality we can assume that all the polynomials are monic.

Sufficiency.

Let $P(X)=X^n+a_1X^{n-1}+\cdots+a_n$ be any polynomial with $a_i>0$.

Case 1. n odd. Let $n=2p+1$. For $p=0$ the result is obvious. We apply induction in the case $p>0$. Assume it is true for $p-1$. Let a be the minimum of $a_2, a_3, \cdots, a_n$. Let ϵ be the minimum of

$$1, \frac{a}{n!2}, \frac{a}{n!2a_1}, \frac{a_1}{2}.$$

Let $\delta=\epsilon/(n-1)$. Let $Q(X)=(X+\delta)^{n-1}\ (X+a_1-\epsilon)=X^n+b_1X^{n-1}+\cdots+b_n$.

* I wish to thank Mr. H. H. Sun of Cornell University for help in the preparation of this introduction.

Then δ and $a_1-\epsilon$ are both positive. Also $b_1=a_1$ and for all $i>1$ we have $b_i<a_i$.* Therefore $P(X)=Q(X)+cP_1(X)$, where $c>0$, and $P_1(X)$ is a $[2(p-1)+1]$st degree monic positive polynomial.

Hence by assumption

$$P_1(X) = \sum_{i=1}^{p} c_iQ_i(X), \qquad \text{where } c_i > 0,$$

and $Q_i(X)$ are monic polynomials with all roots negative. Therefore

$$P(X) = Q(X) + \sum_{i=1}^{p} cc_iQ_i(X),$$

and the induction is complete.

Case 2. n even. The proof will be similar to that of Case 1 if we prove it for $n=2$. In this case, let ϵ be the minimum of 1, $a_1/3$, $a_2/2$. Let $c=a_1-2\epsilon$ and let $b=(a_2-\epsilon^2)/c$. Then ϵ, c and b are all positive and hence $P(X)=(X+\epsilon)^2+c(X+b)$ is a required decomposition.

Necessity.

For any given n the following nth degree positive polynomial $P_n(X)$ cannot be expressed as a positive linear combination of less than $\bar{n}+1$ positive polynomials with all roots negative, where $\bar{n}+1$ is the least integer greater than $n/2$.

$$P_n(X) = X^n + X^{n-1} + \left\{\binom{n}{2}n^2 + 1\right\} X^{n-2} + X^{n-3} + \left[\binom{n}{2}\left\{n^4 + a_2n^2\binom{n}{2}^2\right\} + 1\right] X^{n-4} + \cdots$$

where a_i, the coefficient of X^{n-i}, is a positive number determined, after a_1, a_2, $\cdots$, a_{i-1} are determined, in the following manner.

$$a_i = \begin{cases} 1 & \text{if } i \text{ is odd or zero.} \\ \left[\binom{n}{i}\sum_{j=0}^{k-1} a_{2j}\left\{n\binom{n}{2j}\right\}^{i-2j}\right] + 1 & \text{if } i = 2k > 0. \end{cases}$$

We now proceed to prove the impossibility of decomposing $P_n(X)$.

For $i=2k$ let $f_i^j=\binom{n}{i}a_{2j}\{n\binom{n}{2j}\}^{i-2j}$ if $i<k$ and let $f_i^k=1$. Also let $\phi_i^k=\sum_{j=p}^{k} f_i^j$. Then $a_i=\phi_i^0$.†

Let $P_n(X)=\sum_{l=1}^{s} c_lQ_l(X)$, with $c_l>0$ and Q_l a monic polynomial with all

* The proof of this inequality is as follows:—

$$b_i = \delta^{i-1}(a_1-\epsilon)\binom{n-1}{i-1} + \delta^i\binom{n-1}{i} \leqq \delta^{i-1}n!(a_1-\epsilon+\delta) \leqq n!\delta a_1 \leqq n!\epsilon a_1$$

$$\leqq n!\,\frac{a}{n!2a_1}\,a_1 = \frac{a}{2} \leqq \frac{a_i}{2} < a_i.$$

† The super indices of f, ϕ, g and d do not represent powers but are merely suffixes.

negative roots, be any decomposition of the required type with $s \leqq n$. Let $Q_l(X) = \sum_{i=0}^{n} g_i^l X^{n-i}$. We shall prove by induction:

(I) For all $p \leqq \bar{n}$ it is possible to choose p polynomials $Q_{l_1}, \cdots, Q_{l_p}$ such that if we let $R_p(X) = P_n(X) - \sum_{j=1}^{p} c_{l_j} Q_{l_j}(X) = \sum_{i=0}^{n} d_i^p X^{n-i}$ then $d_i^p \geqq \phi_i^p$ for all even $i \geqq 2p$.

For $p=0$ we have $R_0(X) = P_n(X)$ and hence (I) is obvious. Assume (I) to be true for some $q < \bar{n}$. By rearranging the subscripts of the terms $c_l Q_l$ we may assume that $R_p(X) = \sum_{l=q+1}^{s} c_l Q_l(X)$. Then $\sum_{l=q+1}^{s} c_l g_{2q}^l = d_{2q}^l \geqq \phi_{2q}^l = 1$. Therefore for some l between $q+1$ and s, which by rearranging the terms of $R_q(X)$ we may take as $l = q+1$, we must have $c_l g_{2q}^l \geqq 1/(s-q) \geqq 1/nk$. From now on we denote c_{q+1} by k. Then

$$g_{2q}^{q+1} \geqq \frac{1}{nk} \cdot \tag{II}$$

Also $k g_{2q+1}^{q+1} \leqq a_{2q+1} = 1$. Therefore

$$g_{2q+1}^{q+1} \leqq \frac{1}{k} \cdot \tag{III}$$

Then, if t is the degree of Q_{q+1}, $n \geqq t \geqq n-2q$. Let $x_1, \cdots, x_t$ be the roots of $Q_{q+1}(X) = 0$. Let $h = 2q + t - n$. Then $h \geqq 0$. Now we divide the argument into two cases.

Case 1. Let $h > 0$. Then $\sum x_{i_1} \cdots x_{i_h} = g_{2q}^{2+1} \geqq 1/nk$. Therefore at least one $x_{v_1} \cdots x_{v_h} \geqq 1/(nk\binom{t}{h})$, say $x_1 x_2 \cdots x_h$. Then for all $u > h$ we have $x_1 x_2 \cdots x_h x_u \leqq \sum x_{i_1} \cdots x_{i_{h+1}} = g_{2q+1}^{q+1} \leqq 1/k$. Therefore for all $u > h$ we have $x_u \leqq n\binom{t}{h} \leqq n\binom{n}{2q}$. But for all $v_1, \cdots, v_h$ we have $x_{v_1} \cdots x_{v_h} \leqq \sum x_{i_1} \cdots x_{i_h} = g_{2q}^{q+1} \leqq a_{2q}/k$. Therefore for all $j > 0$ and for all $v_1, \cdots, v_{h+2j}$ we have

$$x_{v_1} \cdots x_{v_{h+2j}} \leqq \frac{a_{2q}}{k} \left\{ n \binom{n}{2q} \right\}^{2j}.$$

Case 2. Let $h = 0$. Then $g_{2q}^{q+1} = 1$. Hence by (II) we have $1/k \leqq n$. Therefore by (III) we have $g_{2q+1}^{q+1} \leqq n$. Then for all j we have $x_j \leqq \sum x_i = g_{2q}^{q+1} \leqq n$. Hence for all $j > 0$ and for all $v_1, \cdots, v_{h+2j}$ we have $x_{v_1} \cdots x_{v_{h+2j}} \leqq n^{2j}$. But $k = k g_{2q}^{q+1} \leqq a_{2q}$. Therefore

$$n^{2j} \leqq \frac{a_{2q}}{k} \left\{ n \binom{n}{2q} \right\}^{2j}.$$

Thus in both the cases, for all $j > 0$ and for all $v_1, \cdots, v_{h+2j}$ we have $x_{v_1} \cdots x_{v_{h+2j}} \leqq a_{2q} \{ n\binom{n}{2q} \}^{2j}/k$. Therefore for all $j > 0$ we have

$$k g_{2q+2j}^{q+1} = k \sum x_{i_1} \cdots x_{i_{h+2j}} \leqq k \binom{t}{2q+2j} \frac{a_{2q}}{k} \left\{ n \binom{n}{2q} \right\}^{2j}$$

$$\leqq \binom{n}{2q+2j} a_{2q} \left\{ n \binom{n}{2q} \right\}^{2j} = f_{2q+2j}^{q}.$$

But $kg_{2q+2j}^{q+1}+d_{2q+2j}^{q+1}=d_{2q+2j}^{q}\geqq\phi_{2q+2j}^{q}$. Therefore for all even $i\geqq q+1$ we have $d_i^{q+1}\geqq\phi_i^q-f_i^q=\phi_i^{q+1}$.

Thus the induction is complete. Hence after choosing $\bar{n}$ polynomials Q_l, which again by rearranging we can take as $Q_1, Q_2, \cdots, Q_{\bar{n}}$, we are left with $R_{\bar{n}}(X)=\sum_{l=\bar{n}+1}^{s} c_lQ_l(X)$ for which $d_{2n}^{\bar{n}}\geqq 1$; so that $R_{\bar{n}}(X)\not\equiv 0$ and hence $s\geqq n+1$.

ABRIDGED SERIES FOR NUMERICAL EVALUATION

B. K. YOUSE, Memphis State College, Tennessee

If the sequence a_n decreases strictly to zero, then $s_m=\sum_1^m(-1)^{n-1}a_n\rightarrow S=\sum_1^\infty(-1)^{n-1}a_n$. The numerical evaluation of S can be simplified in certain cases.

THEOREM. *If there exist numbers Δ_n $(n=1, 2, 3, \cdots)$ such that*

$$a_{2n}-a_{2n+1}\geqq\Delta_n\left|a_n-a_{2n+1}-a_{2n-1}\right|$$

then, for $n=1, 2, 3, \cdots$,

$$s_{2n+1}-S\geqq\delta_n\left|\sigma_n-S\right|$$

where $\delta_n=\mathrm{glb}_{m>n}\Delta_m$ and $\sigma_n=s_n+(-1)^na_{2n+1}$.

Proof.

$$\begin{aligned} s_{2n+1}-S &= \sum_{m=n+1}^{\infty}(a_{2m}-a_{2m+1}) \\ &\geqq \sum_{m=n+1}^{\infty}\Delta_m\left|a_m-a_{2m+1}-a_{2m-1}\right| \\ &\geqq \sum_{m=n+1}^{\infty}\Delta_m\left|\sigma_m-\sigma_{m-1}\right| \\ &\geqq \delta_n\sum_{m=n+1}^{\infty}\left|\sigma_m-\sigma_{m-1}\right| \\ &\geqq \delta_n\left|\sigma_n-S\right|. \end{aligned}$$

Example. Let $a_n=1/n$ in the series $\sum_{n=1}^\infty(-1)^{n-1}a_n$. The hypothesis is satisfied if $\Delta_n<\frac{1}{2}(2n-1)$, and since Δ_n is increasing $\delta_n=\Delta_{n+1}=n+1/2$. Thus, for $n=49$,

$$\left|\sigma_{49}-S\right|\leqq\frac{2}{99}(s_{99}-S)<\frac{2}{99}(.01)<.00021.$$

Another familiar series to which this method is applicable is the Gregory series.

CLASSROOM NOTES

EDITED BY G. B. THOMAS, Massachusetts Institute of Technology

All material for this department should be sent to G. B. Thomas, Department of Mathematics, Massachusetts Institute of Technology, Cambridge, 39, Mass.

ON A GRAPHICAL SOLUTION OF A CLASS OF DIFFERENTIAL EQUATIONS

M. S. KLAMKIN AND J. P. RUSSELL, Polytechnic Institute of Brooklyn

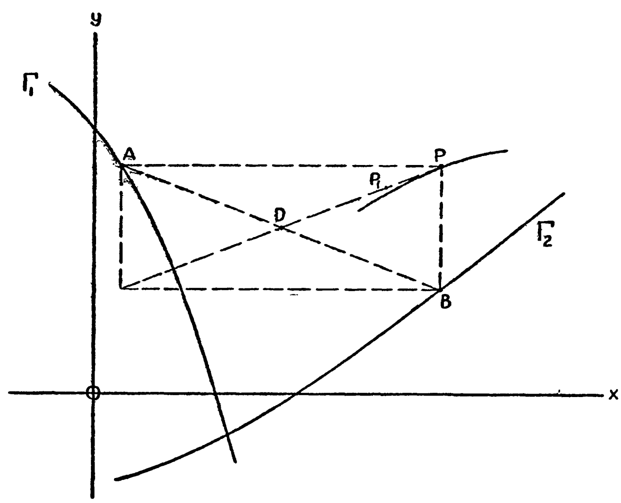
In a previous note (this MONTHLY, vol. 60, 1953, pp. 710–711), J. P. Russell extends the graphical method of Liénard [1] to solving differential equations of the form

$$\frac{dy}{dx} = \frac{\phi(y) + x}{\psi(x) + y}. \tag{1}$$

In this note, the method is extended with slight changes to differential equations of the form

$$\frac{dy}{dx} = \frac{\psi(x) + y}{\phi(y) + x}. \tag{2}$$

Here $\phi(y)$ and $\psi(x)$ are continuous functions.



One first plots the curves Γ_1: $[x = -\phi(y)]$, and Γ_2: $[y = -\psi(x)]$, as in the figure. Through an arbitrary point, $P(x, y)$, a line is drawn parallel to the x axis intersecting Γ_1 at A, and also, a line through P parallel to the y axis is drawn intersecting Γ_2 at B. One then draws the line PD, where D is the midpoint of segment AB. A small section of the integral curve of (2) passing through P is obtained by taking a small segment PP_1 of PD. The procedure is now repeated

starting from point P_1. The proximity of the successive points $P, P_1, P_2, \cdots$ taken will determine the accuracy of the construction.

The justification of the construction is as follows:

The coordinates of points A, B, and D are $[-\phi(y), y]$, $[x, -\psi(x)]$, and $[(x-\phi(y))/2, (y-\psi(x))/2]$, respectively. Thus the slope of PD is

$$\frac{\psi(x)+y}{\phi(y)+x}$$

as is required by equation (2).

Remark. If we take point D such that $AD/AB=K/(K+1)$, then the above method will give a graphical solution of the equation

$$\frac{dy}{dx}=K\frac{\psi(x)+y}{\phi(y)+x}. \tag{3}$$

For the special case $K=-1$, the line PD will be parallel to AB. But in this case, the equation can be solved by quadratures to

$$xy+\int\phi(y)dy+\int\psi(x)dx=0.$$

Reference

1. J. J. Stoker, Non-Linear Vibrations in Mechanical and Electrical Systems, Interscience, New York, 1950, pp. 31–36.

A NOTE ON INDETERMINATE FORMS

L. J. PAIGE, University of California at Los Angeles

It is not surprising that many students suspect the indeterminate form 0^0 to be equal to 1, believing that the elementary rules of algebra will apply. The example $x^{\alpha/\log x}$ immediately dispels this myth.

In attempting to construct examples of the indeterminate form 0^0, we immediately try $\lim_{x\to 0} x^{f(x)}$, where $f(0)=0$ and the derivative $f'(x)$ is assumed to exist in a neighborhood of the origin. But here one is led to an interesting limit and the surprising result that if $\lim_{x\to 0} x^{f(x)}$ exists it must be 1. Perhaps this word of caution will prevent lost time in looking for examples to remedy the students' suspicions.

We attempt to evaluate $\lim_{x\to 0} x^{f(x)}$ by following the text and take the logarithm, obtaining $\lim_{x\to 0}\dfrac{\log x}{1/f(x)}$. Here an application of l'Hospital's rule yields

Consequently,

$$\Delta' = b'de + bd'e + bde' - 2cdd' - c'd^2 - 2aee' - a'e^2$$
$$= 2(c - a)de + be^2 - bd^2 - 2cde + bd^2 + 2ade - be^2 = 0,$$

and so Δ is also invariant with respect to a rotation.

A similar proof of invariance with respect to a translation, $x = x' + x_0$, $y = y' + y_0$, is readily constructed, using partial derivatives with respect to x_0 and y_0. Here, however, the gain in simplicity is not so marked.

THE USE OF INDUCTION IN EXISTENCE PROOFS

E. E. MOISE, University of Michigan

The use of induction in establishing the existence of sequences is illustrated by the proof of the following very well-known theorem:

THEOREM. *Let $p_1, p_2, \cdots$ be a sequence of positive numbers, such that every open interval that includes 0 contains infinitely many of the numbers p_i. Then there is a descending sequence $q_1, q_2, \cdots$, such that*

(1) *$q_1, q_2, \cdots$ converges to 0, and*

(2) *every q_i is a term of the sequence $p_1, p_2, \cdots$.*

LEMMA 1. *For each n, there is exactly one finite sequence Q_n: $q_1, q_2, \cdots, q_n$, satisfying the following conditions:*

(1) *the first term q_1 of Q_n is p_1.*

(2) *if $i < n$, then q_{i+1} is the first term of the sequence $p_1, p_2, \cdots$ which is less than $1/(i+1)$ and less than q_i.*

Proof of lemma. Let P_n be the proposition that there is exactly one sequence Q_n, satisfying (1) and (2) of the lemma. Then P_1 is obviously true. Also, P_n implies P_{n+1}, because given the unique Q_n: $q_1, q_2, \cdots, q_n$, there is exactly one way to define q_{n+1} so that the augmented sequence Q_{n+1}: $q_1, q_2, \cdots, q_n, q_{n+1}$ will satisfy (2) of the lemma. The lemma therefore follows by induction on n.

LEMMA 2. *Let i be a positive integer. Let m and n be integers $\geqq i$. Let Q_m and Q_n be as in Lemma 1. Then the i-th term q_i of Q_m is the same as the i-th term q_i' of Q_n.*

Proof of lemma. Let Q_i be the sequence consisting of the first i terms, $q_1, q_2, \cdots, q_i$ of Q_m. Let Q_i' be the sequence consisting of the first i terms $q_1', q_2', \cdots, q_i'$ of Q_n. By Lemma 1, Q_i is the same as Q_i'. Therefore, in particular, $q_i = q_i'$, which was to be proved.

With the aid of these *unique finite sequences*, satisfying (1) and (2) of Lemma 1, we can now define an infinite sequence, of the sort required in the conclusion of the theorem. For each i, let the i-th term q_i of our infinite sequence be the positive number which is the i-th term q_i of *every* sequence Q_m for which $m \geqq i$.

We know that $q_1, q_2, \cdots$ converges to 0, because (by (2) of Lemma 1) $0 < q_i < 1/i$ for $i > 1$.

In the above discussion, the treatment of sequences is informal. More precisely, a finite sequence Q_n (of n terms) is a function whose domain of definition is the set of all positive integers $\leqq n$; and an infinite sequence Q is a function whose domain of definition is the set of all integers $\geqq 1$. If a function is in turn defined in the usual way as a collection of ordered pairs, then the argument goes as follows: Let $Q = \bigcup Q_n$. By Lemma 1, every integer n occurs as the first term of at least one pair in Q. By Lemma 2, each such n occurs as the first term of at most one pair in Q. Therefore Q is a function defined over the positive integers, which was to be proved.

Note that the idea behind this proof can be stated roughly as follows: If we want to define a sequence, then we should explain what the first term of the sequence is to be, and we should give a rule whereby given the i-th term of the sequence, the $(i+1)$-st term is uniquely determined. This idea is used, without further analysis, even by writers of formidable rigor. The principal merit of the complete argument is not rigoristic but pedagogic. The basic device used in passing from the finite sequences to the infinite sequence is, in essence, the basic idea in the proof of the well-ordering theorem. (See Zermelo, *Beweis, dass jede Menge wohlgeordnet werden kann*, Math. Ann. vol. 59, 1904, pp. 514–516, and *Neuer Beweis für die Möglichkeit der Wohlordnung*, Math. Ann. vol. 65, 1907–08, pp. 107–128; or R. L. Moore, *Foundations of Point Set Theory*, Colloquium Publications of the Amer. Math. Soc. vol. 13, New York, 1932, pp. 1–4). The analogy becomes clearer if we abstract the principle used in the above proof, as follows:

THEOREM. *Let S be a collection of finite sequences, such that* (1) *for each positive integer n, S contains at least one sequence Q_n, having exactly n terms, and* (2) *if sequences Q_m, Q_n belong to S, and $m < n$, then Q_m is the same as the sequence consisting of the first m terms of Q_n. Then there is exactly one infinite sequence Q: $q_1, q_2, \cdots$, such that if Q_n belongs to S, then Q_n is the same as the sequence consisting of the first n terms of Q.*

The proof is trivial. Note that the collection S is very closely analogous to the collection of well-ordered sequences used in the proof of the well-ordering theorem; and note that in both cases the existence of the complete sequence is a consequence of the uniqueness of the partial sequences.

Obviously, none of the above is new. On the contrary, all of it is old, in a much more general form. But the proof of the well-ordering theorem is hard. It is the author's feeling that this proof is hard largely because it comes to grips, in the transfinite case, with a difficulty which in the finite case is almost universally ignored. If so, then rigorous proofs of the existence of simple sequences would be useful in preparing the student for the proof of the well-ordering theorem.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, State University of New York

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Harpur College, Endicott, New York. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1106. *Proposed by C. I. Lubin, University of Cincinnati*

Two non-parallel, non-coincident lines which cut the circle $|z|=r$ in the points a, b and c, d respectively, where a, b, c, d are complex numbers not necessarily all different, intersect in the point z given by

$$z = (a^{-1} + b^{-1} - c^{-1} - d^{-1})/(a^{-1}b^{-1} - c^{-1}d^{-1}).$$

E 1107. *Proposed by Victor Thébault, Tennie, Sarthe, France*

On the edges AB, AC, AD of a tetrahedron $ABCD$ are marked points M, N, P such that $AB=nAM$, $AC=(n+1)AN$, $AD=(n+2)AP$. Show that the plane MNP contains a fixed line as n varies.

E 1108. *Proposed by René Bloch, Humanistisches Gymnasium, Basle, Switzerland*

Let n, k, $a+1$ be three positive integers which are not all odd. Express $\sum_{i=0}^{k}(n+ai)^3$ as a difference of two integer squares.

E 1109. *Proposed by Erich Michalup, Caracas, Venezuela*

Find numbers so that their squares, when reversed, are the squares of the reversed numbers.

E 1110. *Proposed by Edgar Karst, Independence, Missouri*

An *ideal* cryptarithm has only one solution and involves all ten digits. Solve the following ideal addition cryptarithm based upon the name of the daily newspaper, Hannoversche Presse:

$$\begin{array}{cccccc} H & A & N & N & O & V \\ E & R & S & C & H & E \\ \hline P & R & E & S & S & E \end{array}.$$

SOLUTIONS

Making Change

E 1076 [1953, 479]. *Proposed by L. A. Ringenberg, Eastern Illinois State College*

John has a one dollar bill and wishes to change it into cold hard cash, any combination of coins, even perhaps a silver dollar. Jacob has a \$100 bill and wants it changed into smaller bills; he wants at least five ones; he doesn't want all ones and he doesn't want any twos. In how many different ways can a bank teller make change for John? for Jacob?

Solution by Julian Braun, Washington, D. C. Suppose the teller gives John H half dollars, Q quarters, and D dimes. Then the teller can choose to give John any number of nickels N, such that $0 \leqq N \leqq 20-10H-5Q-2D$ after which the number of pennies is automatically determined. Each possible value of N in this case gives rise to a different way John can receive change, a total of $21-10H-5Q-2D$ ways. The maximum value of H is 2, that of Q is $4-2H$, and that of D is $[10-5H-2.5Q]$, where $[x]$ denotes the greatest integer not exceeding x. Thus John can receive change in

$$\sum_{H=0}^{2} \sum_{Q=0}^{4-2H} \sum_{D=0}^{[10-5H-2.5Q]} (21 - 10H - 5Q - 2D) + 1 = 293$$

ways, where the 1 has been added to allow for the silver dollar.

Similar reasoning for Jacob gives

$$\sum_{f=0}^{1} \sum_{w=0}^{4-2f} \sum_{t=0}^{1-5f-2w} (20 - 10f - 4w - 2t) - 1 = 293$$

ways, the same number as before! Here f is the number of fifties, w the number of twenties, and t the number of tens.

Also solved by Leon Bankoff, W. E. Buker, J. E. Darraugh, J. W. Hamblen, A. R. Hyde, Sam Kravitz, D. C. B. Marsh, Azriel Rosenfeld, C. W. Trigg, R. Z. Vause, and the proposer.

Scales of Notation

E 1077 [1953, 479]. *Proposed by J. A. Ward, University of Kentucky*

Let b be an integer greater than 2 and let $n=b^{b-1}-1$. Show that

(1) $n/(b-1)$ when converted to base b is a $(b-1)$-digit integer whose digits are all ones.

(2) $n/(b-1)^2$ when converted to base b is a $(b-2)$-digit integer whose digits from left to right are $1, 2, 3, \cdots, b-4, b-3, b-1$.

Solution by C. W. Trigg, Los Angeles City College. (1) We have

$$N = (b^{b-1} - 1)/(b - 1) = b^{b-2} + b^{b-3} + \cdots + b + 1.$$

To base b, the digits of N are the coefficients of the b^i, so they consist of $b-1$ ones.

(2) To find $M=N/(b-1)$ the division may be accomplished synthetically:

$$\begin{array}{cccccccc|c}
1 & 1 & 1 & 1\cdots & 1 & 1 & 1 & 1 & 1 \\
 & 1 & 2 & 3\cdots b-5 & b-4 & b-3 & b-2 & & \\
\hline
1 & 2 & 3 & 4\cdots b-4 & b-3 & b-2 & \multicolumn{1}{|c}{b-1} & &
\end{array}$$

Since the remainder of the division is $b-1$, the coefficient of the b^0 term in M is $b-2+(b-1)/(b-1)$ or $b-1$. With this modification the third line of the synthetic division gives the $b-2$ digits of M in order.

Also solved by Bonnie Baker, L. F. Boron, D. R. Clutterham, Fred Discepoli, J. R. Enterline, A. L. Epstein, B. A. Hausmann, R. W. Heath, Walter Hoffman, Vern Hoggatt, Douglas Holdridge, A. R. Hyde, John Jones, Jr., Ray Jurgensen, M. S. Klamkin, D. C. B. Marsh, R. J. Mercer, Ancel Mewborn, Marian Moore, Morris Morduchow, E. G. Musch, G. B. Parrish, M. J. Pascual, L. A. Ringenberg, Azriel Rosenfeld, Nathan Schwid, R. E. Shafer, D. R. Sudborough, T. H. Sumner, J. A. Tierney, Donald Trumpler, R. Z. Vause, Chih-yi Wang, Maud Willey, J. W. Young, and the proposer. Late solutions by Sybil Kirk, A. E. Livingston, T. M. Morrow, Margaret Olmsted, S. Parameswaran, S. H. Shugart, and A. V. Sylwester.

Maximizing a Certain Product

E 1078 [1953, 479]. *Proposed by N. Shklov, University of Saskatchewan*

Let a real positive number n be split into x equal parts in such a manner that the product of the parts will be greatest. How many parts will there be?

Solution by C. E. Miller, University of Saskatchewan. We are to find the maximum term of the sequence $\{T_m\}$, where $T_m=(n/m)^m$. It is easily established that the value of the continuous real variable x which makes the function $(n/x)^x$ a maximum is n/e. Hence if n/e is an integer it is the required value of m. Otherwise let $p=[n/e]$. By considering T_{p+1}/T_p we easily see that T_{p+1} is greater than, less than, or equal to T_p according as n is greater than, less than, or equal to $(p+1)^{p+1}/p^p$, and so the required value of m is $p+1$, p, or both, respectively.

Also solved by J. W. Baldwin, Leon Bankoff, L. F. Boron, Julian Braun, I. A. Dodes, Fred Discepoli, A. L. Epstein, Michael Goldberg, R. R. Gutzman, M. S. Klamkin, J. D. Haggard, J. W. Hamblen, J. R. Hatcher and S. L. Morris (jointly), Vern Hoggatt, A. R. Hyde, John Jones, Jr., Ray Jurgensen, J. G. Leghorn, D. C. B. Marsh, Beckham Martin, L. V. Mead, R. J. Mercer, Donald Miller, Morris Morduchow, C. S. Ogilvy, F. D. Parker, G. B. Parrish, M. J. Pascual, L. L. Pennisi and N. C. Scholomiti (jointly), L. A. Ringenberg, Nathan Schwid, R. E. Shafer, O. E. Stanaitis, J. A. Tierney, R. Z. Vause, G. W.

Walker, M. C. Wicht, Arthur Wormser, and the proposer. Late solutions by A. E. Livingston, Margaret Olmsted, and A. V. Sylwester.

Many of these solutions were incomplete.

Algebraic Treatment of Some Trigonometry Formulas

E 1079 [1953, 479]. *Proposed by V. L. Klee, Jr., University of Washington*

Suppose f and g are real functions such that always $g(x-y)=g(x)g(y)+f(x)f(y)$, and such that $f(t)=1$ and $g(t)=0$ for some $t\neq 0$. Prove that always $g(x\pm y)=g(x)g(y)\mp f(x)f(y)$ and $f(x\pm y)=f(x)g(y)\pm g(x)f(y)$.

Solution by T. S. Chihara, Purdue University. From the symmetric roles played by x and y in

$$g(x-y)=g(x)g(y)+f(x)f(y) \tag{1}$$

it follows that g is an even function. Setting $y=t$ in (1) gives

$$g(x-t)=f(x). \tag{2}$$

On the other hand if we set $y=x-t$ in (1) we obtain

$$g(x)g(x-t)+f(x)f(x-t)=0,$$

or, using (2),

$$f(x)[g(x)+f(x-t)]=0.$$

Since this last relation is true for all x we have

$$f(x-t)=-g(x). \tag{3}$$

Thus $f(x)=-g(x+t)=-g(-x-t)=-f(-x)$, so that f is an odd function. Thus the first desired relation follows immediately. To obtain the second relation we replace x by $x\pm y$ in (2) and expand, getting

$$f(x\pm y)=g[x\pm(y\mp t)]=g(x)g(y\mp t)\mp f(x)f(y\mp t).$$

Use of (2) and (3) now gives the desired result.

Also solved by J. W. Baldwin, Leonard Carlitz, P. L. Chessin, J. R. Cox, Edgar Dougherty, D. S. Greenstein, A. S. Gregory, J. R. Hatcher, Vern Hoggatt, Douglas Holdridge, R. W. Huff, C. E. Kerr, M. S. Klamkin, Sidney Kravitz, J. Lehner, E. J. McShane and the proposer (jointly), D. C. B. Marsh, F. D. Parker, G. B. Parrish, M. J. Pascual, L. L. Pennisi, J. D. Reid, B. E. Rhoades, L. A. Ringenberg, Azriel Rosenfeld, Nathan Schwid, O. E. Stanaitis, Donald Trumpler, R. M. Walter, L. E. Ward, Jr., and J. V. Whittaker. Late solutions by A. E. Livingston, Margaret Olmsted, S. Parameswaran, and W. D. Serbyn.

Klamkin and Rosenfeld each pointed out that f and g are periodic of period $4t$. Reid and Gregory noted that the condition $t\neq 0$ is superfluous in the presence of the other hypotheses. Klamkin showed that if we further assume that f and

g are once differentiable, then $f(x)=\sin x$ and $g(x)=\cos x$. Pennisi established the same result under the assumptions that $f(x)$ and $g(x)$ are continuous, $\lim_{x\to 0} f(x)/x=1$, $\lim_{n\to\infty} f(x)x^n/n!=0$ for all values of x.

The problem shows that the usual formulas for $\cos(A\pm B)$ and $\sin(A\pm B)$ follow purely algebraically from the formula for $\cos(A-B)$ and the fact that $\sin \pi/2=1$, $\cos \pi/2=0$.

A Property of the Nine-Point Center

E 1080 [1953, 479]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Let I be the incenter, N the nine-point center, and D the midpoint of side BC of a triangle ABC. Show that one of the common tangents to the circles $I(N)$ and $D(N)$ is parallel to BC.

Solution by Leon Bankoff, Los Angeles, Calif. One of the common tangents to the circles $I(N)$ and $D(N)$ is parallel to BC if the radius of the circle $D(N)$ is equal to the sum of the inradius of triangle ABC and the radius of circle $I(N)$. Since, by Feuerbach's Theorem, $IN=R/2-r$, where $R/2$ is the radius of the nine-point circle and r is the inradius of triangle ABC, the proposition is proved.

Also solved by W. B. Carver (using conjugate coordinates), L. V. Mead, D. M. Seward, and the proposer.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known text books or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4578. *Proposed by N. S. Mendelsohn, University of Manitoba*

Find the number of essentially different eleven card Canasta hands which can be dealt from a 104 card pack. (We ignore red threes since they are always replaced in a hand. The pack consists of 8 aces, 8 twos, 4 threes, 8 fours, 8 fives, $\cdots$, 8 kings and 4 jokers. All cards of a given denomination are considered identical.)

4579. *Proposed by I. J. Schoenberg, University of Pennsylvania*

Let the relation

$$e^z = \left(\sum_{n=0}^{\infty} a_n z^n\right)\left(\sum_{n=0}^{\infty} b_n z^n\right)$$

hold for $|z| < r$, where $a_n \geqq 0$, $b_n \geqq 0$ $(n=0, 1, \cdots)$. Show that the two factors on the right side must be entire functions of the form

$$e^{az+c}, \qquad e^{bz-c}, \qquad (a \geqq 0,\ b \geqq 0,\ a + b = 1).$$

4580. *Proposed by D. J. Newman, Republic Aviation Corporation, Farmingdale, N. Y.*

For what real values of a does the power series expansion of the function $(1-ax+ax^2-x^3)^{-1}$ have all its coefficients non-negative?

4581. *Proposed by Ky Fan, University of Notre Dame*

Let M be a metric space in which every closed spheroid, *i.e.*, set of the form

$$S(z_0; r) = \{z \in M \mid d(z_0, z) \leqq r\}, \qquad (z_0 \in M, r > 0)$$

is compact (d denotes the distance). Let f be a continuous transformation from M into itself. Suppose that there is a point $x_0 \in M$ such that

$$d(x_0, f(x)) < d(x_0, x) \tag{1}$$

for every $x \neq x_0$ of M. Prove

(i) If $f(x_0) = x_0$, then

$$\lim_{n\to\infty} f^n(x) = x_0 \tag{2}$$

for all $x \in M$; where $f^1(x) = f(x)$ and $f^n(x) = f(f^{n-1}(x))$.

(ii) If $f(x_0) \neq x_0$, then for every point $x \in M$, there is a positive integer k, depending on x, such that $f^k(x) = x_0$.

4582. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn*

Show that

$$\text{(a)} \qquad \sum_{n=1}^{p} \frac{S_{1,n}^2}{n} \sim \frac{5}{3}\sum_{n=1}^{\infty}\frac{1}{n^3} + \frac{1}{3}(\gamma + \log p)^3,$$

$$\text{(b)} \qquad \sum_{n=1}^{p} \frac{S_{2,n}}{n} \sim \frac{\pi^2}{6}(\gamma + \log p) - \sum_{n=1}^{\infty}\frac{1}{n^3},$$

where γ is Euler's constant and

$$S_{m,n} = \sum_{r=1}^{n} \frac{1}{r^m}.$$

SOLUTIONS

A Special Case of Egyptian Fractions

4512 [1952, 640]. *Proposed by E. P. Starke, Rutgers University*

Prove that every positive rational number with odd denominator is a sum of a finite number of distinct terms from the sequence 1/3, 1/5, 1/7, 1/9, $\cdots$.

Solution by Robert Breusch, Amherst College.

LEMMA 1: *If* $A=3^n5$, $n\geqq 1$, *and* $B<3^n$, *then* B/A *is a sum of distinct positive fractions of the form* $1/3^t5$ *or* $1/3^t$ $(t\geqq 1)$.

Proof: Let $B=a_0+3a_1+\cdots+3^{n-1}a_{n-1}$, with $0\leqq a_\nu\leqq 2$. Then

$$\frac{B}{A}=\frac{a_0}{3^n5}+\frac{a_1}{3^{n-1}5}+\cdots+\frac{a_{n-1}}{3\cdot 5}\cdot$$

If $a_{n-1}=2$, write $2/3\cdot 5=1/3^2+1/3^25$, and add the last term to $a_{n-2}/3^25$. Thus we have to consider now $b/3^25$, where $0\leqq b\leqq 3$. If $b=3$, this is $1/15$, a term which in this case has not occurred before. If $b=2$, proceed as before: $2/3^25=1/3^3+1/3^35$, and now combine the last term with $a_{n-3}/3^35$. A continuation of this procedure leads to the desired representation of B/A.

LEMMA 2. *If* $A=3^n5^m$, $n\geqq 2$, $m\geqq 1$, *and* $B<3^n5^{m-1}$, *then* B/A *is a sum of distinct positive fractions of the form* $1/3^r5^s$.

Proof: Let $B=a_0+5a_1+5^2a_2+\cdots+5^{m-1}a_{m-1}$, with $0\leqq a_t\leqq 4$ for $t\leqq m-2$, and $a_{m-1}<3^n$. Then

$$\frac{B}{A}=\frac{a_0}{3^n5^m}+\frac{a_1}{3^n5^{m-1}}+\cdots+\frac{a_{m-2}}{3^n5^2}+\frac{a_{m-1}}{3^n5}\cdot$$

By Lemma 1, the last term is a sum of fractions of the desired kind, and with denominators that contain at most the first power of 5. If a_{m-2} is $4=3+1$, or 3, then $a_{m-2}/3^n5^2$ leads to one or two fractions whose numerator is 1, and whose denominator contains exactly the second power of 5. If $a_{m-2}=2$, then $2/3^n5^2=1/3^{n-2}5^3+1/3^n5^3$. Combining the last term with $a_{m-3}/3^n5^3$, we have to consider now $b/3^n5^3$, $(0\leqq b\leqq 5)$. For $b=5$, this is $1/3^n5^2$, a fraction which in this case has not occurred previously. For $b=4$ or $b=3$, we get one or two fractions whose denominators contain the third power of 5. If $b=2$, proceed as before. Continuing in this way, we get the desired representation of B/A.

LEMMA 3: *Let* $p_1=3, p_2, \cdots, p_n, \cdots$ *be the odd primes. Let* $A=p_1^{s_1}p_2^{s_2}\cdots p_n^{s_n}$, *with* $s_1\geqq 2$, *and* $s_t\geqq 1$ *for* $2\leqq t\leqq n$. *Let* B *be less than* A/p_n. *Then* B/A *is a sum of distinct fractions of numerator* 1, *whose denominators contain no prime factors greater than* p_n.

Proof: by induction on n. For $n=2$ the statement is true by Lemma 2. Assume it has been proved up to $n-1$. Let

$$B = a_0 + a_1 p_n + \cdots + a_{s_n} p_n^{s_n}, \quad \text{with} \quad 0 \leqq a_t < p_n \quad \text{for} \quad t \leqq s_n - 1$$

and $a_{s_n} < A/p^{s_n+1}$.
Thus, writing $A = Cp_n^{s_n}$, $a_{s_n} < C/p_n < C/p_{n-1}$, and

$$\frac{B}{A} = \frac{a_0}{C} \frac{1}{p_n^{s_n}} + \frac{a_1}{C} \frac{1}{p_n^{s_n-1}} + \cdots + \frac{a_{s_n}}{C}.$$

Now each numerator is certainly less than C/p_{n-1}, because $p_n < p_1 p_2 \cdots p_{n-1}$ for $n \geqq 4$; while for $n=3$, $p_n = 7$ is less than $3^2 \leqq C/5$. Thus each of the partial fractions for B/A can be represented in the required form, and no two fractions will have any term in common, because their denominators contain different powers of p_n.

THEOREM: *Let A be odd; then B/A can be written as a sum of distinct positive fractions with odd denominators and unit numerators.*

Proof: Let p_n be the greatest prime factor of A. Write the fraction in such a form that its denominator satisfies the hypothesis of Lemma 3. If the resulting numerator is less than A/p_n, the proof is complete from Lemma 3. Otherwise let $m > n$ be such that

$$\sum_{t=n+1}^{m} \frac{1}{p_t} < \frac{B}{A} < \sum_{t=n+1}^{m+1} \frac{1}{p_t}.$$

Then $B'/A' = B/A - \sum_{t=n+1}^{m} 1/p_t$ satisfies all the conditions of Lemma 3; each term in the representation of B'/A' is less than $1/p_m$, and therefore B/A itself is now expressed in the desired form.

Also solved by E. P. Churchill, and B. M. Stewart.

Editorial Note. It is easy to see that B/A has infinitely many representations as a sum of odd unit fractions since it can be broken up into $B_1/A_1 + B_2/A_2 + \cdots$ in innumerable ways and then the above method applied to each part.

The method rarely provides the simplest sum of unit fractions. For example $2/13 = 1/7 + 1/91$ but the method gives the sum of the reciprocals of 15, 17, 45, 315, 495, 1155, 14015, and 153153. The interesting problem of a "best" representation is not considered here.

The other solvers made the proof depend upon the following theorem. *Let A be any odd number, then there exist infinitely many odd multiples M_i of A, each of which has the property: every integer n such that $2 < n < \sigma(M_i) - 2$, is a sum of distinct positive divisors of M_i.*

Polynomials Suggested by Familiar Integers

4513 [1952, 701]. *Proposed by Leonard Carlitz, Duke University*

Put $\{m\}!=(x^m-1)(x^{m-1}-1)\cdots(x-1)$, $\{0\}!=1$.

A. Show that

$$\frac{\{2m\}!\{2n\}!}{\{m+n\}!\{m\}!\{n\}!} \qquad (m, n \geqq 0) \tag{1}$$

is a polynomial with rational coefficients.

B. The quotient

$$\frac{\{mn\}!}{\{m\}!(\{n\}!)^m}(x-1)^m$$

is not integral for all $m, n \geqq 0$. How can it be modified to be integral and at the same time reduce to $(mn)!/m!(n!)^m$ for $x=1$?

Solution by J. V. Whittaker, University of California, Los Angeles.

A. Let $F_k(x)$ be the (irreducible) cyclotomic polynomial of index k. The highest power of $F_k(x)$ which divides $\{m\}!$ is obviously $[m/k]$, the integral part of m/k. The rational function in question, then, is a polynomial with integral coefficients if, and only if,

$$\left[\frac{2m}{k}\right]+\left[\frac{2n}{k}\right] \geqq \left[\frac{m+n}{k}\right]+\left[\frac{m}{k}\right]+\left[\frac{n}{k}\right],$$

for every k. Setting $\mu=m/k-[m/k]$ and $\nu=n/k-[n/k]$, the above inequality reduces to $[2\mu]+[2\nu]\geqq[\mu+\nu]+[\mu]+[\nu]$ which obviously holds.

B. The highest power of $F_k(x)$ which divides $\{mn\}!/\{m\}!(\{n\}!)^m$ is

$$\left[\frac{mn}{k}\right]-\left[\frac{m}{k}\right]-m\left[\frac{n}{k}\right].$$

Setting $m=ak+r$ and $n=bk+s$, $0\leqq r, s<k$, the above expression reduces to $[rs/k]+as-a$ which is <0 if, and only if, $s=0$, and then it is $-a$. Therefore

$$P(x)=\frac{\{mn\}!(x-1)^m}{\{m\}!(\{n\}!)^m}\prod_{1<k|n}\left\{\frac{F_k(x)}{F_k(1)}\right\}^{[m/k]}$$

is a polynomial with rational coefficients such that $P(1)=(mn)!/m!(n!)^m$.

Also solved by O. E. Stanaitis and the Proposer.

Solutions of an Integral Equation

4514 [1952, 702]. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn*

$\psi(t)=$constant and $\psi(t)=|t|$ are solutions of the integral equation

$$2\psi(t) = \int_{-1}^{1} \psi(2tx)dx,$$

as is easily verified. Are there any other solutions?

Solution by J. E. Wilkins, Jr., Nuclear Development Associates, Inc. White Plains, N. Y. There are others. To find additional solutions, substitute $|t|^a$ for $\psi(t)$. This function will satisfy the integral equation if and only if $2^a=a+1$. This transcendental equation has exactly two real roots, namely 0 and 1, and has infinitely many complex roots

$$\alpha_n \pm i\beta_n(\log 2)^{-1}$$

such that

$$\alpha_n = -1 + \beta_n(\log 2)^{-1} \cot \beta_n,$$

$$2\beta_n(\log 2)^{-1} \exp(-\beta_n \cot \beta_n) = \sin \beta_n,$$

$$2n\pi < \beta_n < (4n+1)\pi/2 \qquad (n = 1, 2, \cdots).$$

For example, $\beta_1=7.454087$, $a_1=3.545368\pm 10.75397i$.

Also solved by G. A. Baker, Jr., R. H. Boyer, A. E. Currier and R. P. Bailey, L. L. Pennisi, Edgar Reich, H. J. Zimmerberg, and the Proposer.

Editorial Note. As shown also by the other solvers, there exist no real solutions of the integral equation, under the assumption that $\psi(x)$ is of class C^1 on $[0, 1]$, other than linear combinations of the solutions cited by the Proposer.

A Number Theory Summation

4515 [1952, 702]. *Proposed by Karl Goldberg, National Bureau of Standards, Washington, D. C.*

Prove

$$\sum x \equiv 2^{-4} + (2^{p-1} - 1)/p \pmod{p},$$

where p is an odd prime and the summation extends over all integers $x<p$ which are greater than $(p-1)/2$ and whose inverses mod p are also greater than $(p-1)/2$.

Solution by J. V. Whittaker, University of California, Los Angeles. The binomial expansion of $(1+1)^p$ gives

$$\frac{2^p-2}{p}=1+\frac{p-1}{2}+\frac{p-1}{2}\cdot\frac{p-2}{3}+\cdots+\frac{p-1}{2}\cdots\frac{p-(p-2)}{p-1}$$

$$\equiv 1-\frac{1}{2}+\frac{1}{3}-\cdots-\frac{1}{p-1}$$

$$\equiv -2\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\cdots+\frac{1}{p-1}\right)$$

$$\equiv -\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{(p-1)/2}\right) \pmod{p}.$$

We have also

$$1+2+3+\cdots+\frac{p-1}{2}=\frac{1}{2}\cdot\frac{p-1}{2}\cdot\frac{p+1}{2}\equiv -\frac{1}{8} \pmod{p}.$$

Hence

$$\frac{2^p-2}{p}+\frac{1}{8}\equiv -\sum\left(x+\frac{1}{x}\right) \pmod{p},$$

where the summation extends over the first half of the integers from 1 to $p-1$. Now if x_1 and $1/x_1$ both lie in the first half, then $x_2 \not\equiv -x_1$ and $1/x_2$ are both in the second half. Moreover, if x_1 is in the first half and $1/x_1$ in the second, then $x_2 \not\equiv 1/x_1$ is in the first half and $1/x_2$ in the second, so that the sum of all terms in $\sum(x+1/x)$ with x in the first half and $1/x$ in the second, is zero. Therefore

$$\frac{2^p-2}{p}+\frac{1}{8}\equiv -2\sum x\equiv 2\sum y \pmod{p},$$

where the first summation extends over integers in the first half with inverses in the first half, and the second summation over integers in the second half with inverses in the second half.

Also solved by Leonard Carlitz and the Proposer.

Editorial Note. Carlitz remarks that the same method leads to the formula

$$\sum x\equiv\frac{1}{8}+\frac{2^{p-1}-1}{2p} \pmod{p},$$

where the summation is restricted to such x that both x and its inverse are odd.

Logarithmic Concavity

4517 [1952, 702]. *Proposed by G. G. Lorentz, Wayne University*

A sequence a_n, $n=0, 1, \cdots$, is called logarithmically concave if

$$a_n^2 - a_{n-1}a_{n+1} \geqq 0, \qquad n = 1, 2, \cdots.$$

Prove that if the sequences a_n, b_n are positive and logarithmically concave, then their convolution $c_n = a_0b_n + a_1b_{n-1} + \cdots + a_nb_0$ has the same properties.

Solution by the Proposer. If a_n is logarithmically concave and positive then a_{n+1}/a_n is decreasing, hence for any such sequence

$$(1) \qquad A_{pq} = a_pa_q - a_{p-1}a_{q+1} \geqq 0, \qquad p \leqq q$$

for $p, q \geqq 0$. If we put $a_n = 0$ for $n < 0$, (1) becomes correct for all $p \leqq q$. Therefore our result is implied by the following: *If the sequences a_n, b_n, $n = \cdots, -1, 0, 1, \cdots$ have the property* (1) *and if all series $c_n = \sum_{\gamma=-\infty}^{+\infty} a_{n-\gamma}b_\gamma$, $n = \cdots, -1, 0, 1, \cdots$ are absolutely convergent, then also the c_n have the property* (1).

From (1) it follows that $A_{pq} \leqq 0$ for $p > q$. We introduce also the B_{pq} for the sequence b_n. Then

$$\begin{aligned} c_n^2 - c_{n-1}c_{n+1} &= \left(\sum_\nu a_{n-\nu}b_\nu\right)^2 - \sum_\nu a_{n-1-\nu}b_\nu \sum_\mu a_{n+1-\mu}b_\mu \\ &= \sum_{\nu,\mu=-\infty}^{+\infty} (a_{n-\nu}a_{n-\mu} - a_{n-\nu-1}a_{n-\mu+1})b_\nu b_\mu \\ &= \sum_{\nu,\mu} u_{\nu\mu} = \frac{1}{2}\sum_{\nu,\mu}(u_{\nu\mu} + u_{\mu-1,\nu+1}) \geqq 0, \end{aligned}$$

since

$$\begin{aligned} u_{\nu\mu} + u_{\mu-1,\nu+1} &= (a_{n-\nu}a_{n-\mu} - a_{n-\nu-1}a_{n-\mu+1})(b_\nu b_\mu - b_{\nu+1}b_{\mu-1}) \\ &= A_{n-\nu,n-\mu}B_{\mu\nu} \geqq 0, \end{aligned}$$

which proves our statement.

The proposed result ceases to be correct if it is assumed only that $a_n \geqq 0$, $b_n \geqq 0$. For instance, if $a_0 = a_5 = b_2 = b_5 = 1$, and $a_n = b_m = 0$ for the remaining n, m, then $c_5 = c_7 = 1$, $c_6 = 0$, and c_n is not logarithmically concave. The statement proved in the solution remains true if we replace (1) by (1′): $A_{pq} \leqq 0$ for $p \leqq q$ and concavity by convexity. On the contrary, the result proposed is not true after this change, even if a_n, b_n are positive. For instance, $a_n = 1$, $b_n = q^n$, $q > 0$, then $c_n = (q^{n+1} - 1)/(q-1)$ is not logarithmically linear and is logarithmically concave by the result, hence c_n is not logarithmically convex.

See also Kaluza, *Math. Zeitschrift* vol. 28, 1928, pp. 161–170, and Davenport and Pólya, *Canad. J. Math.*, vol. 1, 1949, pp. 1–5.

Also solved by Leonard Carlitz and O. E. Stanaitis.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to E. P. Vance, Oberlin College, Oberlin, Ohio.

Theory of Functions of a Complex Variable. By W. J. Thron, New York, John Wiley and Sons, 1953. 9+230 pages. $6.50.

This is a carefully written book which fills a need for a rigorous treatment of functions of a complex variable from a modern point of view. The purpose of the author to write a self-contained volume "in which all results are derived from a simple set of axioms" is admirably achieved. As a result, the book should be very useful as a text for graduate majors in mathematics.

The book is not designed for those people whose interest in complex variables is one of formal manipulation. For this reason it is probably not suitable as an introduction to the subject for students who wish simply to add another "mathematical tool" to their collection. Indeed, the contents will probably be more meaningful to any student who has at least a smattering of the theory already available. Although it is true, as stated, that no previous mathematical knowledge is required of the reader, it is also true that he should have a reasonable amount of mathematical maturity.

This in no sense limits the usefulness of the book for the purpose for which it is intended. Its usefulness may be even broader. Because of the range of topics covered, the book might well serve as a text in courses other than those treating complex variables. The first fourteen chapter headings suggest the possible flexibility in this regard: 1. Fundamental Concepts, 2. Real Numbers, 3. Cardinal Numbers, 4. Complex Numbers, 5. Sums and Products, 6. Hausdorff spaces, 7. Metric Spaces, 8. The Plane of Complex Numbers, 9. Limits, Continuity, Differentiability, 10. Real Functions of Real Variables, 11. Curves and Regions in the Plane of Complex Numbers, 12. Some Combinatorial Topology, 13. Jordan Curves, 14. Rectifiable and Directed Curves.

These constitute less than half the book, the remaining chapters being devoted to subjects more strictly related to complex analysis.

This reviewer would like to see more illustrative examples of such things as Hausdorff spaces which might help initiate the newcomer to the more abstract concepts. It is perhaps asking too much in a book which treats so many subjects, but the object might have been achieved by the inclusion of a wider variety of exercises in addition to those which simply ask for proofs of theorems.

All in all, however, this book should be a welcome addition to the literature both as a text and as a convenient reference.

W. D. MUNRO
University of Minnesota

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

SUMMER CONFERENCE FOR HIGH SCHOOL TEACHERS

Under the sponsorship of the National Science Foundation, the University of Washington will hold a Summer Conference for High School Teachers of Mathematics in Seattle from July 26 to August 20. Principal speakers will be Professor B. W. Jones of the University of Colorado and Professor C. B. Allendoerfer of the University of Washington. A number of short lecture series, discussion groups, and field trips will complete the program. A limited number of financial grants will be available to help participants cover the cost of travel and subsistence. Interested teachers should write to: Office of Short Courses and Conferences, University of Washington, Seattle 5, Wash.

SIXTH ANNUAL INSTITUTE FOR TEACHERS AND PROFESSORS OF MATHEMATICS

The sixth institute sponsored by the Association of Teachers of Mathematics in New England will be held at the Massachusetts Institute of Technology from August 19–26, 1954. Group leaders include Professor Elmer Mode, Boston University; Mr. Edmund Berkeley, Berkeley Associates; Mr. Charles Mergendahl, Newton High School; Mr. Carl Shuster, Trenton State Teachers' College, and Professor Herman Shea, M. I. T. For full details write to Mrs. M. Isabelle Savides, Levi F. Warren Junior High School, West Newton, Mass.

PERSONAL ITEMS

University of Ottawa announces the following appointments: Professor Olivier Biberstein, Professor J. T. Duprat, Assistant Professor Leopold Vachon, Lecturer J. L. Allard, and Lecturer Yvon Grandchamp.

Mr. Walter Abramowitz, formerly research fellow at Polytechnic Institute of Brooklyn, is employed as a physicist at the Solid State Research Institute, New York City.

Mr. A. C. Andersen, who has been teaching at Ben Franklin High School, Rochester, New York, has accepted a position as an instructor at General Motors Institute, Flint, Michigan.

Mr. W. P. Anderson, formerly a graduate student at the University of Minnesota, has been appointed to an instructorship in the General College of the University.

Mr. F. J. Arena of North Dakota Agricultural College has been promoted to an assistant professorship.

Mr. E. H. Auerbach, previously a student at Columbia University, has been appointed Lecturer at the University.

Assistant Professor G. B. Ax of Virginia Military Institute has been promoted to an associate professorship.

Mr. Souren Babikian of Aleppo College, Syria, is teaching at Macalester College on an exchange visitor program.

Mr. Howard Baeumler of the University of Buffalo has been appointed to an assistant professorship at Marshall College.

Mr. J. J. Bailey is now Chairman of the Department of Mathematics of Shattuck School, Faribault, Minnesota.

Mr. H. W. Becker has a position as Transmitter Engineer with Station WOW, Omaha, Nebraska.

Mr. T. H. Bedwell, formerly production manager and engineer with the Freeman Company, Yankton, South Dakota, has been appointed to an instructorship at Southern State Teachers College, Springfield, South Dakota.

Mr. D. C. Benson has been appointed to a research assistantship at Stanford University.

Dr. R. L. Blair has been appointed to an instructorship at the University of California at Davis.

Assistant Professor H. D. Block of Iowa State College has been appointed Visiting Assistant Professor in the Department of Mathematics, Institute of Technology, University of Minnesota.

Mr. A. P. Boblétt has been promoted to the position of Chief of the Statistics Division, United States Naval Ammunition Depot, Crane, Indiana.

Mr. J. E. Bosshart, formerly a lecturer at Western Reserve University, has been appointed to an assistant professorship at the University of Dayton.

Dr. Leila D. Bram, previously a research associate at the University of Chicago, has accepted a position as a mathematician at the Office of Naval Research, Washington, D. C.

Mr. G. U. Brauer has been appointed to an instructorship in the Department of Mathematics of the Institute of Technology, University of Minnesota.

Mr. R. B. Bredemeier, recently a graduate assistant at Oregon State College, is teaching at Hermiston High School, Oregon.

Mr. R. C. Brown, Jr., formerly a part-time instructor at the University of Kentucky, has been appointed to an instructorship at Glenville State College.

Dr. R. J. Buehler has returned to his position as a staff member of the Sandia Corporation, Albuquerque, New Mexico.

Dr. L. P. Burton of the University of California at Davis has been promoted to an assistant professorship.

Mrs. Jewell H. Bushey of Hunter College has been promoted to a professorship.

Dr. G. C. Byers has been appointed to an assistant professorship at Michigan College of Mining and Technology.

Dr. Buchanan Cargal of Iowa State College has accepted a position as Senior Aerophysics Engineer with Consolidated Vultee Aircraft Corporation, Fort Worth, Texas.

Mr. A. J. Carlan has been appointed Research Physicist by the American Optical Company, Southbridge, Massachusetts.

Mrs. Audrey M. Carlan has a position as Mathematician with the American Optical Company, Southbridge, Massachusetts.

Mr. T. F. Carroll, formerly an assistant engineer with Celotex Corporation, Marrero, Louisiana, is now Quality Control Supervisor for Sefton Fibre Can Company, New Orleans, Louisiana.

Associate Professor C. E. Clark of Emory University has accepted a position as Operations Analyst in the Operations Research Office, Johns Hopkins University.

Dr. K. G. Clemans of the University of Oregon has a position as Statistician at the Naval Ordnance Test Station, China Lake, California.

Mr. K. J. Cohen has been appointed Carl Lotus Becker Fellow at Cornell University.

Mr. H. B. Coleman has accepted a position as Senior Engineer in the Research Laboratory of Bendix Aircraft Corporation, Detroit, Michigan.

Miss Margaret J. Cotter, previously a student at State Teachers College, Upper Montclair, New Jersey, is teaching at West Orange High School, New Jersey.

Mr. J. W. Creely, formerly a research chemist with American Cyanamid Company, Bound Brook, New Jersey, has accepted a position as Assistant Technical Director of the Heat and Mass Flow Analyzer Laboratory, Columbia University.

Mr. P. M. Curran of Fordham University has been promoted to an assistant professorship.

Dr. Elizabeth H. Cuthill, formerly a part-time instructor at the University of Maryland, has received an appointment as a mathematician at the David Taylor Model Basin, Carderock, Maryland.

Mr. T. S. Dean is now with Dean and Jordan, Architects, Dallas, Texas.

Mr. E. G. Douglas of the University of South Carolina has been appointed Professor and Head of the Department of Mathematics of Newberry College.

Assistant Professor J. E. Eaton of Queens College has been promoted to an associate professorship.

Mr. Karl Eide, previously at Emmanuel Missionary College, Berrien Springs, Michigan, has been appointed to a professorship at Madison College, Tennessee.

Dr. Paul Erdös has been appointed Visiting Professor at the University of Notre Dame.

Assistant Professor A. B. Farnell of the United States Military Academy has accepted a position as Research Engineer with North American Aviation, Los Angeles, California.

Miss Catherine S. Feeley, who has been a graduate student at the University of Notre Dame, is now an auditor with Northern State Company, Chicago, Illinois.

Dr. I. K. Feinstein of the University of Illinois, Navy Pier, Chicago, has been promoted to an assistant professorship.

Associate Professor F. N. Fisch is on leave of absence from Colorado State College of Education for the academic year 1953–54 and is at George Peabody College for Teachers.

Dr. W. T. Fishback of the University of Vermont has been appointed to an assistant professorship at Ohio University.

Dr. Harley Flanders of the University of California has been promoted to an assistant professorship.

Mr. G. C. Francis has been appointed Lecturer at Columbia University.

Miss Frances Freese of Mount Union College has been appointed Assistant Dean of Women and Assistant Professor of Psychology at Texas Technological College.

Mr. A. L. Gilmore, Jr., formerly an instructor at Pearl River Junior College, Poplarville, Mississippi, is now Radar Instructor at Keesler Air Force Base, Biloxi, Mississippi.

Reverend F. J. Ginivan has been appointed to an associate professorship at St. Francis Xavier University, Nova Scotia.

Mr. M. A. Glatt of the Ballistic Research Laboratories, Aberdeen Proving Ground, Maryland, has accepted a position as a mathematician with General Electric Advanced Electronics Center, Ithaca, New York.

Mr. B. T. Goldbeck, Jr., of Texas Christian University has been appointed to an instructorship at the University of Oklahoma.

Miss Bernice Goldberg, previously a mathematician at the Air Force Research Center, Cambridge, Massachusetts, is now Technical Engineer in Numerical Analysis with the General Electric Company, Evandale, Ohio.

Mr. M. L. Goldwater, formerly an electronics engineer with Hughes Aircraft Company, Culver City, California, has accepted a position as Engineer with Librascope, Inc., Glendale, California.

Mr. K. E. Gorsline, who has been teaching at East High School, Denver, Colorado, has been appointed Assistant Principal of Merrill Junior High School, Denver, Colorado.

Dr. Arthur Grad, previously at the Office of Naval Research, Washington, D. C., has been appointed Mathematician at the Institute of Mathematical Sciences, New York University.

Mr. D. S. Greenstein, who has been a graduate student at the University of Pennsylvania, has been appointed to an assistant instructorship in the Moore School of Electrical Engineering of the University.

Mrs. Florence N. Greville, formerly an analyst in the Department of Defense, Washington, D. C., is teaching at Escala Americana de Rio de Janeiro.

Miss Louisa S. Grinstein, recently a graduate student at the University of Buffalo, has been appointed to a teaching fellowship at the University of Michigan.

Assistant Professor B. H. Gundlach of the University of Arkansas has been promoted to an associate professorship.

Mr. J. W. Hamblen has been appointed to a research assistantship in the Statistical Laboratory of Purdue University.

Mr. E. E. Hammond, Jr., has been appointed to an instructorship at Phillips Academy, Andover, Massachusetts.

Associate Professor Louise F. Hanson of Olivet College has been promoted to a professorship.

Assistant Professor Heinz Helfenstein of Stanford University has been appointed Lecturer at the University of Alberta.

Dr. I. N. Herstein, who has been with the Cowles Commission, University of Chicago, has been appointed to an assistant professorship at the University of Pennsylvania.

Mr. V. E. Hoggatt, Jr., has been appointed to an instructorship at San Jose State College.

Assistant Professor F. L. Holzhauser of Kent State University is teaching now at Shaker Heights High School, Cleveland, Ohio.

Dr. Burrowes Hunt has been appointed to an assistant professorship at Reed College.

Mrs. Verba W. Iturralde, who has been teaching at Bowie High School, El Paso, Texas, has been given an appointment at El Paso Technical Institute.

Mr. R. D. Johnson, Jr., has been appointed to a part-time instructorship at the University of Virginia.

Mr. W. W. Johnson of Huntington Polytechnic Institute has been appointed to an instructorship at Fenn College.

Mr. Raymond Kassler, previously mathematician at Evans Signal Laboratory, Belmar, New Jersey, has accepted a position as Development Engineer with R.C.A. Victor Division, Camden, New Jersey.

Miss Dora E. Kearney has been appointed to an associate professorship at Upper Iowa University, Fayette, Iowa.

Associate Professor E. S. Kennedy of American University of Beirut, Lebanon, was Visiting Professor at Brown University for the first semester of 1953–54 and is a member of the Institute for Advanced Study for the second semester.

Mr. R. W. Klopfenstein of Iowa State College has accepted a position as Research Engineer at R.C.A. Laboratories, Princeton, New Jersey.

Miss Mary B. Lieberknecht of Iowa State College has a position with the Boeing Airplane Company, Seattle, Washington.

Mr. F. H. Lloyd has been appointed to an instructorship at Missouri School of Mines and Metallurgy.

Mr. L. S. Lockingen of the University of Houston has been awarded a National Science Foundation Fellowship and is at the University of Texas.

Mr. H. C. Mayer, Jr., has been appointed Assistant Supervisor of Education, General Electric Company, Richland, Washington.

Assistant Professor Janet McDonald of Vassar College has been promoted to an associate professorship.

Mr. K. A. McGown, previously chairman of the Department of Mathematics of Wallington High School, New Jersey, has been appointed to an instructorship at Lafayette College.

Mr. R. L. Mentzer of North Dakota Agricultural College has been appointed to an instructorship at Teachers College of Connecticut.

Mr. J. W. Mettler, recently with the Educational Testing Service, Princeton, New Jersey, has accepted a position with the De Laval Steam Turbine Company, Trenton, New Jersey.

Mr. V. A. Miculka, formerly a graduate assistant at the University of Oklahoma, has been appointed Chairman of the Department of Mathematics of Frank Phillips College.

Mr. Akeley Miller has been appointed to an instructorship in the Physics Department of the University of South Dakota.

Associate Professor W. I. Miller of Bucknell University has been promoted to a professorship.

Assistant Professor Emeritus W. M. Miller of Washington and Lee University has been appointed to an associate professorship at Roanoke College.

Mr. L. J. Montzingo, Jr., of Roberts Wesleyan College has been promoted to an assistant professorship.

Professor G. E. Moore of the University of Illinois has been appointed Associate Dean of the College of Liberal Arts.

Miss Marian A. Moore, recently an instructor at Purdue University, is now Chairman of the Department of Mathematics, Glenbrook High School, Northbrook, Illinois.

Dr. K. B. Morgan has been appointed Chairman of the Department of Mathematics of Mount Kisco High School, New York.

Mr. J. T. Morse has been awarded a National Science Foundation Fellowship and is at Massachusetts Institute of Technology.

Mr. P. M. Moskowitz, previously a research engineer with Hiller Helicopters, Palo Alto, California, is now a dynamics engineer with Sikorsky Aircraft, Bridgeport, Connecticut.

Mr. G. R. Mott, formerly a student at Hofstra College, has accepted a position as Research Engineer with Republic Aviation Company, Farmingdale, New York.

Assistant Professor B. H. Mount, Jr., of the University of Pittsburgh has a position as Engineer with Westinghouse Electric Corporation, East Pittsburgh, Pennsylvania.

Assistant Professor J. H. Mulligan of New York University has been promoted to the position of Professor and Chairman of the Department of Mathematics.

Mr. H. E. Murray of American College, Tarsus, Turkey, has been promoted to the position of Chairman of the Department of Mathematics.

Mr. A. B. Neale, previously with Marquardt Aircraft Company, Van Nuys, California, is employed now as Mechanical Engineer by Hydro-Aire, Inc., Burbank, California.

Mr. J. A. Nickel has been appointed to an instructorship at Willamette University.

Mr. M. L. Norden, formerly mathematician in the Operations Research Office, Johns Hopkins University, has been appointed Senior Associate of Dunlap and Associates, Inc., Stamford, Connecticut.

Mr. L. R. Norwood, recently at the United States Army Signal Corps Laboratory, Fort Monmouth, New Jersey, is now Senior Engineer at the Electronic Defense Laboratory, Mountain View, California.

Mr. H. A. Osborn has been appointed Research Associate at the University of Chicago.

Mr. A. D. Pierson of Kansas City Junior College has retired.

Mr. M. F. Pollack of Union Linden High School, California, has retired.

Dr. D. H. Porter of Marion College has been promoted to the position of Professor of Mathematics and Physics.

Mr. W. O. Portmann has been appointed to a part-time instructorship at Case Institute of Technology.

Mr. L. B. Rall has been appointed to a research fellowship at Oregon State College.

Mr. H. L. Randolph, previously with the Bank of America, Long Beach, California, has a position as Engineering Analyst with Research-Garrett Corporation, Los Angeles, California.

Assistant Professor O. M. Rasmussen has been promoted to the position of Chairman of the Department of Mathematics, University of Denver.

Mr. W. A. Reid, recently a graduate student at Michigan State College, is now Applied Science Representative with International Business Machines, New York City.

Dean P. H. Renton of West Virginia Institute of Technology is teaching at Valley Regional High School, Deep River, Connecticut.

Assistant Professor C. L. Riggs of East Texas State Teachers College has been appointed to an associate professorship at Texas Technological College.

Miss Jean E. Sammet, formerly with Metropolitan Life Insurance Company, New York City, is now Assistant Project Engineer, Sperry Gyroscope Company, Great Neck, New York.

Mr. J. H. Sartain, previously with Edwin Shields Hewitt and Associates, Libertyville, Illinois, has accepted a position as Assistant Operations Analyst, Armour Research Foundation, Chicago, Illinois.

Dr. W. R. Seugling, formerly with Yoh Engineering Inc., Los Angeles, California, has a position as an engineering specialist with Allstates Engineering Company, Trenton, New Jersey.

Mr. F. A. C. Sevier, who has been a research mathematician at Bell Aircraft Corporation, Niagara Falls, New York, has been appointed Chairman of the Department of Mathematics of the College of South Jersey, Rutgers University.

Mrs. Dorothy B. Shaffer has been promoted to the position of Associate Mathematician at Cornell Aeronautical Laboratory, Buffalo, New York.

Mr. J. M. Shaheen has been appointed to an assistant professorship at Tri-State College.

Associate Professor L. W. Sheridan, recently a research engineer with the Minneapolis-Honeywell Regulator Company, Minnesota, has accepted a position as Senior Physicist with General Mills, Engineering Research and Development Division, Minneapolis, Minnesota.

Miss Bettie L. Shipman, formerly a teaching fellow at Texas Technological College, is teaching at Abernathy High School, Texas.

Mr. L. E. Sigler has been appointed Lecturer at Columbia University.

Mr. D. B. Singer has a position as Structural Research Engineer with Armour Research Foundation, Illinois Institute of Technology.

Mr. F. M. Sioson, previously at the Geophysical Observatory, Quezon City, Philippines, has been appointed to an instructorship at the College of Agriculture, University of the Philippines.

Sister Mary Stephanie of Georgian Court College has been promoted to an assistant professorship.

Dr. Michael Skalskyj of Xavier University has been promoted to an assistant professorship.

Mr. Abe Sklar, formerly a graduate student at the University of Chicago, has been appointed to a research assistantship at California Institute of Technology.

Associate Professor R. E. Smith of Duquesne University has been promoted to the position of Chairman of the Department of Mathematics.

Mr. T. C. Smith of Phillips University has been appointed to an instructorship in the Department of Engineering Mechanics, University of Nebraska.

Mr. R. L. Snider has been appointed to an instructorship at York Junior College, Pennsylvania.

Assistant Professor R. H. Sorgenfrey of the University of California at Los Angeles has been promoted to an associate professorship.

Mr. H. L. Steinberg has a position with Rich's, Stamp and Coin Department, Atlanta, Georgia.

Dr. R. G. Stoneham of the University of California has been appointed to an assistant professorship at San Diego State College.

Mr. P. Y. Tani, previously with Minneapolis-Honeywell Regulator Company, Minneapolis, has accepted a position as Research Engineer with North American Aviation, Los Angeles, California.

Mr. Liston Tatum, Jr., is now Sales Representative for International Business Machines, Cincinnati, Ohio.

Assistant Professor Takashi Terami of the College of St. Thomas has been promoted to an associate professorship.

Mr. G. T. Thompson, recently a graduate assistant at Oregon State College, has a position as Applied Mathematician at Jet Propulsion Laboratory, California Institute of Technology.

Associate Professor George Tunell of the Department of Geology, University of California at Los Angeles, has been promoted to a professorship.

Assistant Professor William Wallace of Northeastern University has been promoted to an associate professorship.

Mr. C. R. Wampole is now Head of the Department of Mathematics of Eastern Military Academy, Cold Spring Harbor, New York.

Dr. W. H. Warner, formerly a research assistant at Carnegie Institute of Technology, is now a research associate in the Graduate Division of Applied Mathematics, Brown University.

Dr. F. J. Weyl has been appointed Director of the Mathematical Sciences Division, United States Office of Naval Research, Washington, D. C.

Assistant Professor Herbert Wolf of the University of South Carolina has been appointed to an associate professorship at Tri-State College.

Mr. F. G. Wolfort, recently a student at St. John's University, has been appointed to a research assistantship at Massachusetts Institute of Technology.

The following appointments to graduate assistantships have been announced: Catholic University of America, Mr. D. B. J. Tomiuk; Lehigh University, Mr. E. K. Dorff; Ohio State University, Mr. P. J. Nikolai; Purdue University, Mr. R. E. Dowds, Mr. D. G. Johnson; University of Georgia, Mr. L. H. Williams; University of Skoplje, Yugoslavia, Miss Kovina Milosevich.

The following appointments to teaching assistantships have been announced: Cornell University, Mr. G. E. Collins; University of British Columbia, Mr. C. A. Swanson; University of California, Mr. Melvin Pomerantz; University of Oklahoma, Mr. J. E. Hoffman; University of Virginia, Mr. J. J. Greever, III.

Professor Emeritus E. M. Berry of State Teachers College, Chadron, Nebraska, died on April 26, 1953; he had been a member of the Association for thirty-three years.

Professor Emeritus G. R. Livingston of San Diego State College died on September 2, 1953; he had been a member of the Association for thirty-four years.

President Emeritus C. W. Smith of Wisconsin State College, Superior, died on November 10, 1953; he was a charter member of the Association.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE THIRTY-SEVENTH ANNUAL MEETING OF THE ASSOCIATION

The thirty-seventh annual meeting of the Mathematical Association of America was held at Johns Hopkins University, Baltimore, Maryland on Thursday, December 31, 1953, in conjunction with the annual meeting of the American Mathematical Society. About six hundred and forty persons were registered, including the following three hundred and sixty members of the Association:

J. C. Abbott, M. I. Aissen, C. B. Allendoerfer, R. D. Anderson, H. A. Antosiewicz, K. J. Arnold, L. A. Aroian, Sholom Arzt, W. L. Ayres, Frances E. Baker, B. J. Ball, N. H. Ball, W. E. Barnes, I. A. Barnett, Grace E. Bates, E. G. Begle, W. J. Bellmer, T. J. Benac, A. A. Bennett, R. H. Bing, Z. W. Birnbaum, I. E. Block, I. L. Bossler, T. A. Botts, S. G. Bourne, Julia W. Bower, J. W. Brace, Leila D. Bram, H. W. Brinkmann, A. R. Brown, Jr., B. H. Buikstra, R. S. Burington, J. H. Bushey, Jewell H. Bushey, G. H. Butcher, S. S. Cairns, E. A. Cameron, R. H. Cameron, H. H. Campaigne, J. F. Canu, L. Virginia Carlton, Dorothy I. Carpenter, W. B. Carver, J. W. Cell, E. W. Cheney, P. L. Chessin, B. H. Chovitz, D. E. Christie, G. F. Clanton, C. E. Clark, F. E. Clark, F. Marion Clarke, G. R. Clements, A. H. Clifford, A. H. Cockshott, E. W. Coffin, E. J. Cogan, L. W. Cohen, Teresa Cohen, Nancy Cole, A. H. Copeland, J. B. Cornelison, W. H. H. Cowles, V. F. Cowling, J. H. Curtiss, Wayne Dancer, R. L. Davis, L. J. Deck, R. C. Di Prima, F. G. Dressel, Nelson Dunford, W. L. Duren, Jr., W. H. Durfee, W. F. Eberlein, R. P. Eddy, H. P. Edmundson, D. F. Eliezer, Joanne Elliott, Arthur Erdélyi, Paul Erdös, D. H. Erkiletian, Jr., W. H. Fagerstrom, William Feller, F. A. Ficken, N. J. Fine, W. T. Fishback, Harley Flanders, B. A. Fleishman, S. A. Foote, Gloria C. Ford, L. R. Ford, M. K. Fort, Jr., Tomlinson Fort, A. H. Fox, J. S. Frame, C. H. Frick, R. E. Fullerton, H. M. Gehman, J. J. Gehrig, J. J. Gergen, B. C. Getchell, K. G. Getman, Wallace Givens, A. M. Gleason, Marion Goddard, H. W. Godderz, Casper Goffman, V. D. Gokhale, Michael Goldberg, Oscar Goldman, W. A. Golomski, R. A. Good, A. W. Goodman, W. H. Gottschalk, E. C. Gras, F. L. Griffin, J. S. Griffin, Jr., Emil Grosswald, Arnold Grudin, Samuel Gruenzweig, P. E. Guenther, V. H. Haag, Franklin Haimo, E. E. Hammond, Jr., J. R. Hammond, H. W. Handsfield, H. H. Hartzler, Melvin Hausner, E. A. Hedberg, Marguerite Z. Hedberg, G. A. Hedlund, C. E. Heilman, G. C. Helme, Melvin Henriksen, R. T. Herbst, Coleman Herpel, I. R. Hershner, Jr., I. N. Herstein, P. S. Herwitz, I. I. Hirschman, A. J. Hoffman, J. E. Hoffman, F. E. Hohn, Betty W. Holz, J. R. Holzinger, J. G. Horne, Jr., E. Marie Hove, G. B. Huff, Ralph Hull, D. W. Hullinghorst, Louise S. Hunter, W. A. Hurwitz, M. A. Hyman, R. F. Jackson, S. B. Jackson, H. G. Jacob, Jr., C. A. Johnson, R. E. Johnson, B. W. Jones, F. B. Jones, John Jones, Jr., Mark Kac, Robert Kalin, Hyman Kamel, L. M. Kells, J. B. Kelly, L. M. Kelly, J. H. B. Kemperman, M. R. Kenner, J. R. F. Kent, D. E. Kibbey, E. R. Kiely, H. L. Kinsolving, George Klein, J. R. Kline, Rev. C. F. Koehler, Fulton Koehler, T. L. Koehler, H. L. Krall, R. R. Kuebler, Jr., Stephen Kulik, Harry Langman, Leo Lapidus, Rev. E. H. Larguier, C. G. Latimer, Solomon Lefschetz, Marguerite Lehr, R. A. Leibler, Walter Leighton, R. B. Leipnik, Anne L. Lewis, D. C. Lewis, Jr., E. V. Lewis, F. W. Light, Jr., D. B. Lloyd, Charles Loewner, Lee Lorch, F. W. Lott, Jr., D. B. Lowdenslager, L. L. Lowenstein, C. I. Lubin, Elizabeth C. Lukacs, R. C. Lyndon, Rev. J. J. MacDonnell, C. C. MacDuffee, G. R. MacLane, Saunders MacLane, H. M. MacNeille, W. G. Madow, J. F. Manogue, M. H. Martin, A. P. Mattuck, K. O. May, V. O. McBrien, B. H. McCandless, Sophia L. McDonald, Edith A. McDougle, S. S. McNeary, E. J. McShane, Florence M. Mears, A. E. Meder, Jr., Herman Meyer, Joseph Milkman, D. D. Miller, F. H. Miller, A. K. Mitchell, Deane Montgomery, A. H. Moore, T. W. Moore, W. K. Morrill, C. W. Munshower, F. D. Murnaghan, C. H. Murphy, Jr., W. R. Murray, Morris Newman, C. V. Newsom, R. A. Niemann, A. B. J. Novikoff, C. O. Oakley, Ruth E. O'Donnell, M. W. Oliphant,

Gloria Olive, L. F. Ollmann, E. M. Olson, Morris Ostrofsky, J. C. Oxtoby, O. O. Pardee, G. W. Patterson, III, G. W. Petrie, III, B. J. Pettis, H. P. Pettit, F. A. Phillips, C. F. Pinzka, Everett Pitcher, R. J. Pitts, J. C. Polley, J. W. Popow, G. B. Price, M. H. Protter, F. M. Pulliam, A. L. Putnam, O. J. Ramler, J. F. Randolph, R. M. Rankin, L. T. Ratner, R. S. Raven, G. E. Raynor, Rev. Timothy Reardon, R. W. Rector, C. J. Rees, Mina S. Rees, W. D. Reeve, Haim Reingold, C. N. Reynolds, D. E. Richmond, L. A. Ringenberg, T. J. Rivlin, I. H. Rose, Saul Rosen, R. A. Rosenbaum, P. C. Rosenbloom, Mary E. Rudin, Hans Samelson, Leo Sario, J. B. Scarborough, Robert Schatten, J. A. Schatz, S. A. Schelkunoff, I. J. Schoenberg, Augusta L. Schurrer, Abraham Schwartz, C. H. W. Sedgewick, Raymond Sedney, F. A. C. Sevier, J. M. Shaheen, W. F. Shenton, A. L. Shields, Marlow Sholander, J. A. Silva, L. L. Silverman, Sister Marie Loretta, Sister Mary Raphael, M. F. Smiley, Ernst Snapper, W. S. Soar, Vivian E. Spencer, C. A. Spicer, George Springer, E. R. Stabler, Marion E. Stark, E. P. Starke, C. F. Stephens, Rothwell Stephens, F. M. Stewart, M. F. Stilwell, Ruth W. Stokes, R. R. Stoll, W. J. Strange, P. M. Swingle, Gabor Szego, Choy-tak Taam, T. T. Tanimoto, Mildred E. Taylor, William Clare Taylor, Feodor Theilheimer, G. B. Thomas, Jr., J. M. Thomas, G. L. Thompson, D. L. Thomsen, Jr., R. M. Thrall, J. A. Tierney, John Todd, Marian M. Torrey, J. I. Tracey, A. W. Tucker, J. L. Ullman, Helen E. VanSant, Mary C. Varnhorn, R. J. Walker, A. D. Wallace, J. A. Ward, W. H. Warner, G. C. Webber, J. V. Wehausen, Brother Bernard Alfred Welch, F. G. Werner, D. W. Western, F. J. Weyl, P. M. Whitman, G. T. Whyburn, W. M. Whyburn, H. H. Wicke, D. V. Widder, Mary E. Williams, W. L. Williams, Alberta Wolfe, J. W. Wrench, Jr., R. C. Yates, D. M. Young, Jr., J. A. Zilber.

Sessions of the Association were held on Thursday morning and afternoon in Remsen Hall of Johns Hopkins University, with President E. J. McShane presiding. The Program Committee for the meeting consisted of W. G. Madow, Chairman; David Blackwell, and E. P. Northrop.

INTRODUCTORY NOTE ON THE PROGRAM

The papers which were presented had largely resulted from various projects intended to improve undergraduate mathematical education or to provide a more satisfactory transition from school to college.

Professor Saunders MacLane who was president of the Association in 1951 and 1952, has had a deep interest in attempts to improve mathematical education, and was active in organizing some of the projects which are mentioned below.

Two of the papers dealt with the transition from school to college. Professor H. W. Brinkmann is chairman of the Subcommittee on Mathematics of the School and College Study of Admission with Advanced Standing sponsored by twelve colleges and universities. Professor E. G. Begle was a member of the Mathematics Panel of the School and College Study of General Education sponsored by three schools and three universities. Both of these studies were aided by grants from the Fund for the Advancement of Education established by the Ford Foundation.

As part of its continuing program in the mathematical training of social scientists, the Social Science Research Council sponsored an eight-week intensive summer session at Dartmouth College, known as the 1953 Summer Institute in Mathematics for Social Scientists. All students in this Institute were college graduates; about half had received their doctorates in a social science. A large

part of the curriculum consisted of topics, usually taught in advanced undergraduate or beginning graduate courses in mathematics, that were believed to be suitable for undergraduate mathematical training. The program was conducted with the help of a grant from the Behavioral Sciences Division of the Ford Foundation.

The session on New College Mathematics was the result of three related projects. Professors W. L. Duren, Jr., G. B. Price, and A. W. Tucker are members of the Association's Committee on the Undergraduate Program in Mathematics. Dr. C. V. Newsom is chairman of the Joint Committee on Teacher Training in Mathematics sponsored by the Mathematical Association of America and the National Council of Teachers of Mathematics. Professor B. W. Jones was director of an eight-week Summer Conference in Collegiate Mathematics that met at the University of Colorado under the sponsorship of the National Science Foundation and the University of Colorado.

FIRST SESSION OF THE ASSOCIATION

Round Table: "Mathematics for Social Scientists," by the faculty of the 1953 Summer Institute: Professor W. G. Madow, University of Illinois, Professor R. M. Thrall, University of Michigan, Professor R. R. Bush, Harvard University, and Professor Howard Raiffa, Columbia University.

Retiring Presidential Address: "Of Course and Courses," by Professor Saunders MacLane, University of Chicago.

SECOND SESSION OF THE ASSOCIATION

"Secondary School Mathematics for the Exceptional Student," by Professor H. W. Brinkmann, Swarthmore College.

"A Study of the Integration of School and College Mathematics," by Professor E. G. Begle, Yale University.

New College Mathematics

A symposium exploring some aspects of the problems presented by efforts to introduce major reforms into the undergraduate program in mathematics.

Introductory remarks, by Professor G. B. Price, University of Kansas.

"What should we teach in college mathematics?," by Dr. C. V. Newsom, Associate Commissioner for Higher Education, State of New York.

"What is the role of the set concepts?," by Professor W. L. Duren, Jr., Tulane University.

"What kind of mathematics will social scientists need?," by Professor A. W. Tucker, Princeton University.

"Can our present teachers convert to a new program? A report on the 1953 Summer Conference at Boulder," by Professor B. W. Jones, University of Colorado.

MEETING OF THE BOARD OF GOVERNORS

The Board met on Wednesday morning in the Lounge of Levering Hall, with seventeen members present. Among the more important items of business transacted were the following:

Professor J. F. Randolph of the University of Rochester was re-elected as a member of the Finance Committee for the four-year term 1954–1957.

Approval was given to the appointment by President McShane of the following Nominating Committee for 1954: A. L. Putnam, Chairman; P. S. Jones, and E. E. Moise.

It was voted to hold the thirty-eighth annual meeting at the University of Pittsburgh, Pittsburgh, Pennsylvania, on Thursday, December 30, 1954.

On the recommendation of the Finance Committee, the Board voted that $500 be transferred from the General Fund to the Chauvenet Fund. It is expected that this will leave a sufficiently large balance in the Chauvenet Fund, so that hereafter the payments of the Chauvenet Prize may be made from income alone.

The Board also voted that the expenses of the second edition of "Professional Opportunities in Mathematics" be charged against the Chace Fund and that sales of this pamphlet be credited to the Chace Fund.

Professor J. S. Frame reported for the Committee on Employment Opportunities that the Council of the American Mathematical Society had also voted its approval of the establishment of a register of positions open to mathematicians at the joint meetings of the Society and the Association. The first such register was available at Levering Hall on Wednesday afternoon and Thursday.

ANNUAL BUSINESS MEETING OF THE ASSOCIATION

The annual business meeting of the Association was held on Thursday, December 31, 1953, at 2:00 P.M. in Remsen Hall of Johns Hopkins University, Baltimore, Maryland. Vice-President F. L. Griffin presided.

The Secretary announced the results of the balloting for officers, in which 1486 votes were cast. H. S. M. Coxeter of the University of Toronto was elected First Vice-President for the two-year term 1954–1955. Philip Franklin of the Massachusetts Institute of Technology and C. E. Springer of the University of Oklahoma were elected Governors for the three-year term 1954–1956.

As chairman of the Committee on the 1953 Chauvenet Prize, Professor C. B. Allendoerfer presented the 1953 Chauvenet Prize to Professor E. J. McShane of the University of Virginia for his paper, "Partial Orderings and Moore-Smith Limits," published in this MONTHLY, volume 59, 1952, pp. 1–11.

MEETINGS OF OTHER ORGANIZATIONS

The sessions of the American Mathematical Society began on Monday, December 28 and continued through Wednesday afternoon. The Josiah Willard Gibbs Lecture entitled "Mathematics in Economics" was delivered on Monday

evening by Professor Wassily Leontief of Harvard University. Invited addresses were delivered by Professors A. D. Wallace of Tulane University and D. V. Widder of Harvard University.

The Pi Mu Epsilon Fraternity held its annual meeting on Monday in Levering Hall.

ARRANGEMENTS, ENTERTAINMENT AND RECREATION

The Committee on Arrangements for the meeting consisted of D. C. Lewis, Chairman; W. L. Chow, A. H. Clifford, L. W. Cohen, H. M. Gehman, E. K. Haviland, W. K. Morrill, Marian M. Torrey.

Registration took place in Levering Hall on the Homewood Campus of Johns Hopkins University. Rooms were available in the Alumni Memorial Dormitory and at the Bradford Apartments from Monday morning until Friday morning. Hotel accommodations were also available in downtown Baltimore. Meals were served in the cafeteria of Levering Hall and at the Faculty Club.

The ladies of the Department of Mathematics of Johns Hopkins University entertained at tea in the main ballroom of Levering Hall on Monday afternoon. Sightseeing tours of Annapolis and of Baltimore were held on Tuesday and Wednesday, respectively.

The banquet for the members of the mathematical organizations and their guests was held on Wednesday evening in Levering Hall. Professor D. C. Lewis, acting as toastmaster, welcomed those in attendance on behalf of Johns Hopkins University. President E. J. McShane of the Association and President G. T. Whyburn of the Society responded on behalf of their respective organizations. Dr. Alan T. Waterman, Director of the National Science Foundation, described the program of the Foundation for grants and scholarships in the field of Mathematics.

Professor Ralph Hull presented a resolution of thanks which was adopted by a hearty and unanimous vote. The resolution expressed appreciation to the members of the local committee and to the authorities of Johns Hopkins University for their efforts in arranging this enjoyable meeting.

H. M. GEHMAN, *Secretary-Treasurer*

OFFICERS AND COMMITTEES AS OF JANUARY 1, 1954

OFFICERS

President, E. J. McShane, University of Virginia (1953–1954)
Honorary President, W. D. Cairns, Oberlin College
First Vice-President, H. S. M. Coxeter, University of Toronto (1954–1955)
Second Vice-President, W. L. Duren, Jr., Tulane University (1953–1954)
Editor, C. B. Allendoerfer, University of Washington (1952–1956)
Secretary-Treasurer, H. M. Gehman, University of Buffalo (1953–1957)
Associate Secretary, Edith R. Schneckenburger, University of Buffalo (1953–1957)

ADDITIONAL MEMBERS OF THE BOARD OF GOVERNORS

Ex-Presidents

L. R. Ford, Illinois Institute of Technology (1949–1954)
R. E. Langer, University of Wisconsin (1951–1956)
Saunders MacLane, University of Chicago (1953–1958)

Governors at Large

D. H. Lehmer, University of California (1952–1954)
W. E. Milne, Oregon State College (1952–1954)
S. S. Cairns, University of Illinois (1953–1955)
A. W. Tucker, Princeton University (1953–1955)
Philip Franklin, Massachusetts Institute of Technology (1954–1956)
C. E. Springer, University of Oklahoma (1954–1956)

Sectional Governors (July 1, 1951–June 30, 1954)

Allegheny Mountain, F. H. Steen, Allegheny College
Indiana, J. C. Polley, Wabash College
Kentucky, Aughtum S. Howard, Kentucky Wesleyan College
Metropolitan New York, T. F. Cope, Queens College
Nebraska, J. M. Earl, University of Omaha
Northern California, E. B. Roessler, University of California at Davis
Oklahoma, J. C. Brixey, University of Oklahoma
Rocky Mountain, C. F. Barr, University of Wyoming
Wisconsin, H. P. Pettit, Marquette University

Sectional Governors (July 1, 1952–June 30, 1955)

Kansas, G. B. Price, University of Kansas
Missouri, R. J. Michel, Southeast Missouri State Teachers College
Ohio, I. A. Barnett, University of Cincinnati
Pacific Northwest, R. D. James, University of British Columbia
Southeastern, F. A. Lewis, University of Alabama
Southwestern, R. F. Graesser, University of Arizona
Upper New York State, C. W. Munshower, Colgate University
New England Region, B. H. Brown, Dartmouth College

Sectional Governors (July 1, 1953–June 30, 1956)

Illinois, E. B. Miller, Illinois College
Iowa, D. L. Holl, Iowa State College
Louisiana-Mississippi, F. A. Rickey, Louisiana State University
Maryland-Dist. of Col.-Virginia, S. B. Jackson, University of Maryland
Michigan, P. S. Jones, University of Michigan

Minnesota, K. O. MAY, Carleton College
Philadelphia, C. O. OAKLEY, Haverford College
Southern California, C. W. TRIGG, Los Angeles City College
Texas, E. R. HEINEMAN, Texas Technological College

STANDING COMMITTEES OF THE ASSOCIATION

FINANCE COMMITTEE

W. B. CARVER (1952–1955), J. F. RANDOLPH (1954–1957), H. M. GEHMAN, *ex officio.*

EDITORIAL COMMITTEE ON CARUS MONOGRAPHS

N. H. MCCOY, *Chairman* (1950–1955), KARL MENGER (1949–1954), I. S. SOKOLNIKOFF (1951–1956), P. R. HALMOS (1952–1957), SAMUEL EILENBERG (1953–1958), TIBOR RADO (1954–1959).

COMMITTEE ON EARLE RAYMOND HEDRICK LECTURES

R. J. WALKER, *Chairman* (1953–1954), IRVING KAPLANSKY (1953–1955), EDWIN HEWITT (1954–1956).

COMMITTEE ON PLACES OF MEETINGS

M. F. SMILEY, *Chairman* (1952–1954), E. R. LORCH (1953–1955), W. M. WHYBURN (1954–1956).

COMMITTEE ON THE PUTNAM PRIZE COMPETITION

R. G. SANGER, *Chairman* (July 1951–June 1954), A. M. GLEASON (July 1952–June 1955), TIBOR RADO (July 1953–June 1956), L. E. BUSH, Director (July 1953–June 1958).

COMMITTEE ON SECTIONS

C. O. OAKLEY (1951–1954), J. C. POLLEY (1953–1956), EDITH R. SCHNECKENBURGER, *ex officio.*

COMMITTEE ON SLAUGHT MEMORIAL PAPERS

P. C. ROSENBLOOM, *Chairman* (1953–1955), B. W. JONES (1952–1954), J. L. KELLEY (1954–1956), C. B. ALLENDOERFER, *ex officio.*

REPRESENTATIVES OF THE ASSOCIATION

On the Policy Committee for Mathematics:
J. S. FRAME (1952–1954), E. J. MCSHANE, *ex officio*, H. M. GEHMAN, *ex officio*
On the National Research Council:
EINAR HILLE (July 1, 1953–June 30, 1956)
On the Council of the American Association for the Advancement of Science:
J. H. CURTISS (1953–1954), KARL MENGER (1954–1955)
On the American Council on Education:
E. J. MCSHANE, *ex officio*, H. M. GEHMAN, *ex officio*
On the A.A.A.S. Cooperative Committee on the Teaching of Mathematics and Science:
DAVID BLACKWELL (1954–1956)
On the Committee on Definitions of Electrical Terms:
S. A. SCHELKUNOFF
On the Committee on the Mathematical Training of Social Scientists:
F. L. GRIFFIN, E. P. NORTHROP
On the Mathematics Committee of the School and College Study of Admission with Advanced Standing:
C. R. PHELPS (1953–1955)
On the U. S. Subcommittee of the International Mathematical Instruction Committee (I.M.U.K.):
A. M. GLEASON (1953–1954), S. S. CAIRNS (1953–1956)

CALENDAR OF FUTURE MEETINGS

Thirty-fifth Summer Meeting, University of Wyoming, Laramie, Wyoming, August 30–31, 1954.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Marshall College, Huntington, West Virginia, May 1, 1954.
ILLINOIS, Knox College, Galesburg, May 14–15, 1954.
INDIANA, Rose Polytechnic Institute, Terre Haute, May 1, 1954.
IOWA, Iowa State College, Ames, April 30–May 1, 1954.
KANSAS, Baker University, Baldwin City, March 27, 1954.
KENTUCKY, University of Kentucky, April 24, 1954.
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK, St. John's University, Brooklyn, March 27, 1954.
MICHIGAN, University of Michigan, Ann Arbor, March 27, 1954.
MINNESOTA, Hamline University, St. Paul, May 8, 1954.
MISSOURI, University of Missouri, Columbia, May 7, 1954.
NEBRASKA, Omaha, April 24, 1954.
NORTHERN CALIFORNIA
OHIO, Ohio State University, Columbus, April 17, 1954.
OKLAHOMA, Oklahoma City University, October, 1954.
PACIFIC NORTHWEST, Reed College, Portland, Oregon, June 18, 1954.
PHILADELPHIA
ROCKY MOUNTAIN, Colorado Agricultural and Mechanical College, Fort Collins, April 30–May 1, 1954.
SOUTHEASTERN, University of South Carolina, Columbia, March 19–20, 1954.
SOUTHERN CALIFORNIA, George Pepperdine College, Los Angeles, March 13, 1954.
SOUTHWESTERN, Arizona State College, Tempe, April 16–17, 1954.
TEXAS, Texas Technological College, Lubbock, April 23–24, 1954.
UPPER NEW YORK STATE, College for Teachers at Albany, May 1, 1954.
WISCONSIN, State Teachers College, Eau Claire, May 8, 1954.

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 61 NUMBER 4

CONTENTS

APRIL 1954

The AMERICAN MATHEMATICAL MONTHLY

(Founded in 1894 by Benjamin F. Finkel)

Carl B. Allendoerfer, *Editor*

EDITORIAL CORRESPONDENCE should be addressed to the Editor, C. B. Allendoerfer, Department of Mathematics, University of Washington, Seattle 5, Washington.

ADVERTISING CORRESPONDENCE should be addressed to J. F. Randolph, Morey Hall, University of Rochester, Rochester 3, N. Y.

NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the Monthly should be sent to the Secretary-Treasurer, H. M. Gehman, University of Buffalo, Buffalo 14, N. Y.

Entered as second class matter at the post office at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

Subscription Price: To Members, $4 a Year, To Others, $5 a Year.

Published by the Association at Menasha, Wisconsin, and Seattle, Wash. during the months of January, February, March, April, May, June–July, August–September, October, November, December.

PRINTED IN THE UNITED STATES OF AMERICA

R. P. Dilworth

1. Ideals in Birkhoff lattices, Trans. Amer. Math. Soc. vol. 49, 1941, pp. 325–353.

O. Frink

1. Representations of Boolean algebras, Bull. Amer. Math. Soc. vol. 47, 1941, pp. 755–756.
2. Topology in lattices, Trans. Amer. Math. Soc. vol. 51, 1942, pp. 569–582.
3. Complemented modular lattices and projective spaces of infinite dimensions, Trans. Amer. Math. Soc. vol. 60, 1946, pp. 452–467.
4. A proof of the maximal chain theorem, Amer. J. of Math. vol. 74, 1952, pp. 676–678.

F. Hausdorff

1. Grundzüge der Mengenlehre, Leipzig, 1914.

V. S. Krishnan

1. The theory of homomorphisms and congruences for partially ordered sets, Proc. Indian Acad. Sci. Sect. A, 1945, pp. 1–19.
2. Extensions of partially ordered sets, J. Indian Math. Soc. vol. 11, 1947, pp. 49–68 and 89–106.
3. Extensions of multiplicative systems and modular lattices, J. Indian Math. Soc. vol. 13, 1949, pp. 49–59.
4. Une théorie générale des idéaux, C. R. Acad. Sci. Paris, vol. 231, 1950, pp. 1397–1399.
5. Les algebres partiellement ordonées et leurs extensions, Bull. Soc. Math. France, vol. 78, 1950, pp. 235–263.

H. MacNeille

1. Partially ordered sets, Trans. Amer. Math. Soc. vol. 42, 1937, pp. 416–460.

E. J. McShane

1. Partial orderings and Moore-Smith limits, this Monthly, vol. 59, 1952, pp. 1–11.

O. Ore

1. Galois connections, Trans. Amer. Math. Soc. vol. 55, 1944, pp. 493–513.

M. H. Stone

1. The theory of representations for Boolean algebras, Trans. Amer. Math. Soc. vol. 40, 1936, pp. 37–111.
2. Topological representations of distributive lattices and Brouwerian logics, Časopis pro pro Pěstování Matematiky a Fysiky, vol. 67, 1937, pp. 1–25.

H. Wallman

1. Lattices and topological spaces, Annals of Math. (2) vol. 39, 1938, pp. 112–126.

M. Zorn

1. A remark on method in transfinite algebra, Bull. Amer. Math. Soc. vol. 41, 1935, pp. 667–670.

REMARK ON MY PAPER 'ON TWO THEOREMS OF PLANE TOPOLOGY'

H. E. Vaughan, University of Illinois

In this paper (this Monthly, vol. 60, pp. 462–468) several remarks were made concerning treatments appearing in *Funktionentheorie* by Konrad Knopp. The references were to the English translation of the 4th German edition. In the subsequent 5th, 6th, and 7th German editions many corrections have been made, and my remarks do not apply to these editions.

CRITERIA FOR IRRATIONALITY OF CERTAIN CLASSES OF NUMBERS*

A. OPPENHEIM, University of Malaya, Singapore

1. Cantor [1] gave a criterion for the irrationality of a real number x given by the infinite series

$$x = a_0 + \frac{a_1}{b_1} + \frac{a_2}{b_1 b_2} + \frac{a_3}{b_1 b_2 b_3} + \cdots \tag{1}$$

where the a_i ($i \geqq 0$) and the b_i ($i \geqq 1$) are integers which satisfy the conditions

$$b_i \geqq 2 \qquad (i = 1, 2, 3, \cdots), \tag{2}$$

$$0 \leqq a_i \leqq b_i - 1 \qquad (i = 1, 2, 3, \cdots), \tag{3}$$

(4) for every integer $q \geqq 1$ there exists an integer n such that q divides $B_n = b_1 b_2 \cdots b_n$ (and therefore q divides all subsequent $B_m (m > n)$).

Cantor showed essentially that *x is irrational if and only if*

$$a_i > 0 \qquad \textit{infinitely often} \tag{5}$$

and

$$a_i < b_i - 1 \qquad \textit{infinitely often.} \tag{6}$$

I extend this theorem in two ways. First, I show that the condition (4) can be given up at the cost of a very slight additional restriction on the sequence a_n/b_n. And secondly I show that criteria for irrationality can be given when the a_i are not ultimately of one sign.

The condition (4) implies that $\overline{\lim}\, b_n = \infty$. The bounded sequence.

$$c_n = \frac{a_n}{b_n}, \qquad 0 \leqq c_n \leqq 1,$$

possesses by the Bolzano-Weierstrass Theorem at least one limit in the closed interval (0, 1). If *one* of these limits is irrational, then x is irrational if (2) and (3) hold.

If the limits are all rational but lim sup $c_n = 1$, then again x is irrational if we assume (2), (3), (6).

If lim inf $c_n = 0$ we can obtain a corresponding result at the cost of assuming that the sequence b_{n_i} for which lim $c_{n_i} = 0$ does itself tend to infinity. The precise statements will be given later.

In extending the criteria to cover negative a_i we may assume that $a_i > 0$ infinitely often and $a_i < 0$ infinitely often, for otherwise the criteria previously obtained can be applied to $-x_i$ where

* *Editorial Note*: A note by M. S. Klamkin containing some of the results in this paper was in the hands of the editors when this paper was received.

$$x_i = \frac{a_i}{b_i} + \frac{a_{i+1}}{b_i b_{i+1}} + \frac{a_{i+2}}{b_i b_{i+1} b_{i+2}} + \cdots, \tag{7}$$

or to $+x_i$ for sufficiently large i.

2. Without any restriction on the integers a_i, b_i other than that the series (1) is convergent, it is clear that the numbers

$$x, x_1, x_2, \cdots, x_i, \cdots$$

are all rational or all irrational since

$$x = a_0 + x_1, \qquad b_i x_i = a_i + x_{i+1}. \tag{8}$$

It is also clear that, if x is a rational number p/q where $q \geqq 1$, then

$$x_i = \frac{r_i}{q} \qquad (i = 1, 2, \cdots) \tag{9}$$

for some integer r_i since (for $i > 1$)

$$b_1 b_2 \cdots b_{i-2} x \equiv x_i \pmod{1}. \tag{10}$$

This proves the lemma below.

LEMMA 1. *A necessary and sufficient condition that x given by the convergent series* (1) *where the a_i are integers and the b_i are integers shall be irrational is that for every integer $q \geqq 1$ we can find an integer r and a subsequence $i_1, i_2, \cdots$ such that*

$$\frac{r}{q} < x_{i_n} < \frac{r+1}{q} \qquad (n = 1, 2, 3, \cdots). \tag{11}$$

The criteria to be given for irrationality depend simply on giving sufficient conditions for the inequalities in (11).

3. Inequalities for the x_i

LEMMA 2. *Given* (2) *and* (3) *then*

$$0 \leqq x_i \leqq 1, \qquad \frac{a_i}{b_i} \leqq x_i \leqq \frac{a_i + 1}{b_i} \qquad (i \geqq 1). \tag{12}$$

Given (5), (6) *as well, then* (12) *holds with strict inequality in each place.*

For clearly $x_i > 0$ unless each $a_n = 0$ $(n \geqq i)$: and

$$x_i \leqq \frac{b_i - 1}{b_i} + \frac{b_{i+1} - 1}{b_i b_{i+1}} + \cdots = 1,$$

with *strict* inequality unless each $a_n = b_n - 1$ $(n \geqq i)$.

The first part of (12) together with the gloss for strict inequality is proved.

As for the second part we apply the inequality just proved for x_i to x_{i+1} in the identity (8).

THEOREM 1. (Cantor) *Given* (2), (3), x *is irrational if* (4), (5), (6) *are satisfied.*

Since (4) holds, x_i is an integer for large enough i if x is rational. But since (5), (6) hold, Lemma 2 shows that $0<x_i<1$, a contradiction which proves Theorem 1.

THEOREM 2. *Given* (2), (3), x *is irrational if there exists an irrational number* ξ *and a subsequence* (i_n) *such that*

$$c_{i_n} = a_{i_n}/b_{i_n} \to \xi$$

as $n \to \infty$.

Plainly $0<\xi<1$: $b_{i_n} \to \infty$: (if b_{i_n} is bounded, c_{i_n} can take only a finite number of different rational values). Hence (5) and (6) hold. By Lemma 2

$$x_{i_n} \to \xi \qquad (n \to \infty).$$

Now for any integer $q \geqq 1$ there exists an integer r such that

$$\frac{r}{q} < \xi < \frac{r+1}{q}.$$

Hence for all $n \geqq n_0$ we shall have

$$\frac{r}{q} < x_{i_n} < \frac{r+1}{q}.$$

Lemma 1 now gives Theorem 2.

THEOREM 3. *Given* (2), (3), x *is irrational if* (6) *holds and if there is a subsequence* (i_n) *such that*

$$c_{i_n} \to 1 \qquad (n \to \infty).$$

If $b_{i_n} \leqq b$, then $c_{i_n} \leqq (b-1)/b<1$. Hence b_{i_n} is not bounded. Hence a_{i_n} is not bounded and therefore (5) holds. By Lemma 2 for $i=i_n$,

$$\frac{a_i}{b_i} < x_i < \frac{a_i+1}{b_i} \leqq 1.$$

Hencé for any given integer $q \geqq 1$ there exists n_0 such that for all $n \geqq n_0$

$$\frac{q-1}{q} < x_{i_n} < 1.$$

Lemma 1 now yields Theorem 3.

THEOREM 4. *Given* (2), (3), x *is irrational if* (5) *holds and if there is a subse-*

quence (i_n) *such that*

$$b_{i_n} \to \infty \quad \textit{and} \quad c_{i_n} \to 0 \qquad (n \to \infty).$$

The condition that $b_{i_n} \to \infty$ is needed in order to exclude $\underline{\lim}\ c_n = 0$ arising from $a_{i_n} = 0$ for a bounded set b_{i_n}

Since $b_{i_n} \to \infty$ and $c_{i_n} \to 0$ it follows that $a_i < b_i - 1$ infinitely often. Thus (6) holds. By Lemma 2 for $i = i_n$

$$0 \leqq \frac{a_i}{b_i} < x_i < \frac{a_i}{b_i} + \frac{1}{b_i}\cdot$$

For a given integer q, n_0 exists such that

$$0 < x_{i_n} < \frac{1}{q}$$

for all $n \geqq n_0$. Lemma 1 now gives Theorem 4.

4. We consider now the case when a_i may have either sign. We assume that

$$|a_i| \leqq b_i - 1 \qquad (i = 1, 2, 3, \cdots), \tag{13}$$

and that

$$a_m a_n < 0 \tag{14}$$

for some $m > i$, $n > i$ when i is any assigned integer.

Lemma 2 is now replaced by Lemmas 3 and 4.

LEMMA 3. *Given* (2), (13), (14), *then*

$$-1 < x_i < 1, \qquad \frac{a_i - 1}{b_i} < x_i < \frac{a_i + 1}{b_i} \qquad (i = 1, 2, \cdots).$$

The proof is identical with that for Lemma 2 since a_n must change sign infinitely often.

LEMMA 4. *No* x_i *can vanish if* (2), (13), (14) *hold. If* $a_i \neq 0$ *then* x_i *has the sign of* a_i. *If* $a_i = 0$, $a_{i+1} = 0, \cdots, a_{i+m-1} = 0$, $a_{i+m} \neq 0$ $(1 \leqq m < \infty)$, *then* x_i *has the sign of* a_{i+m}.

If $a_i \geqq 1$, Lemma 3 yields $x_i > 0$. If $a_i \leqq -1$, then $x_i < 0$.

If $a_i = 0$, then a_{i+m} is the first non-zero a_j following a_i (existent by (14)) and

$$\begin{aligned} x_i &= \frac{a_{i+m}}{b_i b_{i+1} \cdots b_{i+m}} + \frac{a_{i+m+1}}{b_i \cdots b_{i+m+1}} + \cdots, \\ x_i &= \frac{x_{i+m}}{b_i b_{i+1} \cdots b_{i+m-1}}, \end{aligned} \tag{15}$$

and the result follows by applying the first part of the proof to x_{i+m}.

THEOREM 5. *Given* (2), (13), (14), *x is irrational if* (4) *holds.*

For if x is rational and (4) holds then x_i is an integer for all large enough i. But Lemmas 3 and 4 give

$$-1 < x_i < 1, \qquad x_i \neq 0 \qquad (i = 1, 2, \cdots),$$

a contradiction which proves Theorem 5.

THEOREM 6. *Given* (2), (13), *x is irrational if there exists a subsequence (i_n) and an irrational number θ such that*

$$c_{i_n} = a_{i_n}/b_{i_n} \to \theta \qquad (n \to \infty).$$

Here $-1<\theta<1, \theta\neq 0$. The proof is essentially that of Theorem 2.

THEOREM 7. *Given* (2), (13), (14), *x is irrational if there exists a subsequence (i_n) such that*

$$c_{i_n} \to 1(-1) \qquad (n \to \infty).$$

The proof follows the same lines as that of Theorem 3.

THEOREM 8. *Given* (2), (13), (14), *x is irrational if there exists a subsequence (i_n) such that*

$$b_{i_n} \to \infty \quad \text{and} \quad c_{i_n} \to 0 \qquad (n \to \infty).$$

By Lemmas 3 and 4 and (15),

$$x_{i_n} \to 0 \quad \text{and} \quad x_{i_n} \neq 0.$$

Hence for any given integer $q \geqq 1$ there exists an integer n_0 such that for all $n \geqq n_0$,

$$0 < |x_{i_n}| < \frac{1}{q}$$

and hence there is a subsequence tending to infinity such that

$$0 < x_{i_n} < \frac{1}{q} \quad \text{or} \quad -\frac{1}{q} < x_{i_n} < 0.$$

Lemma 1 now proves Theorem 8.

5. As an example we have

THEOREM 9: *The number*

$$x = \sum_{n=1}^{\infty} \frac{a_n}{r^n (n!)^b} \tag{16}$$

is irrational if $r \geqq 1$, $b \geqq 1$ are fixed integers, a_n is an integer such that

$$|a_n| \leqq rn^b - 1, \qquad (n \geqq n_0), \tag{17}$$

and a_n differs from each of $0, rn^b-1, -(rn^b-1)$ *infinitely often.*

Plainly this result follows from Theorem 5 or Theorem 1 if the a_i are ultimately of one sign. Spiegel [2] proved that x in (16) is irrational with a more stringent condition on a_n in place of (17) which amounts to

$$|a_n| < n^{b-\epsilon} \qquad (\text{some } \epsilon > 0,\ n > n_1(\epsilon))$$

and of course $a_n \neq 0$ infinitely often.

As another example the numbers

$$\sum_1^\infty \frac{\epsilon_n d(n)}{n!}, \qquad \sum_1^\infty \frac{\epsilon_n \phi(n)}{n!}$$

are irrational where $\epsilon_n = \pm 1$, $d(n)$ is the number of divisors of n and $\phi(n)$ is Euler's function.

An example of Theorem 8 is afforded by

$$\sum_{n=1}^\infty \frac{\epsilon_n d(b_n)}{b_1 b_2 \cdots b_n}$$

if $\lim b_n = \infty$, since $d(m) = O(m^\epsilon)$ for every positive ϵ so that $d(b_n)/b_n \to 0$. If the ϵ_n are ultimately all of one sign Theorem 4 must be used.

6. I give now another theorem which is not included in the preceding theorems and allows a_i to be of higher magnitude than b_i. The notation

$$c_n \to \xi \ (\text{mod } 1) \qquad (n \to \infty)$$

means that the fractional part of c_n tends to ξ as $n \to \infty$.

So $x_m = o(b_m)$ (mod b_m) will mean that the fractional part of x_m/b_m tends to 0 as $m \to \infty$.

THEOREM 10. *Suppose that* (i) $b_i \geq 1$, a_i *are integers such that the series*

$$x_1 = \sum_1^\infty a_i/b_1 \cdots b_i$$

is convergent, (ii) $c_i = a_i/b_i \to \xi$ (mod 1) *as $i \to \infty$ through some subsequence* (i_n) *where ξ is irrational,* (iii) $x_{i+1} = o(b_i)$ (mod b_i) *for this subsequence, then x_1 is irrational.*

It may be noted that ξ in (ii) may be rational provided we assume further that $x_i \not\equiv \xi$ (mod 1) for the members of the subsequence (i_n).

We have, when i belongs to (i_n),

$$\begin{aligned} x_i &= \frac{a_i}{b_i} + \frac{x_{i+1}}{b_i} \\ &= N_i + \xi + \epsilon_i + \delta_i \end{aligned}$$

where N_i is an integer and ϵ_i, $\delta_i \to 0$ as $i \to \infty$ in (i_n).

Now for any integer $q \geqq 1$ there exists an integer r such that

$$\frac{r}{q} < \xi < \frac{r+1}{q}.$$

We now choose n_0 so large that for $i = i_n \geqq i_{n_0}$

$$\frac{r}{q} < \xi + \epsilon_i + \delta_i < \frac{r+1}{q}.$$

It follows that x_{i_n} cannot be a rational number with denominator q. Hence x is irrational. And this proves Theorem 10. The modification required when ξ is rational and $x_i \not\equiv \xi \pmod 1$ is obvious.

THEOREM 11. *The number x of Theorem* 10 *is irrational if we assume* (i), (4), *and that, for i in some subsequence* (i_n),

$$0 < x_{i_n} < 1 \pmod 1.$$

The proof proceeds as before.

As an example, *the number*

$$\sum_1^\infty \frac{\epsilon_n \sigma(n)}{n!}$$

is irrational for any choice of $\epsilon_n = \pm 1$ where $\sigma(n)$ denotes the sum of the divisors of n. For plainly

$$\sigma(n) = n \sum_{d|n} 1/d = O(n \log n)$$

so that trivially

$$x_n = O(n^{1/2}).$$

Take $n = 3^s$ so that

$$\frac{|a_n|}{b_n} = \frac{\sigma(3^s)}{3^s} = \frac{1}{2} - \frac{1}{2 \cdot 3^s},$$

$$x_n = \frac{a_n}{b_n} + \frac{x_{n+1}}{b_n} \to \frac{1}{2} \pmod 1$$

as $n = 3^s \to \infty$. Theorem 11 applies.

References

1. G. Cantor, Über die einfachen Zahlensysteme, *Zeit. für Math.* vol. 14, 1869, pp. 121–128; *Coll. Papers*, Berlin, 1932, pp. 35–42.

2. M. R. Spiegel, On a class of irrational numbers, this MONTHLY, vol. 60, 1953, pp. 27–28.

THE CONFERENCE ON TRAINING IN APPLIED MATHEMATICS

F. E. HOHN, Bell Telephone Laboratories and the University of Illinois

I. The Nature of the Conference

This conference, which was held at Columbia University, New York, October 22–24, 1953, was sponsored by the American Mathematical Society and the National Research Council under a contract with the National Science Foundation. The details were arranged by Dr. F. J. Weyl of the National Research Council's Committee on Applied Mathematics. Invited addresses by leaders from a wide variety of educational, industrial, and governmental organizations were presented and discussed. Apart from describing the specific natures of their own organizations, the speakers treated various relevant general issues. Their opinions showed remarkable uniformity. In fact, many of the ideas recorded below were stated in essentially the same form by a number of the speakers and participants in the discussions. The concern of the speakers was primarily with training at the Ph.D. level, although training at lower levels was also discussed. The principles summarized below are appropriate to either case.

Complete proceedings of this Conference are being submitted to the National Science Foundation as a report under the contract mentioned above. Inquiries regarding the availability of copies should be sent to L. W. Cohen, Program Director for Mathematics, National Science Foundation, Washington, D.C., prior to April 30, 1954. Final arrangements for distribution will depend upon the anticipated demand.

II. Reasons for the Current Interest in Applied Mathematics

It was admitted that an individual's interest in "applied mathematics" is sometimes an artificial or a temporary one, inspired primarily by a desire to "get in on the gravy train." However, by and large, the widespread, current interest seems to be due to other, more constructive factors such as the following:

1. A growing realization that there are vital, challenging problems to be solved in the various fields where mathematics may be applied. These problems arise from the increased complexity of modern technology, a complexity resulting from its increasing concern with "systems" rather than "mechanisms."
2. An increased appreciation in industrial and governmental research agencies of the usefulness of fundamental mathematical approaches and of specific, advanced mathematical tools.
3. Improved opportunities for permanent employment resulting from (2).
4. A renewed understanding of the facts that the psychological and historical origins of mathematics are in the "real world," that the postulational approach is not the beginning of mathematics but is rather a final

step in its development, and that therefore mathematics stands always to learn and grow through its applications.

5. A realization that applications of mathematics today often demand the best that a mathematician has to offer rather than calling only on inferior abilities, so that the role of applied mathematician is once again to be regarded as a dignified, fascinating, creditable activity for a scholar.

III. Pure vs. Applied Mathematics

The consensus was that mathematics cannot be separated precisely or usefully into the categories "pure" and "applied," that all mathematics, as such, is "pure." When used as tools in the investigation of problems related to the real world, the techniques of pure mathematics become "applied" mathematics. So many mathematical disciplines have ultimately been applied in this way that a distinction on the basis of what has been applied and what has not would change almost from day to day and hence could only be artificial.

A distinction that *can* be drawn is one between pure and applied mathematicians. The difference here lies neither in the mathematical techniques used, nor in the level of creative ability. It is rather a difference based on interests, attitudes, and special aptitudes.

IV. The Characteristics of an Applied Mathematician

As identified by those who have had experience in employing or training mathematicians for work with applications, the important characteristics of an (ideal) applied mathematician include the following:

1. He is first and foremost a scientist devoted to high scientific standards.
2. He possesses a high degree of independent creative ability and drive rather than being a mere technician acting primarily on orders.
3. He has the ability and common sense necessary for developing useful mathematical abstractions or models of the problems with which he must work.
4. He has had a broad training in the fundamental methods and processes of mathematical and scientific thought so that he is equipped to cope with the problems of today as well as to adapt himself to the needs and problems of tomorrow.
5. He possesses many easily aroused interests.
6. He has a flair for oral and written expression.
7. He enjoys cooperative research, such as work with a team of specialists.
8. He has the maturity, the superior ability, and the personality to inspire confidence in his advice and conclusions.
9. He should have a temperament which permits him, when it is necessary, to be satisfied with the approximate rather than the ideal, and to work immediately with the problems that need to be solved, rather than to work only when and as he sees fit with only those problems that interest him particularly.

V. The Training of Applied Mathematicians

The characteristics of an applied mathematician just listed have, of course, their implications for training. Programs for training applied mathematicians will naturally vary from one institution to another, according to the particular facilities available and according to the specific needs a given program is designed to meet. The following characteristics, however, seem to belong to any good program:

1. The student should receive a broad, thorough training in fundamental mathematics so that he learns what a mathematical problem is and has available the proper tools with which to solve it.
2. The student's training should include probability and statistics because of their specific usefulness and because of the attitudes toward approximate data and experimentation characteristic of the latter. (The intent here was not to ascribe any transcendent importance to probability and statistics, but rather to emphasize that they should not be neglected.)
3. The program should include a broad training in other sciences than mathematics, in order to introduce the student to the languages, problems, and methods of areas in which mathematics is applied.
4. Experience with significant applications should be introduced early in the program in order to create high initial interest.
5. Interest should be maintained by competent instruction and by glimpses of the rewards and of the stimulating nature of a career in applied mathematics.

One of the great problems of the universities is to find enough able and inspiring teachers of potential applied mathematicians. This is a critical point since often a promising student is turned from his interests in the applications of mathematics by the influence of teachers devoted to mathematics as a pure science but holding its applications in disdain. Increased respect and recognition is in order for those who are gifted in the training of applied mathematicians.

Another difficulty is the fact that advanced training of applied mathematicians often requires a longer than average period of study, because of both subject-matter and maturity requirements. The resulting financial needs of students must be recognized and met. Meeting them could be expected to increase the number of students in the area, of whom there are at present altogether too few.

VI. Responsibilities of Government and Industry

Extensive demand for applied mathematicians in government and industry has developed only during the last ten or twelve years. If the demand continues, interest will surely increase, so that the gap between supply and demand

will be narrowed. However there are specific ways in which government and industry can help to close the gap and to improve the quality of the product:

1. The interest of able people can be aroused by assuring prospective mathematical employees a challenging and genuinely scientific atmosphere in which to work.
2. The mathematician should be given reasonable opportunity to pursue significant mathematical investigations which may be suggested by the applications with which he works, whether or not these investigations seem to have any immediately obvious practical value.
3. Competent teachers should be given inspiration and recognition by temporary but broadening and remunerative positions in government and industry.
4. Competent students should be given inspiration and support by means of subsidies, fellowships, contract work and well-paid summer employment.

Fortunately, a number of progressive organizations have already recognized the importance of these four factors and have acted accordingly. It is hoped that others will follow suit.

VII. Conclusions

The conference left the clear impression that fundamental problems and responsibilities are well-understood by many mathematical leaders in industry, government, and education. One would expect that as their influence makes itself felt, and as cooperation among concerned groups develops, the applied mathematician will increasingly assume his proper role in the scientific life of our time.

THE PRINCIPAL TERM IN THE ASYMPTOTIC EXPANSION OF THE LEBESGUE CONSTANTS*

LEE LORCH, Fisk University

It is well known that there exist continuous periodic functions whose Fourier series diverge at a given point. This was first established in 1876 by du Bois-Reymond [1] by the explicit construction of such a function. Later authors, particularly Fejér [2], simplified his construction and furnished simpler examples.

Lebesgue [7, p. 86] introduced a different approach. He showed that the

* This work was done with the assistance of a grant from the Research Corporation.

divergence of a certain sequence of constants (1), since known by his name, implies the existence of such a function and also that of another continuous function whose Fourier series converges everywhere but not uniformly in the neighborhood of a preassigned point [7, p. 88]. The nth Lebesgue constant is the integral of the absolute value of the Dirichlet kernel which appears in the familiar integral representation of the partial sums of a Fourier series. It may be written as follows:

$$L_n = \frac{2}{\pi}\int_0^{\pi/2} \frac{|\sin(2n+1)t|}{\sin t}\, dt, \qquad n = 1, 2, \cdots. \tag{1}$$

If the Dirichlet kernel is replaced in (1) by its transform under a given summation method, then the non-boundedness of the resulting integral is necessary and sufficient for the existence of (i) a continuous function whose Fourier series is non-summable by the method in question at a given point, and (ii) another whose Fourier series is summable everywhere, but not uniformly in the neighborhood of a prescribed point. This approach has been used frequently in the study of summability questions.

Lebesgue's method is also applied to the study of similar convergence and summability questions arising in the study of developments other than Fourier series, but the corresponding constants will not be considered here.

The divergence of the sequence $\{L_n\}$, all that was necessary for his purposes, was verified by Lebesgue and more precise results were given by quite a few later authors. References to much of the literature are in [5] and [8].

The principal term and the constant term in the asymptotic expansion of L_n as a function of n were found first by Fejér [2]. Short derivations of the principal term, (6) below, are given, for example, in [6, p. 52] and [11, p. 172]. Szegö [10] has provided a particularly elegant analysis of $\{L_n\}$, showing, *e.g.*, that it is completely monotonic.

The purpose of this note is to present a new elementary derivation of (6), along different lines, together with some allied results. The main idea is first to replace $\sin t$, or, more generally, the periodic function $g(t)$, by its average value and then to show that the resulting error is bounded. In greater detail and with more apparatus, this approach has been used [8] to derive more precise results.

The desired formulae are all corollaries of the following theorem:

THEOREM. *Let $g(t)$ be an integrable function of period p such that*

$$\int_0^1 \frac{g(t)}{t}\, dt \tag{2}$$

exists in some sense. Then, for $0 < \tau(\xi) = o(\xi)$ $(\xi \to \infty)$,

$$L_\xi(g) \equiv \int_0^{1/\tau(\xi)} \frac{g(\xi t)}{t}\, dt = m \log\left\{\frac{\xi}{\tau(\xi)}\right\} + O(1), \tag{3}$$

where

$$m = \frac{1}{p}\int_0^p g(t)dt.$$

In the proof of the theorem the following result is used:

$$\int_1^{\xi/\tau(\xi)} \frac{1}{t}\{m - g(t)\}dt = O(1), \qquad (\xi \to \infty). \tag{4}$$

To prove the latter, the second mean-value theorem is applied, yielding

$$\int_1^{\xi/\tau(\xi)} \frac{1}{t}\{m - g(t)\}dt = \int_1^{\eta}\{m - g(t)\}dt, \qquad 1 < \eta < \frac{\xi}{\tau(\xi)}.$$

The right-hand integral is bounded for all η, since m is the mean value of the periodic function $g(t)$ over its complete period. Thus, (4) is proved.

Proof of theorem.

$$\begin{aligned}\int_0^{1/\tau(\xi)} \frac{g(\xi t)}{t}dt &= \int_0^{\xi/\tau(\xi)} \frac{g(t)}{t}dt = \int_1^{\xi/\tau(\xi)} \frac{g(t)}{t}dt + \int_0^1 \frac{g(t)}{t}dt \\ &= \int_1^{\xi/\tau(\xi)} \frac{g(t)}{t}dt + O(1) \\ &= \int_1^{\xi/\tau(\xi)} \frac{m}{t}dt - \int_1^{\xi/\tau(\xi)} \frac{1}{t}\{m - g(t)\}dt + O(1) \\ &= m\log\left\{\frac{\xi}{\tau(\xi)}\right\} + O(1),\end{aligned}$$

which completes the proof.

COROLLARY 1. *If, further, $g(t)$ is bounded, then*

$$\int_0^{\pi/2} \frac{g(\xi t)}{\sin t}dt = m\log\xi + O(1), \qquad (\xi \to \infty). \tag{5}$$

Proof. Put $\tau(\xi) = \frac{1}{2}\pi$ in (3) and then consider the difference between the integrals in (3) and (5). The conclusion then follows from the boundedness of $g(t)$ and of

$$\frac{1}{\sin t} - \frac{1}{t}, \qquad 0 \leqq t \leqq \tfrac{1}{2}\pi.$$

COROLLARY 2 (The Lebesgue Constants).

$$L_n = \frac{4}{\pi^2}\log n + O(1), \qquad (n \to \infty). \tag{6}$$

Proof. Here $g(t)=(2/\pi)|\sin t|$, $\xi=2n+1$, $p=\pi$, and

$$m = \frac{1}{\pi}\int_0^\pi \frac{2}{\pi}|\sin t|\,dt = \frac{4}{\pi^2}\cdot$$

Replacing $|\sin(2n+1)t|$ by $\sin^2(2n+1)t$ in the sequence $\{L_n\}$ gives a sequence $\{F_n\}$, introduced by Fejér [2], who pointed out that its unboundedness has the same consequences as does that of the sequence $\{L_n\}$.

In this instance, $g(t)=(2/\pi)\sin^2 t$, $\xi=2n+1$, $p=\pi$ and

$$m = \frac{1}{\pi}\int_0^\pi \frac{2}{\pi}\sin^2 t dt = \frac{1}{\pi}\cdot$$

This gives

COROLLARY 3 (The Fejér Constants).

$$F_n = \frac{2}{\pi}\int_0^{\pi/2}\frac{\sin^2(2n+1)t}{\sin t}dt = \frac{1}{\pi}\log n + O(1), \qquad (n\to\infty). \tag{7}$$

Gronwall [3] made a complete and very simple analysis of Fejér's constants.

Finally, Hardy [4] observed that the divergence of the function

$$F_B(\xi) = \frac{2}{\pi}\int_0^{\xi^{-1/2}}\frac{\sin^2\xi t}{t}dt, \qquad (\xi\to\infty)$$

implies the existence of a continuous function whose Fourier series is not even Borel summable at a given point. The same is true of

$$L_B(\xi) = \frac{2}{\pi}\int_0^{\xi^{-1/2}}\frac{|\sin\xi t|}{t}dt, \qquad (\xi\to\infty).$$

Taking into account that $\tau(\xi)$ now equals $\xi^{\frac{1}{2}}$, the theorem gives the following additional relations:

COROLLARY 4 (Borel Summability).

$$F_B(\xi) = \frac{1}{2\pi}\log\xi + O(1), \qquad (\xi\to\infty), \tag{8}$$

and

$$L_B(\xi) = \frac{2}{\pi^2}\log\xi + O(1), \qquad (\xi\to\infty). \tag{9}$$

Strictly speaking, $F_B(\xi)$ and $L_B(\xi)$ are not the Fejér and Lebesgue constants, respectively, for Borel's methods, but, rather, for a method equivalent to his for the Fourier series of functions for which

$$f(x+t)+f(x-t)-2s\to 0, \qquad (t\to 0),$$

in particular, of continuous functions. In the light of Hardy's observation (above), Corollary 4 establishes the existence of a continuous function whose Fourier series is not Borel summable everywhere, a phenomenon discovered first by Moore [9].

References

1. P. du Bois-Reymond, Untersuchungen über die Convergenz und Divergenz der Fourier'schen Darstellungsformeln, Abhandl. Akad. d. Wissens. München, vol. 12, 1876, pp. 1–103.
2. L. Fejér, Lebesguesche Konstanten und divergente Fourierreihen, J. f. d. reine u. angew. Math., vol. 138, 1910, pp. 22–53.
3. T. H. Gronwall, Über die Lebesgueschen Konstanten bei den Fourierschen Reihen, Math. Ann., vol. 72, 1912, pp. 244–261.
4. G. H. Hardy, Remarks on some points in the theory of divergent series, Annals of Math., series 2, vol. 36, 1935, pp. 167–182.
5. G. H. Hardy, Note on Lebesgue's constants in the theory of Fourier series, J. London Math. Soc., vol. 17, 1942, pp. 4–13.
6. G. H. Hardy and W. W. Rogosinski, Fourier Series, Cambridge Tracts in Mathematics and Mathematical Physics, number 38, Cambridge, 1944.
7. H. Lebesgue, Leçons sur les Séries Trigonométriques, Paris, 1906.
8. L. Lorch, Asymptotic expressions for some integrals which include certain Lebesgue and Fejér constants, Duke Math. J., vol. 20, 1953, pp. 89–103.
9. C. N. Moore, On the application of Borel's method to the summation of Fourier's series, Proc. Natl. Acad. Sci., vol. 11, 1925, pp. 284–287.
10. G. Szegö, Über die Lebesgueschen Konstanten bei den Fourierschen Reihen, Math. Zeits., vol. 9, 1921, pp. 163–166.
11. A. Zygmund, Trigonometrical Series, Warsaw, 1935.

MATHEMATICAL NOTES

EDITED BY F. A. FICKEN, University of Tennessee

Material for this department should be sent to F. A. Ficken, University of Tennessee, Knoxville 16, Tenn.

REMARK ON ORTHONORMAL SETS IN $\mathfrak{L}_2(a, b)$

EDWIN HEWITT, University of Washington

H. P. McKean has recently given a short proof that the Hermite functions form a complete set of orthonormal functions in $\mathfrak{L}_2(-\infty, \infty)$ [this MONTHLY, vol. 59, 1952, pp. 621–622.]. His proof, however, uses a decidedly non-elementary theorem due to N. Wiener. We give here a brief and reasonably elementary proof of a theorem which implies the completeness of the Hermite functions and the Laguerre functions as special cases.

THEOREM. *Let $-\infty \leqq a < b \leqq +\infty$. Let $p(x) \in \mathfrak{L}_2(a, b)$ be different from zero almost everywhere and also be $O(e^{-\alpha|x|})$ for some $\alpha > 0$ as $|x| \to \infty$ (if $a = -\infty$ or*

$b=+\infty$). *Then the functions* $\phi_1, \phi_2, \cdots, \phi_n, \cdots$ *formed from the linearly independent set* $\{x^n p(x)\}_{n=0}^{\infty}$ *by the Gram-Schmidt orthonormalization process are a complete set in* $\mathfrak{L}_2(a, b)$.

Let $z=u+iv$ be a complex variable and let f be an arbitrary element of $\mathfrak{L}_2(a, b)$. Let $F(z)=\int_a^b e^{izx}p(x)\overline{f(x)}dx$. For $-\infty<a<b<+\infty$, it is evident that $F(z)$ is an entire function and that

$$F^{(n)}(z) = i^n \int_a^b e^{izx}x^n p(x)\overline{f(x)}dx \qquad (n = 0, 1, 2, 3, \cdots). \tag{1}$$

For $a=-\infty$ or $b=+\infty$, a simple computation, which we omit, shows that $F(z)$ is analytic in the strip $-\alpha<v<\alpha$, and that $F^{(n)}(z)$ has the representation (1). Now, if

$$\int_a^b x^n p(x)\overline{f(x)}dx=0 \quad (n=0, 1, \cdots),$$

it follows that $F^{(n)}(0)=0$ $(n=0, 1, 2, \cdots)$. This implies that $F(z)=0$ identically in its domain of definition. Hence in particular $F(u)=\int_a^b e^{iux}p(x)\overline{f(x)}dx=0$ for $-\infty<u<\infty$. The function $p(x)\overline{f(x)}$ is thus a function in $\mathfrak{L}_1(a, b)$ whose Fourier transform vanishes identically. By the classical uniqueness theorem for Fourier transforms, $p(x)\overline{f(x)}=0$ almost everywhere, and since $p(x)$ is different from 0 almost everywhere, $\bar{f}$ and f are 0 almost everywhere.

Taking $a=-\infty$, $b=+\infty$, and $p(x)=e^{-x^2/2}$, we see that the Hermite functions are a complete orthonormal set. Taking $a=0$, $b=+\infty$, and $p(x)=e^{-x/2}$, we see that the Laguerre functions form a complete orthonormal set.

A NECESSARY CONDITION FOR THE CONVERGENCE OF $\int_a^\infty f(x)dx$

A. E. LIVINGSTON,* The Institute for Advanced Study

Let $\log^{(0)} x=x$ and $\log^{(k)} x=\log [\log^{(k-1)} x]$ for $k=1, 2, \cdots$. Consider the following statement:

$P(n)$: *If* $f(x)\prod_0^{n-1} \log^{(k)} x$ *is non-increasing in a neighborhood of infinity and if* $\int_a^\infty f(x)dx$ *is convergent, then* $\lim_{x\to\infty} f(x)\prod_0^n \log^{(k)} x=0$.

That $P(0)$ is true is well-known, and M. J. Norris has shown (*Some necessary conditions for convergence of infinite series and improper integrals*, this MONTHLY, vol. 60, 1953, pp. 96–97) that $P(n)$ is true for $n=1, 2, \cdots$. He then used this fact to obtain the corresponding result for infinite series.

It is the purpose of this note to prove the following

THEOREM. *Let* $G(x)=\int_A^x [g(t)]^{-1}dt$ *be strictly increasing to* $+\infty$. *If* $f(x)g(x)$ *is non-increasing in a neighborhood of infinity and if* $\int_a^\infty f(x)dx$ *is convergent, then* $\lim_{x\to\infty}f(x)g(x)G(x)=0$.

Proof: It is clearly no restriction to assume that all neighborhoods of in-

* The author is a National Science Foundation Fellow.

finity involved are the interval $a<x<\infty$. Note that the hypothesis that $G(x)$ be strictly increasing in a neighborhood of infinity is nothing more than the requirement that $g(x)$ be positive and finite in this neighborhood.

Let $H(y)$ be the function inverse to $G(x)$: $H[G(x)]=x$ and $G[H(y)]=y$. Then

$$\int_a^{\infty} f(x)dx = \int_{G(a)}^{\infty} f[H(y)]g[H(y)]dy,$$

since $H(y)$ is strictly increasing to $+\infty$. By hypothesis, $f(t)g(t)$ is a non-increasing function of t. Since $H(y)$ increases with y, it follows that $f[H(y)]g[H(y)]$ is a non-increasing function of y. Hence, by $P(0)$, $\lim_{y\to\infty} f[H(y)]g[H(y)]y=0$. The proof is completed by setting $y=G(x)$.

There is, of course, a corresponding result for infinite series.

Norris's result follows from the above theorem by taking $g(x)=\prod_0^{n-1}\log^{(k)} x$ and $A=\exp_{n-1} 1$, where $\exp_0 x=x$ and $\exp_k x=\exp[\exp_{k-1} x]$.

Specializing $g(x)$ to be $[2\log^{(n)} x]^{-1}\prod_0^{n-1}\log^{(k)} x$, $n\geqq 1$, and A to be $\exp_{n-1} 1$ gives $G(x)=[\log^{(n)} x]^2$. Hence, if $f(x)[\log^{(n)} x]^{-1}\prod_0^{n-1}\log^{(k)} x$ is nonincreasing and $\int_a^\infty f(x)dx$ is convergent, then $\lim_{x\to\infty} f(x)\prod_0^n \log^{(k)} x=0$. The conclusion is that of $P(n)$, but the hypothesis is considerably weaker.

A NOTE ON A PAPER OF GROSSWALD

H. W. Gould, Portsmouth, Virginia

In his paper [1] through use of Legendre Polynomials and the Hypergeometric Function, Grosswald has shown that the following sum is valid:

$$\begin{aligned}\sum_{k=0}^{n-r}(-1)^k\binom{n-r}{k}\binom{n+k+r}{n}\frac{1}{2^k}\\ =(-1)^{\frac{1}{2}(n-r)}2^{r-n}\left\{\begin{matrix} n \\ \frac{n+r}{2}\end{matrix}\right\}\binom{n+r}{n}\frac{1}{\binom{n}{r}} \qquad \text{where } n-r=2m.\end{aligned} \tag{1}$$

Written in this form the series may appear slightly different than the same result in Grosswald's paper, but with minor changes of notation the series are identical. The reason for the present notation is that we shall evaluate the sum in question by a third method. Carlitz in his note [2], has given a second method of finding (1). But both these methods involve use of the Hypergeometric Function, which is not needed, for the series (1) may be found quite simply from the definition of the Legendre Polynomials alone. This also allows us to introduce some remarks about standard forms for this important polynomial. Most textbooks on the subject mention one or two, but seldom more standard forms of $P_n(x)$. We shall develop one of these which is neglected by many writers but which has been the inspiration of many discoveries of supposedly new summations, some of which are in fact new to the literature.

Therefore the writer feels they should receive more attention. (Note Rainville's treatment [3].)

Beginning with the definition of $P_n(x)$ by the relation

$$P_n(x) = \frac{1}{2^n n!} D_x^n (x^2 - 1)^n, \tag{2}$$

we have the usual result:

$$\begin{aligned} P(x) &= \frac{1}{2^n n!} D_x^n \sum_{k=0}^{n} (-1)^k \binom{n}{k} x^{2n-2k} \\ &= \frac{1}{2^n} \sum_{k=0}^{[n/2]} (-1)^k \binom{n}{k} \binom{2n-2k}{n} x^{n-2k}. \end{aligned}$$

Hence

$$D_x^r P_n(x) = \frac{r!}{2^n} \sum_{k=0}^{[n/2]} (-1)^k \binom{n}{k} \binom{2n-2k}{n} \binom{n-2k}{r} x^{n-2k-r},$$

and

$$\begin{aligned} D_x^r P_n(x) \big]_{x=0} &= r! \frac{(-1)^{(n-r)/2}}{2^n} \binom{n}{\frac{n+r}{2}} \binom{n+r}{n} \qquad \text{for } n \equiv r \pmod 2, \\ &= 0 \text{ otherwise.} \end{aligned} \tag{3}$$

But now, from (2) we may also write the following:

$$\begin{aligned} P_n(x) &= \frac{1}{2^n n!} D_x^n \{ (x-1)^n (x+1)^n \} \\ &= \frac{1}{2^n n!} \sum_{k=0}^{n} \binom{n}{k} D_x^k (x-1)^n D_x^{n-k} (x+1)^n \\ &= \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k}^2 (x-1)^{n-k} (x+1)^k \\ &= \left(\frac{x-1}{2}\right)^n \cdot \sum_{k=0}^{n} \binom{n}{k}^2 \left(\frac{x+1}{x-1}\right)^k. \end{aligned}$$

Now this last sum may be found quite easily, in fact reference to a problem in the Elementary Problem Section will show that it has been the subject of much inquiry before. In problem E 799 (1948, p. 30) it was brought out that we have in fact

$$\sum_{k=0}^{n} \binom{n}{k}^2 x^{n-k} = \sum_{k=0}^{n} \binom{n}{k} \binom{2n-k}{n} (x-1)^k \tag{4}$$

which is a polynomial identity in x, each side being of degree n. From this we find therefore:

$$P_n(x) = \sum_{k=0}^{n} \binom{n}{k} \binom{2n-k}{n} \left(\frac{x-1}{2}\right)^{n-k}, \tag{5}$$

and this will be the third form we touch on in this note. It follows immediately from this relation that:

$$\begin{aligned} D_x^r P_n(x) &= \sum_{k=0}^{n} \binom{n}{k}\binom{2n-k}{n} \frac{1}{2^{n-k}} D_x^r (x-1)^{n-k} \\ &= r! \sum_{k=0}^{n} \binom{n}{k}\binom{2n-k}{n} \frac{1}{2^{n-k}} \sum_{\alpha=0}^{n-k} (-1)^{n-k-\alpha} \binom{n-k}{\alpha}\binom{\alpha}{r} x^{\alpha-r} \\ &= r! \sum_{\alpha=0}^{n} (-1)^{\alpha} \binom{\alpha}{r} x^{\alpha-r} \sum_{k=\alpha}^{n} (-1)^k \binom{n}{k}\binom{n+k}{n}\binom{k}{\alpha}\frac{1}{2^k}. \end{aligned}$$

We now only have to let $x=0$, obtaining

$$D_x^r P_n(x)\big]_{x=0} = r!(-1)^r \sum_{k=r}^{n} (-1)^k \binom{n}{k}\binom{n+k}{n}\binom{k}{r}\frac{1}{2^k}.$$

Since

$$\binom{n}{k}\binom{k}{r} = \binom{n}{r}\binom{n-r}{k-r},$$

we have

$$\begin{aligned} D_x^r P_n(x)\big]_{x=0} &= r!(-1)^r \sum_{k=r}^{n} \binom{n}{r}\binom{n-r}{k-r}\binom{n+k}{n}\frac{(-1)^k}{2^k} \\ &= r!(-1)^r \binom{n}{r} \sum_{k=0}^{n-r} \binom{n-r}{k}\binom{n+k+r}{n}\frac{(-1)^{k+r}}{2^{k+r}} \\ &= \frac{r!}{2^r}\binom{n}{r}\sum_{k=0}^{n-r}(-1)^k\binom{n-r}{k}\binom{n+k+r}{n}\frac{1}{2^k}. \end{aligned} \tag{6}$$

But the series here is exactly (1) which we were looking for. Equating (6) to the equivalent form of the derivative evaluated at x equal zero as given by relation (3) yields exactly the desired result. Therefore the series (1) results directly from one of the normal forms of the polynomials $P_n(x)$.

References

1. Emil Grosswald, On sums involving binomial coefficients, this MONTHLY, vol. 60, 1953, pp. 179–181.

2. L. Carlitz, Note on a formula of Grosswald, this MONTHLY, vol. 60, 1953, p. 181.

3. E. D. Rainville, Symbolic relations among classical polynomials, this MONTHLY, vol. 53, 1946, p. 299.

CLASSROOM NOTES

EDITED BY G. B. THOMAS, Massachusetts Institute of Technology

All material for this department should be sent to G. B. Thomas, Department of Mathematics, Massachusetts Institute of Technology, Cambridge 39, Mass.

ON THE COMPUTATION OF DEFINITE INTEGRALS

L. M. WEINER, DePaul University

A disturbing problem which confronts the integral calculus student in the use of the fundamental theorem of integral calculus,

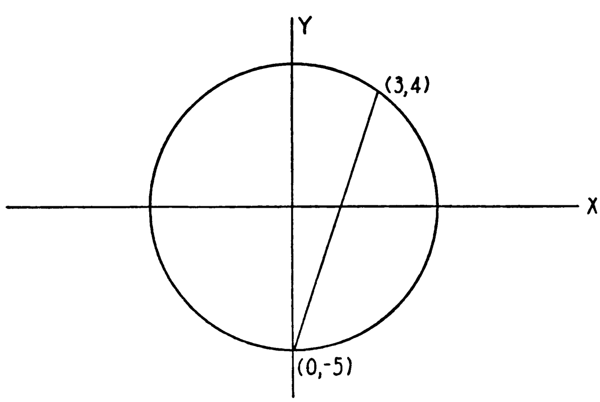
$$\int_a^b f(x)dx = F(b) - F(a), \tag{1}$$

to compute the area between the curve $y=f(x)$ and the x axis from $x=a$ to $x=b$ is the case where $F(x)$ is not a single-valued function. As an example, one finds by use of (1) that the area between the curve $y\sqrt{1-x^2}=1$ and the x axis from $x=-\frac{1}{2}$ to $x=\frac{1}{2}\sqrt{2}$ is equal to arc sin $\frac{1}{2}\sqrt{2}-$arc sin $(-\frac{1}{2})$. The question is which values should be chosen for arc sin $\frac{1}{2}\sqrt{2}$ and arc sin $(-\frac{1}{2})$ in order to obtain the correct area.

To answer this question, one traces back the proof of (1) and finds that it is based directly upon the mean value theorem:

$$F(b) - F(a) = (b-a)f(c), \qquad a \leqq c \leqq b. \tag{2}$$

This theorem does not hold when $F(x)$ is multiple-valued as may be seen by considering the circle, $F(x)=\pm\sqrt{25-x^2}$ and choosing $a=0$, $b=3$, $F(a)=-5$, and $F(b)=4$.



The standard textbooks require that $F(x)$ be single-valued, and the proofs given are seen to hold for multiple-valued functions provided we limit ourselves to a single branch of the curve. In the particular example mentioned above, there is

no difficulty if we agree to limit ourselves to the principal value of the function arc sin x; *i.e.*, $-\pi/2 \leqq y \leqq \pi/2$. As a working rule for calculus students, who are concerned primarily with continuous functions, one might state, "Choose a point on the curve $y=F(x)$ with abscissa a and proceed along the curve constantly to the right if $b \geqq a$ or constantly to the left if $b \leqq a$ until the first point with abscissa b is reached. The ordinate of this point along with the ordinate of the original point when substituted in (1) will then give the correct value of the area for the curve $y=f(x)$."

In applying this rule to the example given above, we must remember that $\int dx/\sqrt{1-x^2} = \text{arc sin } x$ only for those branches of $y = \text{arc sin } x$ for which the tangent has a positive slope, and $\int dx/\sqrt{1-x^2} = -\text{arc sin } x$ for those branches for which the tangent of $y = \text{arc sin } x$ has a negative slope.

DETERMINANTS AND PLANE EQUATIONS

R. R. Stoll, Oberlin College

In an elementary exposition of solid analytic geometry it is quite common to find the equation of the plane determined by three noncollinear points (x_i, y_i, z_i), $i=1, 2, 3$, written in the form

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0.$$

That determinants can also be used to advantage in deriving the equations of planes satisfying other standard conditions, seems to have been overlooked.

For example, the equation of the plane through the points (x_i, y_i, z_i), $i=1, 2$, and perpendicular to the plane with equation $ax+by+cz=d$ is

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ a & b & c & 0 \end{vmatrix} = 0. \tag{1}$$

Indeed, the only novelty of the statement, *viz.* the perpendicularity of the given plane and (1), follows from the relation

$$\begin{vmatrix} a & b & c & 0 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ a & b & c & 0 \end{vmatrix} = 0$$

upon expanding the determinant by the elements of the first row, since the coefficients of a, b, c are the coefficients of x, y, z, respectively in (1).

Again, the equation of the plane through (x_1, y_1, z_1) and perpendicular to each of the planes $a_ix+b_iy+c_iz=d_i$, $i=1, 2$ (or in other words, perpendicular to a given line) is

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ a_1 & b_1 & c_1 & 0 \\ a_2 & b_2 & c_2 & 0 \end{vmatrix} = 0.$$

Finally, it is amusing and sometimes profitable to discuss in a beginning class the equation

$$\begin{vmatrix} x & y & z & 1 \\ a_1 & b_1 & c_1 & 0 \\ a_2 & b_2 & c_2 & 0 \\ a_3 & b_3 & c_3 & 0 \end{vmatrix} = 0$$

which is suggested by the foregoing as a possibility for the equation of a plane which is perpendicular to each of the three planes $a_ix+b_iy+c_iz=d_i$, $i=1, 2, 3$

DEVELOPMENTS IN POWER SERIES

R. C. YATES, U. S. Military Academy

The purpose of this note is to suggest an expeditious method of obtaining power series expansions of certain real functions which show repetitive characteristics in their derivatives. A case in point is the function tan x whose successive derivatives are increasingly complicated but nevertheless somewhat repetitive. It seems that the method might be used profitably in the beginning course in calculus.

The spirit of the approach is illustrated with the trigonometric functions. Consider first the expansion of $f(x)=\sin x$ into the form

$$(1) \qquad f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots .$$

We have $f'(x)=\cos x$ and $f''(x)=-\sin x=-f(x)$. The function $\sin x$ then is defined by

$$(2) \qquad f'' = -f$$

together with the conditions $f(0)=0, f'(0)=1$.

Equation (2) provides a *recurrence formula* for all derivatives. Thus

$$f''(0) = 0;\ f'''(0) = -f'(0) = -1;\ f^{\text{iv}}(0) = -f''(0) = 0;\ etc.$$

Accordingly, with (1):

$$\sin x = x - \frac{x^3}{3!} + \cdots .$$

The cosine function is defined by

$$f'' = -f, \qquad f(0) = 1, \qquad f'(0) = 0,$$

and evolves in the same fashion.

Consider

$$f(x) = \tan x.$$

Here

$$f' = \sec^2 x = 1 + f^2$$

and this recurrence relation provides the succession:

$$f'' = 2ff'; \qquad f''' = 2(ff'' + f'^2); \qquad f^{\text{iv}} = 2(ff''' + 3f'f'');$$
$$f^{\text{v}} = 2(ff^{\text{iv}} + 4f'f''' + 3f''^2);\ etc.,$$

so that

$$f(0) = 0; \quad f'(0) = 1; \quad f''(0) = 0; \quad f'''(0) = 2; \quad f^{\text{iv}}(0) = 0; \quad f^{\text{v}}(0) = 16; \quad etc.$$

The expansion (1) for $f(x) = \tan x$ is thus

$$\tan x = x + \frac{2}{3!}x^3 + \frac{16}{5!}x^5 + \cdots .$$

For

$$f(x) = \sec x$$

we have

$$f' = \sec x \tan x; \qquad f'' = \sec^3 x + \sec x \tan^2 x = 2f^3 - f.$$

Thus

$$f''' = 6f^2f' - f'; \qquad f^{\text{iv}} = 6f^2f'' + 12ff'^2 - f'';$$
$$f^{\text{v}} = 6f^2f''' + 36ff'f'' + 12f'^3 - f'''; \quad etc.,$$

so that

$$f(0) = 1; \qquad f'(0) = 0; \qquad f''(0) = 1; \qquad f'''(0) = 0; \qquad f^{\text{iv}}(0) = 5;\ etc.$$

Accordingly,

$$\sec x = 1 + \frac{1}{2!} x^2 + \frac{5}{4!} x^4 + \cdots .$$

It is to be noted that the point of embarkation is the formation of the differential equation which together with initial conditions defines the function.

The method is, of course, of general use in determining solutions to some differential equations in the form of power series. For example, the equation and condition

$$y' = x + y^2, \qquad y(0) = 1$$

yield to the method. Here

$$y'' = 1 + 2yy'; \qquad y''' = 2(yy'' + y'^2); \text{ etc.},$$

which lead to

$$y = 1 + x + \tfrac{3}{2}x^2 + \tfrac{4}{3}x^3 + \cdots .$$

This is a recognizable economy in comparison to the more prevalent substitution of a series with undetermined coefficients directly into the differential equation.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, State University of New York

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Harpur College, Endicott, New York. This department welcomes problems believed to be new and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1111. *Proposed by P. L. Chessin, Westinghouse Electric Corporation*

Oúr good friend and eminent numerologist, Professor Euclide Paracelso Bombasto Umbugio, has been busily engaged testing on his desk calculator the $81 \cdot 10^9$ possible solutions to the problem of reconstructing the following exact long division in which the digits indiscriminately were each replaced by x save in the quotient where they were almost entirely omitted.

```
                      8
              ---------------
x x x ) x x x x x x x x
        x x x
        -----
          x x x x
            x x x
            -----
              x x x x
              x x x x
              -------
```

Deflate the Professor! That is, reduce the possibilities to $(81 \cdot 10^9)^0$.

E 1112. *Proposed by L. C. Graue, Sacramento State College*

Consider two families of circles, one tangent at the origin to the x-axis and the other tangent at the point (1,1) to a line of slope m. Find the locus of the points of tangency of the two families.

E 1113. *Proposed by Peter Treuenfels, Ballistic Research Laboratories, Aberdeen Proving Ground.*

Let $y(x)$ denote that solution of the differential equation $dy/dx = x^2 + y^2$ which passes through the origin. Show that $y(1) < 23/53$.

E 1114. *Proposed by Viktors Linis, University of Saskatchewan*

What is the diameter of the smallest circular plate on which a semicircular pie can be placed if the pie is allowed to be cut in sectors of the same radius as the pie?

E 1115. *Proposed by R. M. Gordon, China Lake, California*

(1) Let $Q_1Q_2Q_3Q_4$ be a (not necessarily convex) plane quadrilateral. On its sides construct similar isosceles triangles $Q_vP_vQ_{v+1}$ (with $Q_5 \equiv Q_1$), having arbitrary base angles θ. The angles $Q_{v+1}Q_vP_v$ $(=\theta)$ are oriented alternately clockwise and counterclockwise from the adjacent sides, $Q_{v+1}Q_v$, of the quadrilateral. Show that P_1, P_2, P_3, P_4, the vertices of the isosceles triangles, are the vertices of a parallelogram.

(2) Let $P_1P_2P_3P_4$ be a plane quadrilateral. On its vertices construct similar isosceles triangles $Q_vP_vQ_{v+1}$ (with $Q_5 \equiv Q_1$) having vertex angles $Q_vP_vQ_{v+1}$ and base angles θ. The angles $Q_{v+1}Q_vP_v$ $(=\theta)$ are oriented alternately clockwise and counterclockwise in adjacent triangles. Show that, for arbitrary θ, the bases of the isosceles triangles are the sides of infinitely many quadrilaterals $Q_1Q_2Q_3Q_4$ provided that $P_1P_2P_3P_4$ is a parallelogram, and that if $P_1P_2P_3P_4$ is not a parallelogram then there exists a unique quadrilateral $Q_1Q_2Q_3Q_4$.

SOLUTIONS

A Symmetric Function

E 1081 [1953, 550]. *Proposed by A. E. Livingston, University of Washington*

Let $f(x_1, \cdots, x_n)$ be the nth order determinant $|a_{ij}|$ with $a_{ii}=x_i$ and $a_{ij}=1$ for $i\neq j$. Clearly, $f(x_1, \cdots, x_n)$ is symmetric in $x_1, \cdots, x_n$. Find its representation in terms of the elementary symmetric functions in $x_1, \cdots, x_n$.

Solution by O. E. Stanaitis, St. Olaf College, Northfield, Minnesota. Perform the operation $\text{col}_k-\text{col}_n$ for $k=1, 2, \cdots, n-1$ and take out the factors $x_1-1, x_2-1, \cdots, x_n-1$. Now add the $n-1$ rows to the last row to get a triangular determinant with diagonal elements $1, \cdots, 1, 1+\sum_{i=1}^{n} 1/(x_i-1)$. Hence the determinant can be written in the form

$$\begin{aligned} f(x_1, \cdots, x_n) &= \prod_{i=1}^{n} (x_i - 1)\left[1 + \sum_{i=1}^{n} 1/(x_i - 1)\right] \\ &= p_n - p_{n-2} + 2p_{n-3} - 3p_{n-4} + \cdots + (-1)^{n-1}(n-1) \\ &= \sum_{i=0}^{n} (-1)^{i-1}(i-1)p_{n-i}, \qquad (p_0 = 1), \end{aligned}$$

as required.

Also solved by J. W. Baldwin, L. F. Boron, S. H. Eisman, David Ellis, H. M. Feldman, N. J. Fine, A. D. Fleshler, R. E. Greenwood, A. S. Gregory, Zoran Ivkovic, M. S. Klamkin, J. A. Nickel, A. M. Ostrowski, M. J. Pascual, G. C. Smith, J. V. Wittaker, and the proposer. Late solutions by John Jones, Jr., S. Parameswaran, and W. D. Serbyn.

Feldman remarked that the problem appears in Polya and Szegö, *Aufgaben und Lehrsätze*, vol. 2, p. 99, prob. 7, and, in slightly modified form, in Burnside and Patton, *Theory of Equations*, vol. 2, p. 57, prob. 20.

A Consequence of Stirling's Formula

E 1082 [1953, 551]. *Proposed by L. L. Pennisi, University of Illinois*

Show that $\lim_{n\to\infty} n(\sqrt{n}/n!)^{1/n}=e$.

Solution by O. E. Stanaitis, St. Olaf College, Northfield, Minnesota. If Stirling's approximation for factorials, $n!=\sqrt{2\pi n}\, n^n e^{-n}$, is used, it follows that

$$\lim_{n\to\infty} n(\sqrt{n}/n!)^{1/n} = \lim_{n\to\infty} e/(2\pi)^{1/2n} = e.$$

Also solved by Leon Bankoff, P. M. Berry, L. F. Boron, Julian Braun, W. E. Briggs, P. L. Chessin, M. V. Davis, H. M. Feldman, N. J. Fine, A. D. Fleshler, D. S. Greenstein, A. S. Gregory, Emil Grosswald, J. W. Hamblen, Frank Harary, J. R. Hatcher, A. R. Hyde, C. E. Kerr, M. S. Klamkin, A. E. Livingston, David Mandelbaum, Jerome Manheim, D. C. B. Marsh, H. R. Menkes, Erich Michalup, J. D. Miller, Morris Morduchow, Leo Moser, C. S.

Ogilvy, M. V. Pai, M. J. Pascual, J. D. Reid, William Small, R. P. Smith, E. J. Steinberg, J. A. Tierney, J. V. Whittaker, R. R. Williams, Jr., and the proposer. Late solutions by Joseph Muskat, S. Parameswaran, W. D. Serbyn, and C. W. Trigg.

An Inequality Arising from the Sum of nth Powers

E 1083 [1953, 551]. *Proposed by F. J. Duarte, Caracas, Venezuela*

Suppose that $x^n+y^n=z^n$, where n is a positive integer and x, y, z are positive real numbers. Show that $(xy/z^2)^n<2/5$.

I. *Solution by D. P. Richardson, University of Arkansas.* We have, for any real number n, $(x^n-y^n)^2=z^{2n}-4x^ny^n\geqq 0$, whence $x^ny^n/z^{2n}\leqq 1/4<2/5$.

II. *Solution by Kovina Milosevich, Skoplje, Jugoslavia.* Set $(x/z)^n=a$, $(y/z)^n=b$; then $a+b=1$. The maximum of the expression $(xy/z^2)^n=ab=a-a^2$ is $1/4$. Therefore $(xy/z^2)^n\leqq 1/4<2/5$. Note that there is no need to restrict n to be integral.

Also solved by A. N. Aheart, J. W. Baldwin, Leon Bankoff, L. F. Boron, Julian Braun, W. E. Briggs, P. L. Chessin, K. W. Crain, Monte Dernham, Fred Discepoli, I. A. Dodes, Edgar Dougherty, A. L. Epstein, H. M. Feldman, N. J. Fine, A. D. Fleshler, L. A. Fulk, D. S. Greenstein, A. S. Gregory, Emil Grosswald, J. W. Hamblen, J. R. Hatcher, Dudley Herschbach, M. H. Hoehn and A. Steger (jointly), Vern Hoggatt, A. R. Hyde, Ray Jurgensen, C. E. Kerr, M. S. Klamkin, Rose Lariviere, Mark Leum, A. E. Livingston, D. C. B. Marsh, H. H. Martens, G. T. Miller, J. D. Miller, Morris Morduchow, C. S. Ogilvy, Arthur Pancoe, M. J. Pascual, J. D. Reid, B. E. Rhoades, L. A. Ringenberg, Azriel Rosenfeld, F. C. Smith, G. C. Smith, R. P. Smith. O. E. Stanaitis, E. J. Steinberg, D. R. Sudborough, J. H. Wahab, L. E. Ward, Jr., Chih-yi Wang, R. H. Wilson, Jr., and the proposer. Late solutions by S. Parameswaran, W. D. Serbyn, and C. W. Trigg.

A Sharpened Inequality

E 1084 [1953, 551]. *Proposed by R. E. Shafer, University of California*

Trygve Nagell in his book *Elementary Number Theory* shows that

$$\sum_{p\leqq x}(1/p) > \log\log x - 1,$$

where the summation is over all primes p not exceeding x. Establish the sharper inequality

$$\sum_{p\leqq x}(1/p) > \log\log x + 1 - \pi^2/6.$$

I. *Solution by Leo Moser, University of Alberta.* Following Nagell's treatment (*Elementary Number Theory*, pp. 57–59) we set $P_x = \prod_{p \leqq x} (1-1/p)^{-1}$ and find

(1) $$P_x > \log x.$$

On the other hand

$$\log P_x = -\sum_{p \leqq x} \log (1 - 1/p) = \sum_{p \leqq x} (1/p + 1/2p^2 + 1/3p^3 + \cdots).$$

Now

(2) $$1/2p^2 + 1/3p^3 + \cdots < (1/2)(1/p^2 + 1/p^3 + \cdots) = 1/2p(p-1),$$

so that

(3) $$\sum_{p \leqq x} (1/2p^2 + 1/3p^3 + \cdots) < (1/2)\sum_{n=1}^{\infty} 1/n(n-1) = 1/2.$$

Combining (1), (2), (3) yields

$$\sum_{p \leqq x} (1/p) > \log \log x - 1/2,$$

which is better than the required result.

It is clear that this result can be further improved. It is well known that

$$\sum_{p \leqq x} (1/p) = \log \log x + B + O(1/\log^q x)$$

for every q (Landau, *Primzahlen*, p. 201), and it has been shown that

$$B = 0.2614972128476427837554 \cdots$$

(H. M. Terril and L. Sweeney, *J. Franklin Inst. 239*, pp. 242–243 [1945]).

II. *Solution by J. R. Hatcher, Fisk University.* Writing $\log(1-1/p)$ as a power series in $1/p$ shows that $-\log(1-1/p) - 1/p < 1/p^2$. Therefore, since

$$\sum_{p \leqq x} p/(p-1) > \log x \quad \text{and} \quad \sum_{2}^{\infty} 1/p^2 = \pi^2/6 - 1,$$

we have

$$\sum_{p \leqq x} (1/p) > \log \log x + 1 - \pi^2/6.$$

Also solved by L. F. Boron, W. E. Briggs, N. J. Fine, J. W. Hamblen, W. D. Serbyn, J. A. Tierney, and the proposer.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4572 [1954, 52]. Correction. *Proposed by Paul Erdös, University of Notre Dame*

Let b_k be a sequence of non-negative integers such that

$$\overline{\lim}\ \frac{1}{n}\sum_{k=1}^{n} b_k < \infty.$$

Let $f(n) \to \infty$ denote the number of b_k's with $1 \leqq k \leqq n$ for which $b_k > 0$. If $\underline{\lim}\ f(n)/n = 0$, show that

$$c = \sum_{k=1}^{\infty} \frac{b_k}{2^k}$$

is irrational.

4583. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn*

For $a_n = 1$, $n = 1, 2, \cdots$,

$$S = \sum_{r=1}^{\infty} [r!]^{-a_r}$$

is transcendental. Find other non-decreasing sequences $\{a_n\}$ such that S is transcendental.

4584. *Proposed by W. F. Cheney, Jr., University of Connecticut*

The equation, $15x^2 - 78x + 97 = 0$, has a root $3.14160+$, and the equation, $27x^2 - 55x - 50 = 0$, has a root $2.71829+$. It is required to find quadratic equations with integer coefficients, each less than 100 in absolute value, which have roots closest to π and e, respectively.

4585. *Proposed by H. F. Sandham, Trinity College, Ireland*

Denoting the number of partitions of n by $p(n)$, prove

$$\sum_{n=1}^{\infty} \frac{np(n)}{\cosh \pi\sqrt{(2n - \frac{1}{4})}} = \frac{1}{4\pi}.$$

4586. *Proposed by Robert Steinberg and F. A. Valentine, University of California at Los Angeles*

Consider a bounded closed convex set S_0 in the Euclidean plane. Let S_r be the parallel convex set to S_0, that is, the set of all points whose minimum distance from S_0 is at most r. Designate the centroid of S_r by g_r.

Show that the set of points g_r $(0 \leqq r \leqq \infty)$ is a point or an arc of an hyperbola (possibly degenerate) lying in S_0. Does g_∞ have a simple geometric relationship with S_0?

4587. *Proposed by A. E. Livingston, University of Washington, and Harry Pollard, Cornell University*

Show that

$$-(2n+1)\pi \int_{(2n+1)\pi}^{\infty} x^{-1} \sin x dx \uparrow 1.$$

(A similar statement holds for

$$2n\pi \int_{2n\pi}^{\infty} x^{-1} \sin x dx.)$$

SOLUTIONS

A Polynomial Assuming Prescribed Values

4508 [1952, 640]. Correction. In the solution of this problem in the February, 1954, issue, by Chih-yi Wang, the solver points out the following misprint: on page 128, line 2, for

$$\left(\frac{r-1}{k}\right) \quad \text{read} \quad \binom{r-1}{k}.$$

An Irrational Number

4518 [1953, 47]. *Proposed by Paul Erdös, University of Notre Dame, and Mark Kac, Cornell University*

For each integer n let $\sigma_k(n)$ be the sum of the kth powers of its divisors. It is a natural conjecture that

$$\sum_{n=1}^{\infty} \frac{\sigma_k(n)}{n!}$$

is irrational. Prove this for the case $k=2$. (The case $k=1$ is problem 4493 [1953, 557].)

Solution by Robert Breusch, Amherst College. Let p be a prime greater than b, and also greater than 20, and assume

$$\sum_{n=1}^{\infty} \frac{\sigma_2(n)}{n!} = \frac{a}{b},$$

with a, b integers. Multiply by $(p-1)!$; then

$$S \equiv \frac{\sigma_2(p)}{p} + \frac{\sigma_2(p+1)}{p(p+1)} + \sum_{n=p+2}^{\infty} \frac{\sigma_2(n)}{p(p+1)\cdots n}$$

is an integer. Now $\sigma_2(p) = p^2+1$; and for every $n > 20$,

$$n^2 < \sigma_2(n) < n^2 \sum_{\nu=1}^{n} 1/\nu^2 < n^2\pi^2/6 < n(n-1)\cdot 7/4.$$

Thus

$$\sigma_2(n) = n(n-1)(1+\alpha_n), \quad \text{with} \quad 0 < \alpha_n < 3/4.$$

Now the following expression must be an integer:

$$S - p - 1 = \frac{\sigma_2(p)}{p} - p + \frac{\sigma_2(p+1)}{p(p+1)} - 1 + \sum_{n=p+2}^{\infty} \frac{(1+\alpha_n)n(n-1)}{p(p+1)\cdots n}$$

$$= \frac{1}{p} + \alpha_{p+1} + \sum_{n=p+2}^{\infty} \frac{1+\alpha_n}{p(p+1)\cdots(n-2)}.$$

But

$$0 < S - p - 1 < \frac{1}{p} + \frac{3}{4} + 2\left(\frac{1}{p} + \frac{1}{p(p+1)} + \cdots\right)$$

$$< \frac{1}{p} + \frac{3}{4} + \frac{2}{p-1} < 1$$

for $p > 20$, which is a contradiction and establishes the proposition.

Also solved by Richard Brauer and Oscar Goldman, Leonard Carlitz, and the Proposer.

A Summation

4519 [1953, 47]. *Proposed by H. F. Sandham, Trinity College, Ireland*

Prove that

$$1 + \frac{1-n}{1+n}\cdot\frac{1}{3} + \frac{(1-n)(2-n)}{(1+n)(2+n)}\cdot\frac{1}{5} + \cdots = \frac{1}{4n}\cdot\left\{\frac{2\cdot 4\cdots 2n}{1\cdot 3\cdots(2n-1)}\right\}^2.$$

Solution by the Proposer. The following somewhat more general result will be proved:

$$\sum_{m=0}^{\infty} \frac{(1-n)_m(1-x)_m}{(1+n)_m(1+x)_m} = \frac{1}{2}\cdot\frac{n!(\frac{1}{2}+x)_{n-1}}{(\frac{1}{2})_n(1+x)_{n-1}},$$

where, as usual, we put $(a)_r=a(a+1)\cdots(a+r-1)$, $(a)_0=1$. The case $x=\frac{1}{2}$ is that the problem proposed.

When x is an integer it is easily verified that the formula is symmetric in n and x.

Assume the formula is true for $n=1, 2, \cdots, p-1$. Now take $n=p$ and multiply across by $(x+1)_{p-1}$, and it is required to prove that two polynomials of degree $p-1$ in x are identical. Because of the symmetry between n and x and the assumption made on n, these polynomials are already equal for $x=1, 2, \cdots, p-1$. If they are proved equal for one more value of x they must be identical and the formula will be established. Take $x=1-p$ and all terms of the summation vanish except the one for which $m=p-1$, whence it remains only to show that

$$\frac{(1-p)_{p-1}(p)_{p-1}}{(1+p)_{p-1}}=\frac{1}{2}\cdot\frac{p!(3/2-p)_{p-1}}{(\frac{1}{2})_p},$$

but this is immediate.

Also solved by Leonard Carlitz, H. W. Gould, O. E. Stanaitis, Chih-yi Wang, and J. V. Whittaker.

Editorial Note. Carlitz remarks that the formula is true also for non-integral n if the right member is put in the form

$$\frac{2^{4n}}{4n}\cdot\frac{(\Gamma(n+1))^4}{(\Gamma(2n+1))^2}.$$

See W. N. Bailey, *Generalized Hypergeometric Series*, Cambridge, 1935, p. 96. The left member is

$$\int_0^1 F(1-n, 1; n; x^2)dx.$$

An Inequality Involving Moments

4520 [1953, 48]. *Proposed by N. S. Mendelsohn, University of Manitoba*

Let $\phi(x)$ be a Lebesgue measurable function in the interval $0\leqq x\leqq 1$ such that

$$\int_0^1 \phi(x)dx = 0, \qquad \int_0^1 x\phi(x)dx = 1.$$

Show, that $|\phi(x)|\geqq 4$ in some set of measure greater than zero.

Solution by R. P. Agnew, Cornell University. If $|\phi(x)|<4$ almost everywhere, then

$$1=\int_0^1 \phi(x)(x-\tfrac{1}{2})dx<\int_0^1 4\,|x-\tfrac{1}{2}|\,dx=1,$$

so that the result follows. If $|\phi(x)|$ does not exceed 4 in some set of measure greater than zero, then $|\phi(x)| = 4$ almost everywhere.

This proof illustrates the fact that if $\phi(x)$ has given moments

$$\mu_k = \int_0^1 x^k \phi(x) dx, \qquad k = 0, 1, \cdots, n,$$

not all zero, and if $|\phi(x)| < M$ almost everywhere in $0 \leqq x \leqq 1$, then information about M can be obtained from the inequality

$$\left| \sum_{k=0}^n c_k \mu_k \right| = \left| \int_0^1 \phi(x) \sum_{k=0}^n c_k x^k dx \right| < M \int_0^1 \left| \sum_{k=0}^n c_k x^k \right| dx$$

which holds whenever $c_1, \cdots, c_n$ are not all zero. For the case of the special problem proposed, the inequality reduces to

$$|c_1| < M \int_0^1 |c_0 + c_1 x| dx \quad \text{or} \quad 1 < M \int_0^1 |x + c_0/c_1| dx$$

and the best estimate of M is obtained when $c_0/c_1 = -\frac{1}{2}$. The conditions of the problem are satisfied if $\phi(x) = -4$ when $0 \leqq x \leqq \frac{1}{2}$ and $\phi(x) = 4$ when $\frac{1}{2} < x \leqq 1$.

Also solved by P. T. Bateman, Richard Bellman, R. P. Boas, Jr., Harry Furstenberg, S. L. Jamison, P. G. Kirmser, M. S. Klamkin, A. E. Livingston, E. A. Maier, H. S. Shapiro, J. G. Wendel, Albert Wilansky, J. E. Wilkins, Jr., E. M. Wright, and the Proposer. Late solutions by S. Aljančić and R. Bojanić.

Editorial Note. Boas remarks that a similar proof, using Hölder's inequality, establishes the result

$$\left\{ \int_0^1 |\phi(x)|^p dx \right\}^{1/p} \geqq 2(q+1)^{1/q},$$

where $1 < p < \infty$ and $q = p/(p-1)$. The proposed problem is the limiting case $p = \infty$, $q = 1$. The other limiting case gives

$$\int_0^1 |\phi(x)| dx \geqq 2.$$

Shapiro refers also to a more general inequality involving moments, due to Riesz, as given in Banach's *Théorie des Opérations Linéaires*, pp. 74–77. Bateman solves the problem by the method of the paper *On Moment Spaces*, Bellman and Blackwell, *Annals of Mathematics*, 1951.

Klamkin remarks that if $\phi(t)$ is considered to be an acceleration the problem can be restated: If a particle starts and ends at rest traversing one foot in one second, then its acceleration must have been numerically $\geqq 4$ ft./sec.^{2} for some set

of positive measure. See Courant *Differential and Integral Calculus*, Vol. I, p. 556.

Limiting Value of a Definite Integral

4522 [1953, 48]. *Proposed by O. E. Stanaitis, St. Olaf College, Northfield, Minn.*

Evaluate

$$\lim_{k\to\infty} \int_{2k\pi}^{(2k+1)\pi} \phi(t^\alpha) \sin t dt$$

where $\phi(x) = x - [x]$, $\alpha > 1$, and $[x]$ denotes the greatest integer in x.

Solution by E. M. Wright, University of Aberdeen, Scotland. We write

$$\psi(x) = \phi(x) - \tfrac{1}{2}, \quad \beta = 1/\alpha, \quad u = t^\alpha, \quad c = (2k\pi)^\alpha, \quad d = (2k+1)^\alpha \pi^\alpha,$$
$$f(u) = \beta u^{\beta-1} \sin (u^\beta).$$

We have then

$$\int_{2k\pi}^{(2k+1)\pi} \phi(t^\alpha) \sin t dt = 1 + \int_{2k\pi}^{(2k+1)\pi} \psi(t^\alpha) \sin t dt$$
$$= 1 + \int_c^d \psi(u) f(u) du.$$

Now, for all $u \geqq c$,

$$\Psi(u) = \int_c^u \psi(v) dv = O(1),$$
$$f(u) = O(u^{\beta-1}), \qquad f'(u) = O(u^{2\beta-2}) = O(c^{\beta-1}u^{\beta-1}),$$

and, on integration by parts,

$$\int_c^d \psi(u) f(u) du = [\Psi(u) f(u)]_c^d - \int_c^d \Psi(u) f'(u) du$$
$$= O(c^{\beta-1}) \left\{ 1 + \int_c^d u^{\beta-1} du \right\}$$
$$= O(c^{\beta-1}) \{ 1 + d^\beta - c^\beta \} = O(c^{\beta-1}) \to 0$$

as $k \to \infty$. Hence the value of the required limit is 1.

Also solved by M. I. Aissen and Albert Novikoff, P. G. Kirmser, and the Proposer.

Editorial Note. Kirmser proves the following generalization: *If $f(t)$ is continuous in $a \leqq t \leqq b$, $\phi(x) = x - [x]$, $\alpha > 1$, then*

$$\lim_{\theta\to\infty}\int_a^b \phi\{(t+\theta)^\alpha\}f(t)dt = \frac{1}{2}\int_a^b f(t)dt.$$

Aissen and Novikoff established the theorem: *Let $f(t)$ be defined and be convex and monotone increasing for $a<t<\infty$, and let*

$$\lim_{t\to\infty}\{f(t+1)-f(t)\} = \infty.$$

Further, let $g(t)$ be periodic with period T and properly Riemann-integrable in $[0, T]$. Then

$$\lim_{A\to\infty}\int_A^{A+T}\phi\{f(t)\}g(t)dt = \frac{1}{2}\int_0^T g(t)dt.$$

Linear Congruence in Real, Projective Space

4523 [1953, 123]. *Proposed by H. S. M. Coxeter, University of Toronto*

Given a point P in real projective space, and four skew lines a, b, c, d, find a linear construction for the line through P of the linear congruence determined by a, b, c, d.

Solution by A. P. Dempster, University of Toronto. Every line of the given congruence is linearly dependent on (*i.e.*, belongs to the regulus determined by) three lines which are each linearly dependent on three of the four given lines. We seek a line through P which will be dependent on f, c, d, where f is dependent on a, b, c.

The line f is found as follows. The transversal from P to c and d, say, $e=Pc\cdot Pd$, meets the quadric abc in one known point $C=c\cdot Pd$ and therefore meets it in another point F. The plane Pc, touching the quadric, meets it in c and a second generator g. The sections of the generators a and b by this plane lie on the quadric and therefore on g. Thus g is the join $(Pc\cdot a)(Pc\cdot b)$, and the desired point F is

$$g\cdot e = (Pc\cdot a)(Pc\cdot b)\cdot Pd.$$

Since F is on the quadric, there is a member f, through F, of the regulus abc, *viz.*, the transversal from F to any two transversals of the three lines a, b, c. Now the line e passes through P and is a transversal of f, c, d. Therefore we can draw a member h, through P, of the regulus fcd, *viz.*, the transversal from P to any two transversals of the three lines f, c, d. This h is the line we were asked to construct.

This particular construction breaks down if the points C and F coincide, *i.e.*, if PC touches the quadric abc at C. But in that case we can use $Pa\cdot Pd$ or $Pb\cdot Pd$ instead of $Pc\cdot Pd$ .The only case in which the method would fail completely is when all these three transversals are tangents. But then Pd would be the tangent plane at P, and the problem would be solved by taking the generator through P of the regulus abc.

Also solved by W. R. Andress.

Consecutive Integers Whose Totients are Multiples of n

4524 [1953, 123]. *Proposed by E. C. Milner, University of Malaya, Singapore*

Prove that for any positive integers n, N there are blocks of consecutive integers of length greater than N, with the property that each of their totients is divisible by n.

Solution by W. E. Briggs, University of Colorado. If an integer is divisible by a prime p its totient is divisible by $p-1$.

By Dirichlet's theorem there are infinitely many primes congruent to 1 modulo n. Let $p_0, p_1, \cdots, p_N$ be any $N+1$ of the primes in this progression. By the Chinese remainder theorem, the following system of congruences is solvable:

$$T \equiv 0 \pmod{p_0}, \quad T+1 \equiv 0 \pmod{p_1}, \cdots, T+N \equiv 0 \pmod{p_N}.$$

Therefore, for each solution T, of which there are infinitely many, each number of the block of consecutive integers $T, T+1, \cdots, T+N$, is divisible by some p_i, and its totient is divisible by p_i-1 which is a multiple of n.

Also solved by P. T. Bateman, Leonard Carlitz, S. H. Gould, J. B. Kelly, Leo Moser, and the Proposer.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to E. P. Vance, Oberlin College, Oberlin, Ohio.

College Algebra. By R. H. Bardell and Abraham Spitzbart. Cambridge, Addison-Wesley Publishing Company, Inc., 1953. ix+197 pages. $3.50.

It is a rare and welcome pleasure, these days, to encounter the slim and compact freshman algebra text that achieves so admirably the goals intended by the authors. It is an extremely flexible and teachable text which allows of departures and amplifications without upsetting the continuity of the material. Clearly presented, concisely written, uncluttered by the irritating confusions of over-cautious development, yet it lacks none of the rigour necessary to such a text.

An intensive and rapid review of High School Algebra, terminating in a set of well chosen review exercises, opens the book, enabling the instructor to check the capacities of his students and the students to check the solidity of their background.

Linear and quadratic functions, polynomials of higher degree and systems of equations are treated mainly from the centralizing concept of function and graphical representation, emphasis being placed upon the connection between zeros of a function and roots of an equation. A chapter on determinants is sufficient to introduce the student to their use and properties and gives him ample material upon which to base further exploration in this field should it be required.

Probably the most exciting chapter in this fine little book is the one on Mathematical Induction and the Binomial Theorem. The method of induction is very lucidly presented and numerous examples are provided whereby the student may gain confidence in the use of this method of proof.

The concluding chapters, Exponential and Logarithmic Functions, Progressions, Permutations, Combinations and Probability, round out and complete as unified and concise a text as I have seen.

It is readily adaptable (as the authors point out) to a three, four or five semester hour course, with every expectation from the instructor of completing the text in this time without the necessity of omitting anything.

The examples and review exercises, which intersperse the book, are carefully chosen to amplify and supplement the textual material. Answers to odd numbered examples are given at the end of the book (answers to even numbered examples may be obtained from the publisher), as are also the conventional tables.

M. A. OLIVER
Bennington College

College Algebra. By J. R. Britton and L. C. Snively. New York and Toronto, Rinehart and Co., 1953. x+502 pages. $5.00.

In the opinion of the reviewer this book ranks among the best of a number of excellent texts in college algebra which have appeared in recent years. The material is exceptionally well organized and is presented clearly and concisely. That thoroughness and rigor characterize the theoretical developments is demonstrated, for example, in the treatment of division by zero, the theory of exponents, and extraneous roots of an equation. The exercises are well selected and graded and include more than the usual number drawn from geometry. In general the topics covered are those which commonly appear in such a text. Exceptions are the inclusion of a chapter on approximate numbers, highly to be commended, and one on finite differences. Topics not included which do appear in several recent texts are infinite series and elementary statistics. Following the text material are a table of powers, roots, and reciprocals, a five place table of common logarithms, a table of natural logarithms, a set of answers to the odd-numbered problems, and an index.

Among the many features worth noting with approval are two in particular. One is the extent to which vectors are applied in connection with the discussion

of complex numbers. The other is that proofs are given for the rules of operation for partial fractions, whereas they are omitted in most college algebra texts.

In the nature of adverse criticism are the following three observations, all on relatively minor matters. On page 2, in the statement of "The Commutative Law of Addition," the expression "a given set of numbers" might better be replaced by "two numbers" unless a sufficient number of illustrative examples are given in support of the statement. The change would also make the statement of the law consistent with that of the corresponding law for multiplication. It would also be preferable to avoid introducing the expression "real number system" (page 108) until imaginary numbers are discussed. On page 270, Property 2 of complex numbers should for completeness be stated as a biconditional property as is Property 1.

J. C. POLLEY
Wabash College

Elements of Algebra. By V. B. Caris, Boston, Ginn and Company, 1953. vi+307 pages. $3.30.

This book covers the material which customarily appears in intermediate algebra. The first ten chapters include the usual topics in algebra through quadratic equations in one unknown, and the remaining four chapters are: (11) Quadratic Systems; (12) Ratio, Proportion, and Variation; (13) Logarithms; (14) Progressions. The chapter on progressions includes the binomial theorem for the case involving positive integral exponents, and a brief introduction to annuities. The text proper is followed by a table of squares and square roots of the numbers 1 through 150, a four place table of logarithms, a set of answers to odd-numbered problems, and an index. Answers to even-numbered problems are available in a separate pamphlet.

The book is designed for use in a college course for students who have had only one year of algebra in secondary school and for the dual purpose of serving as a text for either a terminal course for those needing or desiring only a second course in algebra, or a course which aims to bridge the gap between elementary and college algebra.

Clarity and simplicity of expression are characteristic of the development of the topics. The format of the text is excellent. The use of boldface type in the statement of rules, laws, and special forms, and, in particular, the liberal interpolation of the statements in agate type with appropriate headings such as NOTE, CAUTION, and HINT, are features highly to be commended. The exercises are well selected and are more than adequate from the point of view of both variety and quantity.

There are a few instances of confusing or inaccurate statements. On page 12, in the paragraph defining "root of an equation," the second sentence needs rewriting for the sake of clarity and accuracy. The first sentence of the last paragraph of Section 52, page 88, should also be rewritten. In the first sentence of

Section 114, page 225, the words "either integral or fractional" should be deleted for the sake of accuracy. In the first paragraph of Section 42, page 73, treating "factored form of an expression" and "factors of an expression" as synonymous is unfortunate.

On the basis of comparison with a representative selection of texts in the area, this text rates with the best and, for the purposes for which it is designed, it is superior to most.

J. C. POLLEY
Wabash College

College Algebra. By R. R. Middlemiss. New York, McGraw-Hill Book Company, 1952. 20+344 pages. $3.50.

This text comprises the usual topics to be found in a college algebra including a chapter on Compound Interest and Annuities. It is carefully written, and due attention has been paid to the order of the topics. For example, complex numbers are introduced along with quadratic equations. The author's aim is to give the student the necessary training in technique with due regard to the logic of the procedures involved. He carefully distinguishes between a theorem and its converse, reverses the usual order in making a proof by mathematical induction, and stimulates the interest of the better student by references to where one might find a proof of the fundamental theorem of algebra and a more thorough discussion of irrational numbers.

Such matters as "division by zero" and equivalent equations are carefully discussed, but the student might conceivably get the impression that such equations as $x-4=0$ and $(x-4)^2=0$ are equivalent. Cramer's Rule is taken up in the chapter on Determinants, and both the method of successive linear approximations and Horner's method are discussed in the Theory of Equations. The "proof" of the Remainder Theorem possesses the usual weakness of assuming the identity: $P(x)\equiv(x-r)Q(x)+R$ after establishing it for one or two isolated cases, and absolute values are not used in stating the properties of logarithms.

All in all, however, this reviewer finds the text unusually well-written, and adaptable for a liberal arts course in College Algebra or a moderate engineering course. The exercises are carefully chosen, and a wise choice has been made of "word" problems. We agree with the statement on the back "flap" that "the problems in the text require the student to analyze his procedures carefully, to respect accuracy, and to know the reason behind the problem's solution." Answers are given for all problems excepting where simple checks are available or where it has been found wiser to omit the answer. An adequate index is included with four-place tables of common logarithms and the natural and logarithmic trigonometric functions. Included also are tables of powers and roots, natural logarithms, exponential and hyperbolic functions, $(1+i)^n$, $(1+i)^{-n}$, amount of annuity of 1 per period, and present value of annuity of 1

per period. The teacher would also be interested in the forty-eight lesson arrangement including topics and problem assignments.

H. S. GRANT
Rutgers University

Algebra for College Students. By R. R. Middlemiss, New York, McGraw-Hill Book Company, 1953. 10+394 pages. \$3.75.

This book differs from the author's *College Algebra* only in the fact that the first three chapters, for which eleven lessons were scheduled, have been replaced by five chapters covering essentially the same material, namely, the real number system, polynomials and rational functions, linear and quadratic equations. This enables the instructor to review this elementary material more thoroughly for those students who need it. The revised schedule calls for a maximum of twenty-two lessons to cover this material as compared with only eleven lessons in *College Algebra*. The lessons would be shorter and contain more purely "drill" exercises. However, each instructor might adjust the schedule for these early chapters to meet the needs of his class.

This reviewer feels that Professor Middlemiss has succeeded somewhat in making clear to the student the vast difference between mere mechanical routine and straight, honest thinking, and he has done this without departing from traditional subject matter. There is much of value in the topics one usually associates with College Algebra, and Professor Middlemiss has succeeded in emphasizing the necessity for striking a proper balance between the mastery of essential techniques and the understanding of the logic behind the procedures involved.

H. S. GRANT
Rutgers University

Intermediate Algebra for Colleges. By J. B. Rosenbach and E. A. Whitman. Boston, Ginn and Company, 1951. 11+219+22 pp. \$3.00.

This little book seems ideal for review and necessary drill work with polynomials, factoring, fractions, functions, graphs, exponents and radicals, linear and quadratic equations, variation, the binomial theorem, progressions, and logarithms. Since the book is intended for college students, it includes something in the way of formal definitions and proofs. The exercises are carefully graded and copious and do not contain long lists of repetitious drill problems, but enough such problems to strengthen the weaker student. The illustrative examples are excellent, and throughout the text are warnings against certain common errors, and both supplemental and historical notes.

The book is intended not only as a review and further training in the techniques of algebra preparatory to a sound course in college algebra, but also as a terminal course in algebra for students interested in such things as natural or social science, or business administration. Whether or not it could be used for

this latter purpose would of course depend on the aims and purposes of those adopting the text. It contains no probability or statistics, but it does contain sufficiently challenging material for those whose secondary school training in algebra has been inadequate. The authors use statement problems to link algebra with many other fields without the use of other than simple algebraic manipulations.

The book is clearly and concisely written, thoroughly teachable and readily understood. Answers are given to the odd-numbered problems only, there is quite an adequate index, and a four-place table of common logarithms, as well as tables of powers, roots, and reciprocals, Napierian logarithms, and important constants and their logarithms are included. Throughout the text applications to other fields of endeavor are constantly kept in mind, and painstaking care is taken in the illustrative examples and elsewhere to make the solution of certain specific problems which constantly arise in practice crystal clear to the student. The long experience of the authors in the field of teaching has taught them where most pitfalls lie in working these problems, but whoever might use this book must not expect that the student has been warned of every pitfall that may arise.

In summary then, it is the opinion of this reviewer that here is a text that gives adequate review and necessary drill in the fundamental techniques of intermediate algebra preparatory to a strong course in college algebra. The main part of a course using this text would consist in working most of the excellent problems presented. Although the statements in the text are for the most part concise and clear, the student would derive little benefit from them without working a fair number of the exercises. The illustrative examples seem quite adequate for the early exercises, but some of the later problems will challenge the better students. Although its treatment of the theory involved is adequate for the purposes in mind, this text is definitely one in which the student learns the theory by the practice.

H. S. GRANT
Rutgers University

NEW BOOKS RECEIVED

Stability Theory of Differential Equations. By Richard Bellman. New York. McGraw-Hill Book Company, 1953. 13+166 pages. $5.50.

Principles of Numerical Analysis. By A. S. Householder. New York, McGraw-Hill Book Company, 1953. 10+274 pages. $6.00.

Tables of Coefficients for the Numerical Calculation of Laplace Transforms. By H. E. Salzer (Bureau of Standards Applied Mathematics Series 30), 1953. 36 pages. $.25.

Simultaneous Linear Equations and the Determination of Eigenvalues. Edited by L. J. Paige and Olga Taussky (Bureau of Standards Applied Mathematics Series 29), 1953. 126 pages. $1.50.

Tables of Natural Logarithms for Arguments Between Zero and Five to Sixteen

Decimal Places. (Bureau of Standards Applied Series 31), 1953. 201 pages. $3.25.

Faster Than Thought. A Symposium on Digital Computing Machines. Edited by B. V. Bowden. London, Putman and Sons, Ltd., 1953. 19+416 pages. $4.75.

Elementary Differential Equations. New Fourth Edition. By L. M. Kells. New York, McGraw-Hill Book Company, 1954. 10+266 pages. $4.00.

Math Is Fun. By Joseph Degrazia. New York, Emerson Books, Inc., 1954. 159 pages. $2.75.

First Course in Abstract Algebra. By R. E. Johnson. New York, Prentice-Hall, Inc., 1953. 8+257 pages. $5.50.

Calculus. By G. B. Thomas. Cambridge, Addison-Wesley Publishing Company, 1953. 614 pages. $6.50.

Higher Transcendental Functions, Volumes I and II. By the Bateman Project Staff. Editor, A. Erdélyi. New York, McGraw-Hill Book Company. Inc., 1953. Vol. I, 26+302 pages. $6.50. Vol. II, 17+396 pages. $7.00.

Tables of 10^x. (Bureau of Standards Applied Series 27), 1953. 543 pages. $3.50.

Mathematics, Plus and Minus. By L. K. Field. New York, Pageant Press, 1954. 15 pages. $2.00.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

MEETING OF MATHEMATICS DIVISION OF A.S.E.E.

The Mathematics Division of the American Society for Engineering Education has planned an instructive program for the A.S.E.E. meetings to be held at the University of Illinois, Urbana, Illinois, June 14–18, 1954. Dr. R. S. Burington, Bureau of Ordnance, Navy Department, is Chairman of the Mathematics Division and is in charge of the program.

The sessions on June 16 will be devoted to a panel discussion on "Problems of Training in the Use of Computing Equipment," led by Dr. C. V. L. Smith of the Office of Naval Research. Members of the panel include Professor W. H. Boghesian, University of Pennsylvania; Dr. J. W. Carr, University of Michigan; Dr. W. J. Eckert, Watson Scientific Computing Laboratory; Professor M. R. Hestenes, University of California; Professor P. M. Morse, Massachusetts In-

stitute of Technology; Professor F. J. Murray, Columbia University. The panel will deal with the problems of training mathematicians, physicists, statisticians and engineers in the utilization of automatic computing equipment, both analogue and digital. Particular attention will be devoted to the type of training that can be provided in typical engineering schools and universities, only a few of which have, or can afford to have, any large scale installations. Attention will be given to the problems involved in the operation, maintenance and use of such machines, as well as those relating to the formulation of problems in a manner suited for machine operation.

On June 17, Dr. C. V. Newsom, Associate Commissioner of Higher Education, State of New York, will give his retiring address on "The Introductory College Course for Engineers."

In addition, at sessions on Thursday and Friday, there will be lectures by the following: Professor G. B. Price, University of Kansas, who will speak on "The Mathematical Needs of Engineering Students" and Professor B. E. Meserve, University of Illinois, who will lecture on "Strengthening the Teaching of Mathematics in Engineering Schools;" Professor Karl Menger, Illinois Institute of Technology; Professor C. O. Oakley, Haverford College; Dr. L. W. Cohen, National Science Foundation; and Dr. F. J. Weyl, Office of Naval Research.

"The Place of Mathematical Statistics and Probability Theory in Engineering Curricula" is the subject of a panel discussion to be held Thursday afternoon under the leadership of Dr. W. P. Pabst, Bureau of Ordnance, Navy Department. The members of this panel have been selected to reflect various points of view in industry, government and education. The panel includes: Mr. J. H. Davidson, General Electric Co.; Dr. Forest Blanding, Standard Oil Development Co.; Professor I. W. Burr, Purdue University. This discussion will be of considerable interest to engineering educators and mathematicians alike.

CALIFORNIA CONFERENCE FOR TEACHERS OF MATHEMATICS

The California Conference for Teachers of Mathematics is holding its fourth annual meeting on the Los Angeles Campus of the University of California during the period July 6–16, 1954. The Conference is sponsored by the University in cooperation with the California Mathematics Council. General sessions include a wide variety of lectures, panel discussions, and campus tours. The choice of study groups will satisfy a wide range of individual interests. Of special interest are the laboratory groups in elementary and secondary mathematics where teachers may actually learn to make many of the teaching aids which are so necessary in our modern schools. Two units of credit may be earned by those participating in the Conference. A moderate fee is charged. For further information write to Clifford Bell, Mathematics Extension, University of California, Los Angeles 24, California.

RECORDINGS OF RADIO PROGRAMS OF THE UNIVERSITY OF OKLAHOMA

Seventeen 15-minute radio broadcasts on mathematics have been given during the current academic year by members of the staff of the University of Oklahoma. Anyone wishing a tape recording of one, or more, of these broadcasts may secure them in the following manner:

Your magnetic tape, in a single reel film shipping case, or packed between sheets of corrugated cardboard and wrapped in heavy paper, can be mailed to *Tapes for Teaching, Educational Materials Services, University of Oklahoma, Norman, Oklahoma.*

A service charge of 50 cents for each 15-minute broadcast, plus postage charges, will be made for duplication of our master tape onto your tape. Be sure to state whether your recorder runs at $3\frac{3}{4}$ or $7\frac{1}{2}$ inches per second! All recordings will be single track; however they will play on double track recorders. Each 15-minute broadcast will require a minimum of 300 feet of tape at $3\frac{3}{4}$ in./sec. or 600 feet at $7\frac{1}{2}$ in./sec. Standard tape lengths are 600 feet (5 inch diameter) and 1200 feet (7 inch diameter) reels. Order tapes by giving the date on which the talk was given, the complete title, and the name of the speaker.

As a convenience to those who do not wish to send in a tape, recordings may be rented at $1.50 per 15-minute recording for a five day period.

Programs

October 5—*Mathematics—Our Great Heritage*, C. E. Springer
October 12—*How Mathematics Started*, J. O. Hassler
October 19—*The Struggle for a Number System*, J. O. Hassler
October 26—*Butter and Eggs Mathematics*, R. C. Dragoo
November 2—*Mathematical Pastimes*, J. C. Brixey
November 9—*Codes and Ciphers*, R. V. Andree
November 16—*Proving the Impossible*, A. F. Bernhart
November 23—*Misinterpretation of Statistical Data*, B. T. Goldbeck
November 30—*Quality Control*, J. C. Brixey
December 7—*Running Around in Circles*, N. A. Court
December 14—*Entertaining Mathematics*, N. A. Court
December 21—*Small Observatories*, B. S. Whitney
December 28—*New Light on an Old Problem*, R. V. Andree
January 4—*What Should a High School Student Expect from Mathematics?*, Miss Dora McFarland
January 11—*Mathematics and Stamp Collecting*, W. N. Huff
January 18—*Random Walk and Gambler's Ruin*, P. W. M. John
January 25—*Why Study Geometry?* J. O. Hassler

SUMMER SESSIONS

The following institutions announce advanced courses in mathematics for the summer of 1954:

Boston University. June 1 to July 10: Professor Scheid, numerical analysis.

July 12 to August 21: Professor Blackett, point set topology; Professor Noether, statistical methods in research.

The Catholic University of America. June 28 to August 7: Professor Ramler, college geometry, synthetic projective geometry, differential equations; Professor Finan, higher algebra II; Professor Rice, advanced calculus II; Professor Moller, higher algebra I; Professor Lepson, advanced calculus I; Professor Mendousse, applied mathematics with engineering applications; Professor Taam, partial differential equations of mathematical physics.

Columbia University, Teachers College. July 5 to August 13: Professor Fehr, teaching arithmetic in the elementary school, current problems in teaching secondary school mathematics; Professors Fehr and Rosskopf, research and departmental seminar in teaching mathematics; Professor Rosskopf, the history of mathematics, survey of higher mathematics for teachers; Professor Shuster, field work in mathematics, teaching non-academic mathematics in high school; Mr. Rourke, teaching of algebra in secondary schools, professionalized subject matter in advanced secondary school mathematics; Professor Kays, materials and models in mathematical education. July 12 to 16: Professor Fehr and special lecturers, workshop in teaching mathematics.

De Paul University. June 28 to August 4: Professor DeCicco, analytic dynamics, polygenic functions; Professor Caton, theory of equations. June 14 to August 6 (evenings): Professor Caton, potential theory I.

Massachusetts Institute of Technology. July 12 to July 16: Professors Wiener and Lee will conduct a one week special program entitled "Mathematical Problems of Communication Theory."

New York University. July 6 to August 13: Professor Bradley, history of mathematics; Dr. Dodes, the program of mathematics in general eduation; Dr. Harrison, the teaching of arithmetic; Dr. Kinsella, mathematics for graduate students of education.

Stanford University. June 21 to August 17: Professor Bacon, ordinary differential equations; visiting professors (to be announced later), selected topics from the theory of functions of a complex variable, hydrodynamics.

University of Buffalo. July 6 to August 13: Professor Gehman, infinite series; Professor Schneckenburger, non-euclidean geometry; Professor Montague, advanced methods for teachers of mathematics.

University of California, Department of Mathematics. June 21 to July 30: Professor McDonald, coordination in mathematics for secondary schools.

University of California, Statistical Laboratory. June 21 to July 31: Professor Rao, multivariate analysis; Professor Neyman, individual research leading to higher degrees. August 2 to September 11: Professor Neyman, individual research.

University of Colorado. June 14 to July 20 and July 22 to August 24: Mr. McLeod, foundations of analysis; Professor Sniveley, vector analysis; Professor Stahl, Fourier series and boundary value problems; Mr. Zirakzadeh, fundamental concepts of geometry (first term); Mr. Smith of St. Louis Public Schools,

teaching of secondary school mathematics, mathematics workshop in teaching problems (first term).

University of Florida. June 11 to August 7: Professor Cowan, Fourier series; Professor Ellis, theory of groupoids and/or foundations of mathematics; Professor Gormsen, foundations of geometry; Professor Hadlock, advanced topics in calculus; Professor Hutcherson, differential geometry; Professor Kokomoor, history of mathematics; Professor Lang, calculus of variations; Professor Moore, theory of groups of finite order; Professor Phipps, vector analysis; Professor Pirenian, advanced college geometry; Professor Rohde, tensor analysis.

University of Illinois. June 18 to August 14: Professor Bateman, theory of rings; Professor Schoenfeld, measure and integration.

University of Michigan. June 21 to August 13: Professor Bartels, intermediate differential equations and Fourier analysis; Mr. Bicknell, mathematics of life insurance; Professor Coe, mechanics; Professor Craig, analytic theory of frequency functions; Professor Darling, theory of probability and theory of mathematical statistics; Professor Hay, plates and shells and vector analysis; Professor Hildebrandt, linear integral equations; Professor Jones, history of geometry, teaching of collegiate mathematics; Professor Leisenring, modern geometry; Professor Lyndon, structure of rings and theory of matrices; Dr. McLaughlin, analytic projective geometry; Professor Myers, normed algebras; Professor Reade, operational mathematics and functions of complex variable with applications; Professor Samelson, functions of real variable and general spaces; Professor Thrall, higher algebra; Professor Young, foundations of mathematics.

University of Minnesota. June 14 to July 17: Professor Wilcox, intermediate calculus, vector analysis; Professor Kalisch, theory of numbers, elementary theory of summability; Professor Loud, calculus of finite differences, integral equations. July 19 to August 21: Professor Koehler, advanced calculus; Mr. Slye, mathematical reasoning and theory of equations; Professor Gelbaum, mathematics of transient analysis, intermediate differential equations.

University of Nebraska. June 8 to July 30: Professor Doole, advanced euclidean geometry; Professor Basoco, differential equations; Professor Leavitt, solid analytic geometry; Professor Halfar, topics in analysis.

University of North Carolina. June 10 to July 17: Professor Winsor, introduction to higher geometry; Professor Jones, fundamental concepts; Professor Mackie, theory of equations; Professor Linker, differential equations; Professor MacNerney, topics in analysis; Professor Brauer, elementary theory of numbers. July 19 to August 25: Professor Garner, history of mathematics; Professor Hill, elementary mathematical statistics; Professor Lasley, elementary analytic geometry from a higher standpoint; Professor Hoyle, advanced calculus.

Twelve fellowships of $225 plus tuition and fees, provided by a grant from the E. I. du Pont Company, are available for qualified secondary school teachers in the Southeast.

University of Oklahoma. June 14 to August 9: Professor Bernhart, college geometry, vector analysis; Professor Huff, elementary differential equations; Professor Lafon, ordinary and partial differential equations; Professor Brixey, theory of numbers; Professor Foote, advanced partial differential equations.

University of Pittsburgh. June 7 to July 16 and July 19 to August 27: Professor Blumberg, advanced calculus; Professor Levine, modern algebraic theories; Professor Taylor, geometry of the complex domain; Professor Myers, algebraic geometry; Professor Laush, differential equations (first term), integral equations (second term); Professor Bryson, Laplace transform theory and applications (first term), differential equations (second term). June 14 to August 6 (evenings): Professor Christiano, differential equations for engineering students; Professor Bryson, vector analysis. June 28 to August 6: Professor Teats, geometry for teachers, history of mathematics; Professor Edwards, mathematical theory of statistics; Professor DeSua, recreational mathematics for teachers; Professor Laush, theory of equations.

University of South Carolina. June 15 to August 17: Professor Williams, vector analysis; Professor Hedberg, theory of numbers, theory of equations; Professor Collins, college geometry.

University of Washington, June 21 to August 20: Professor Beaumont, linear algebra; Professor Jerbert, differential equations; Professor Avann, vector analysis; Professor Cramlet, topics in applied analysis; Professor Dekker, foundations of geometry; Professor Aggarwal, statistical inference in applied research.

University of Wisconsin. June 28 to August 20: Professor MacDuffee, abstract algebra; Professor Thomas of Duke University, theory of integral equations, advanced topics in complex variable theory.

University of Wyoming. June 14 to July 16: Professor Barr, seminar in geometry; Professor S. R. Smith, vector analysis, numerical analysis; Professor W. N. Smith, ordinary differential equations; Professor Steen, theory of equations, college geometry. July 16 to August 20: Professor Schwid, partial differential equations, theory of functions of a complex variable; Professor Varineau, higher algebra, advanced calculus, fundamental concepts of mathematics.

West Virginia University. June 2 to July 13: Professor Stewart, differential equations; Professor Vest, Fourier series, partial differential equations; Professor Mamelak, introduction to modern algebra (I). July 14 to August 20: Professor Vehse, calculus of variations; Professor Peters, introduction to modern algebra (II).

PERSONAL ITEMS

Alabama Polytechnic Institute announces the following: Assistant Professors Ernest Ikenberry, and W. A. Rutledge have been promoted to associate professorships; Mr. J. C. Wilson of Louisiana State University and Dr. N. C. Perry of San Jose State College have been appointed to assistant professorships.

Bucknell University reports the following: Professor C. H. Richardson has retired from the position of Chairman of the Department of Mathematics, but is continuing as Professor of Mathematics; Professor W. I. Miller has been elected Chairman of the Department.

At Lehigh University: Dr. R. C. Carson and Mr. S. L. Gulden have been appointed to instructorships; Mr. J. F. Burke and Mr. T. F. Green have been appointed to graduate assistantships.

Ohio University announces: Mr. Jack Elliott, previously a graduate assistant at Michigan State College, Miss Beverly Ferner, and Mr. Floyd Poole, formerly of the Bendix Research Division, have been appointed to instructorships; Instructor Andrew Sterrett is now at Denison University.

University of Pennsylvania announces the following: Dr. Murray Gerstenhaber, formerly at the Institute for Advanced Study, has been appointed to an assistant professorship; Associate Karl Zeller of the University of Tübingen, Germany has been appointed Visiting Lecturer for the year 1953–54.

Dr. Don Alkire of the University of South Dakota has been appointed Instructor and Departmental Supervisor of Teacher Training at Fresno State College.

Dr. M. L. Anthony, formerly a research engineer with Armour Research Foundation, Chicago, Illinois, has accepted a position as Senior Research Engineer with Chicago Midway Laboratories, Illinois.

Adjunct Professor W. E. F. Appuhn of Polytechnic Institute of Brooklyn is Visiting Professor of Mathematics in the College of Pharmacy, St. John's University, for the second semester of 1953–54.

Mr. C. R. Berndtson, formerly with Nuclear Development Associates, White Plains, New York, is teaching at Vinalhaven High School, Maine.

Professor J. G. Bowker of Middlebury College has been named Dean of the Faculty.

Mr. C. L. Bradshaw has accepted a position as a mathematical analyst with Lockheed Aircraft, Marietta, Georgia.

Mr. J. A. Carpenter, previous with Snow and Schule, Inc., Cambridge, Massachusetts, has a position as Mathematician with Ultrasonic Corporation, Cambridge, Massachusetts.

Mr. J. J. Chakalis, recently a student at Northeastern University, has accepted a position as an oceanographer with the Division of Tides and Currents, Department of Commerce, Washington, D. C.

Mr. N. H. Choksy has received an appointment as a research assistant in the College of Engineering, University of Wisconsin.

Miss Laura E. Christman has retired from her position at Senn High School, Chicago, Illinois.

Dr. J. H. Curtiss, formerly of the National Bureau of Standards, Washington, D. C., has been appointed Senior Scientist in the Institute for Mathematical Sciences, New York University.

Reverend F. T. Daly has been appointed to the staff of Regis College, Denver, Colorado.

Miss Carolina D. del Mar, formerly a graduate student of St. Louis University, has been appointed to an instructorship at the University of San Carlos, Cebu City, Philippines.

Miss Betty C. Detwiler, previously a student at Kentucky Wesleyan College, has been appointed to a graduate assistantship at Washington University.

Mr. W. C. Dixon, formerly at the United States Naval Proving Ground, Dahlgren, Virginia, has been appointed Research Associate in the Computation Laboratory, Wayne University.

Mr. F. W. Donaldson, who has been a staff member of the Los Alamos Scientific Laboratory, has accepted a position as a senior aerophysics engineer with the Consolidated Vultee Aircraft Corporation, Ft. Worth, Texas.

Mr. F. A. Downing of North Carolina Utilities Commission, Raleigh, has accepted a position with P. H. Hanes Knitting Company, Winston-Salem, North Carolina.

Dr. William H. Durfee, previously at the National Bureau of Standards, Washington, D. C., has accepted a position in the Operations Research Office, Chevy Chase, Maryland.

Professor J. C. Eaves of Alabama Polytechnic Institute has been appointed Head of the Department of Mathematics of the University of Kentucky.

Assistant Professor Bernard Epstein is on leave of absence from the University of Pennsylvania for the year 1953–54 and is engaged as a research associate at the Institute of Mathematical Sciences, New York University.

Dr. W. S. Ericksen of the United States Forest Products Laboratory has been appointed to a professorship at the United States Air Force Institute of Technology.

Assistant Professor Trevor Evans of Emory University is on leave of absence and has been appointed a research associate at the University of Chicago.

Professor Tomlinson Fort of the University of Georgia has been appointed to a professorship at the University of South Carolina, effective September, 1954.

Mr. T. L. Glahn has accepted a position as a technical engineer with General Electric Company, Evendale, Ohio.

Dr. M. J. Gottlieb of the University of Chicago has been appointed Director of Research, Market Facts, Inc., Chicago, Illinois.

Mr. Donald Greenspan has returned to his position as an instructor at the University of Maryland after a period of military service.

Mr. R. L. Gulley has a position as a mathematician with the Bureau of Ships, Navy Department, Washington, D. C.

Mr. H. H. Harman, formerly in the Statistical Research and Analysis Unit, Department of the Army, Washington, D. C., is engaged now as Social Scientist with the Rand Corporation, Santa Monica, California.

Assistant Professor A. R. Harvey, San Diego College, has been promoted to an associate professorship.

Associate Professor E. E. Haskins of Fenn College has been appointed Professor and Head of the Department of Physics, Norwich University.

Mr. P. S. Herwitz has accepted a position as a research associate at the Institute for Cooperative Research, Johns Hopkins University.

Mr. Ernest Johnston has been appointed to the staff of Wisconsin State College, Whitewater.

Associate Professor T. L. Jordon of Wofford College is now a staff member of the Los Alamos Scientific Laboratory, Los Alamos, New Mexico.

Assistant Professor M. L. Keedy of North Dakota State College has been appointed to an instructorship at the University of Nebraska.

Mr. B. C. Kenny of Lehigh University has accepted a position as a mathematician with the Vitro Corporation of America, Florida.

Mr. Rolando Lara, formerly a graduate student at the University of Oklahoma, has a position with the Seismograph Service Corporation, New Orleans, Louisiana.

Dr. L. S. Laws, recently a graduate assistant in the School of Education, Michigan State College, has been appointed Professor and Dean-Registrar at Southwestern College.

Dr. Walter Leighton, Jr., professor and chairman of the Department of Mathematics, Washington University, and chief of the Mathematics Division, Office of Scientific Research, United States Air Force, has been appointed Professor and Head of the Department of Mathematics, Carnegie Institute of Technology, effective July 1, 1954.

Dr. T. M. Little has accepted a position as Extension Vegetable Specialist at the University of California, Riverside.

Mr. T. C. Littlejohn has a position as a computing analyst with Douglas Aircraft Company, Long Beach, California.

Assistant Professor G. G. Lorentz of the University of Toronto has been appointed to a professorship at Wayne University.

Mr. L. I. Lowell, formerly with Boeing Airplane Company, Seattle, Washington, has accepted a position as a research analyst with Northrop Aircraft, Hawthorne, California.

Mr. J. E. McGaughy, previously a lecturer at Columbia University, is with the United States Civil Service, Eglin Air Force Base, Florida.

Mrs. Ann R. Merzbacher has been appointed to an instructorship at Duke University.

Associate Professor W. H. L. Meyer, Jr., of the College of the University of Chicago, is spending the year 1953–54 as Visiting Associate Professor at the University of California.

Dr. Mabel D. Montgomery of the University of Buffalo has been appointed Supervisor of Credentials, Office of the Registrar.

Mr. P. S. Null has accepted a position with the Elliott Company, Ridgway, Pennsylvania.

Mr. V. D. Nyhoff is teaching at Beloit High School, Kansas.

Mr. E. M. Olson, previously lecturer at Columbia University, is engaged as a mathematician with the Ballistics Research Laboratories, Aberdeen Proving Ground, Maryland.

Dr. A. M. Peiser, formerly chief mathematician with Hydrocarbon Research, Inc., is engaged now in private consulting, with offices in Wantagh, New York.

Mr. P. C. Rapp of Bell Aircraft Corporation, Niagara Falls, New York, has been promoted to the position of Dynamicist.

Assistant Professor P. M. Pepper of Ohio State University has been promoted to an associate professorship.

Mr. V. G. Ryan of St. Joseph's College has been promoted to an assistant professorship.

Mr. Charles Saltzer of Case Institute of Technology has been promoted to an assistant professorship.

Associate Professor Abraham Seidenberg of the University of California has held a Guggenheim Fellowship during the year 1953–54.

Professor H. G. S. Sharma of Tingaraj College, Balgaum, India, is now at Siddaganga Samscrit College, Mysore, India.

Mr. Malcolm Smith, formerly a mathematician at Wright Air Development Center, Dayton, Ohio, has accepted a position as a senior engineer with Cook Research Laboratories, Chicago, Illinois.

Mr. R. K. Smith, previously with American Cyanamid Company, Idaho Falls, Idaho, has accepted a position as an engineer with Boeing Airplane Company, Seattle, Washington.

Miss Margaret O. Taylor, formerly a mathematician with Gulf Research and Development Company, Pittsburgh, Pennsylvania, has accepted a position with Continental Oil Company, Ponca City, Oklahoma.

Dr. Feodor Theilheimer, who has been at the Naval Ordnance Laboratory, Silver Spring, Maryland, has a position with David Taylor Model Basin, Carderock, Maryland.

Mr. V. E. Thomas, previously a graduate student at the University of Massachusetts, has been appointed to an instructorship at West Virginia University.

Mr. W. E. Timon, Jr., has been appointed to an instructorship at Louisiana State University.

Mr. J. D. Tupac, formerly at the United States Naval Air Missile Test Center, Pt. Mugu, California, is now with the Rand Corporation, Santa Monica, California.

Mr. Eugene Usdin, who has been a research engineer with Stanolind Oil and Gas Company, Tulsa, Oklahoma, is now with the Southwestern Computing Service, Tulsa, Oklahoma.

Dr. Jack Warga, recently with Republic Aviation Corporation, Farmingdale, New York, has accepted a position with Consolidated Engineering Corporation, Pasadena, California.

Mr. E. H. Weiss, who has been with the Bureau of the Census, Washington, D. C., is now with the Engineering and Research Corporation, Riverdale, Maryland.

Assistant Professor R. E. Wheeler of Florida State University has been appointed to an assistant professorship at Howard College.

President W. E. Wilson of South Dakota School of Mines and Technology has been appointed Director of the Engineering Sciences Division, Office of Ordnance Research, Durham, North Carolina.

Mr. W. H. Winnis, formerly a student at Iona College, is now a junior actuary with the Union Labor Life Insurance Company, New York City.

Mrs. Elizabeth S. Wolf of the University of South Carolina has been appointed to an instructorship at Indiana Technical College.

Assistant Professor G. N. Wollan of Memphis College has been appointed to an assistant professorship at Purdue University Center, Ft. Wayne, Indiana.

Mr. John Woodward, previously a research assistant at the University of Georgia, is now with the Surveillance Laboratory, Aberdeen Proving Ground, Maryland.

Mr. Paul Yacynych, formerly a student at the University of Pittsburgh, has accepted a position as an electro-mechanical engineer with the Glen L. Martin Company, Baltimore, Maryland.

Dr. J. J. Corliss, chairman of the Mathematics Department, Chicago Undergraduate Division, University of Illinois, died on September 28, 1953. He had been a member of the Association for twenty-three years.

Assistant Professor Jan Kalicki of the Department of Philosophy, University of California at Berkeley, died on November 25, 1953.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

REPORT OF THE TREASURER FOR THE YEAR 1953

Following is a summary of the report of Professor H. M. Gehman as Treasurer of the Association for the year 1953. The complete report has been approved by the Finance Committee and accepted by vote of the Board of Governors. Any member of the Association who wishes the complete report of the Treasurer may obtain it by writing to the office of the Association.

The Board of Governors has authorized the transfer of $500 from the General Fund to the Chauvenet Fund, expecting that hereafter payments of the

Chauvenet prize may be made from income alone.

In connection with the report of the Treasurer, it should be noted that up to the end of 1953 no payments had been made toward the expenses of the Combined Membership List. It is expected that these payments will cause a substantial deficit in the Current Fund of the Association during 1954.

I. TOTAL FUNDS OF THE ASSOCIATION ON JANUARY 1, 1953

M & T Trust Co., Buffalo	$ 9,370.81	Current Fund	$ 9,370.81
Securities	82,355.00	Carus Fund	14,799.23
		Chace Fund	12,520.89
		Houck Fund	8,614.53
		Chauvenet Fund	700.40
		Dunkel Fund	14,225.95
		General Fund	31,494.00
	$91,725.81		$91,725.81

II. CURRENT FUND

Balance, January 1, 1953	$ 9,370.81	MONTHLY	
Dues	20,947.56	Publication	$16,512.90
Initiation fees	890.00	Reprints (net)	284.29
Subscriptions	4,355.76	Editor's Office	1,747.32
Sale of back numbers	1,362.77	Secretary-Treasurer's Office	
Advertisements	3,300.98	Clerical help	7,548.22
Contributions	2.00	Postage and printing	1,826.95
Sale of exchange periodicals	55.50	Office expense	478.03
Interest on General Fund	1,209.83	Bank fees	133.82
Income from Hardy Fund	120.00	Auditing fee	50.00
Charges against Funds	158.70	Board of Governors	2,073.73
Transfer from General Fund	1,360.66	Meetings	447.13
		Committees	456.02
		Subventions	451.17
		Balance, December 31, 1953	$11,124.99

	III. CARUS FUND	IV. CHACE FUND	V. HOUCK FUND
Balance, January 1, 1953	$14,799.23	$12,520.89	$ 8,614.53
Sale of publications	1,297.71	230.45	3.00
Interest	570.08	481.40	329.38
Decrease in values of securities	209.36	176.79	120.96
Less charges and bank fees	186.76	76.76	33.23
Fire Insurance	35.40	—	—
Reprinting Monograph #7	1,402.12	—	—
Balance, December 31, 1953	$14,833.38	$12,979.19	$ 8,792.72

	VI. Chauvenet Fund	VII. Dunkel Fund	VIII. General Fund
Balance, January 1, 1953	$ 700.40	$14,225.95	$31,494.00
Cash from Dunkel Estate	—	1,855.22	—
Interest	28.50	547.90	—
Transfer from General Fund	500.00	—	—
Decrease in value of securities	10.47	201.21	444.30
Less charges and bank fees	2.85	54.77	—
Award of 1953 Chauvenet Prize	100.00	—	—
Transfer to Chauvenet Fund	—	—	500.00
Transfer to Current Fund	—	—	1,360.66
Balance, December 31, 1953	$ 1,115.58	$16,373.09	$29,189.04

IX. Total Funds of the Association on December 31, 1953

Current Fund	$11,124.99	M & T Trust Co., Buffalo	$11,124.99
Carus Fund	14,833.38	Securities	83,283.00
Chace Fund	12,979.19		
Houck Fund	8,792.72		
Chauvenet Fund	1,115.58		
Dunkel Fund	16,373.09		
General Fund	29,189.04		
	$94,407.99		$94,407.99

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 80 persons have been elected to membership by the Board of Governors on applications duly certified.

M. Katherine Alexander, M.A. (Alabama) Teacher, George S. Gardiner High School, Laurel, Miss.

Gloria Andekian, M.A. (Wisconsin) Research Assistant, Radiation Laboratory, Johns Hopkins University.

Jerome Burtt, M.A. (Columbia) Asso. Prof., Tri-State College.

F. C. Calabrese, M.A. (Massachusetts) Junior Development Engineer, Goodyear Aircraft Corp., Akron, Ohio.

R. C. Carson, Ph.D. (Wisconsin) Instr., Lehigh University.

W. F. Cassidy, Ph.D. (Fordham) Prof., St. John's University, Brooklyn, N. Y.

D. L. Clark, B.A. (Eastern Washington College of Ed.) Grad. Assistant, Oregon State College

C. N. Cochran, M.S. (West Virginia) Instr. West Virginia University.

R. E. Cummings, M.S. (Florida State) Mathematician, U. S. Navy Mine Countermeasures Station, Panama City, Fla.

Rev. J. F. Daly, S.J., M.S. (St. Louis) Instr., St. Louis University.

R. A. Dean, Ph.D. (Ohio State) Lieutenant, U. S. Navy.

D. B. Edelen, Instr., McCoy College, Johns Hopkins University.

F. P. Frary, B.S. (St. John's) Asst. Project Engineer, Sperry Gyroscope Co., Great Neck, N. Y.

David Gans, Ph.D. (New York) Asst. Prof., New York University.

A. M. Glicksman, M.A. (Columbia) Teacher, Bronx High School of Science and Polytechnic Institute of Brooklyn.

MARION GODDARD, M.A. (Cornell) Asst. Prof., Elmira College.

W. T. HAMILTON, M.S. (Columbia) Instr., Long Island University.

W. A. HAWKINSON, M.A. (Appalachian S.T.C.) Asst. Prof., Appalachian State Teachers College.

W. A. HEINLY, M.A. (Pittsburgh) Prof., Potomac State College.

S. A. HOFFMAN, B.A. (Pennsylvania) Grad. Student, University of Pennsylvania.

S. P. HOFFMAN, JR., M.A. (Yale) Asst. Prof., Polytechnic Institute of Brooklyn.

W. W. HOOKER, Student, Harvard University.

E. C. HUBBARD, M.S. (Western Illinois S.C.) Mathematician, U. S. Naval Proving Ground, Dahlgren, Va.

C. E. HUMPHREY, Ph.D. (Texas) Research Psychologist, Johns Hopkins University.

STEPHEN JAUREGUI, JR., Student, University of California.

BERNARD KATZ, B.A. (Brooklyn) Design Engineer, Hydro-Aire, Burbank, Calif.

R. B. KRIEGH, M.A. (Nebraska) Instr., University of Colorado.

STEPHEN KRULIK, Student, Brooklyn College.

JOSEPH KRUSKAL, JR., M.S. (Chicago) Student, Princeton University.

A. J. LEINO, A.B. (Fresno S. C.) Grad. Student, University of California.

NORMAN LEVINE, Ph.D. (Ohio State) Asst. Prof., University of Pittsburgh.

W. E. MALBON, M.A. (Virginia) Instr. and Grad. Student, University of Virginia.

A. D. MARTIN, Ph.D. (Washington U.) Instr., Oberlin College.

I. A. MCCOLLUM, M.A. (Northwestern) Teacher, North Carolina College.

MYLES MCCONNON, Ph.D. (Pittsburgh) Head of the Department of Mathematics, Norwich University.

D. O. MCKAY, M.A. (Buffalo) Instr., University of Buffalo.

J. H. MCKAY, Ph.D. (U. of Washington) Instr., University of Washington.

O. T. MCMILLAN, M.A. (Michigan) Instr., General Motors Institute, Flint, Mich.

R. E. MESSICK, M.S. (Illinois) Grad. Student, Carnegie Institute of Technology.

IRENE P. MONAHAN, M.S. (Illinois) Asst. Prof., Keuka College.

D. E. MYERS, B.S. (Kansas S.C.) Grad. Assistant, Kansas State College.

D. J. NEWMAN, PH.D. (Harvard) Research Mathematician, Republic Aviation Corp., Farmingdale, N. Y.

L. A. ONDIS, II, B.S. (Ohio) Junior Scientist, Westinghouse Electric Corp., Pittsburgh, Pa.

ISADOR PARDO, M.A. (Wisconsin) Mathematician, U. S. Naval Aviation Supply Depot, Philadelphia, Pa.

E. J. PELLICCIARO, Ph.D. (North Carolina) Instr., University of Delaware.

P. A. PENZO, B.S. (Youngstown) Grad. Assistant, University of Pittsburgh.

W. A. PIERCE, Ph.D. (Harvard) Asst. Prof., Syracuse University.

R. S. PIETERS, M.A. (Princeton) Instr., Phillips Academy, Andover, Mass.

EMILY C. PIXLEY, Ph.D. (Chicago) Asso. Prof., University of Detroit.

S. I. PLOTNICK, M.E.E. (Delaware) Director of Research, Mathematics Research, Inc., State College, Pa.

HANS RADEMACHER, Ph.D. (Göttingen) Prof., University of Pennsylvania.

H. L. RESNIKOFF, Student, M.I.T.

FILOMENA R. REYES, B.S.E. (Holy Ghost College, Manila) Quezon City, Philippines.

MRS. DOROTHEA J. RHEA, B.A. (Bryn Mawr) Grad. Student, Bryn Mawr College.

R. J. ROHLFS, M.A. (South Dakota) Instr., University of Idaho.

MARABETH ROLLINS, Student, Texas Christian University.

P. T. RYGG, M.S. (Iowa S.C.) Analyst, Department of Defense, Washington, D. C.

L. W. SCHMIDT, M.S. (West Coast) Lab. Engineer, Hydro-Aire, Inc., Burbank, Calif.

R. I. SCIBOR-MARCHOCKI, B.S. (Wayne) Electrical Engineer, Hoffman Laboratories, Los Angeles, Calif.

R. C. SCOTT, B.A. (Amer. International C.) Grad. Student, University of Massachusetts.

R. G. SEYLER, Student, Pennsylvania State University.

HENRY SHARP, JR., Ph.D. (Duke) Asst. Prof., Georgia Institute of Technology.

JEROME SHERMAN, B.Ch.E. (C.C.N.Y.) Engineer, Babcock & Wilcox Co., Research Center, Alliance, Ohio.

NATHAN SHKLOV, M.A. (Toronto) Special Lecturer, University of Saskatchewan, Canada.
JAMES SIAGAS, B.A. (New York) Research Assistant, New York University.
H. R. SMITH, Student, Johns Hopkins University.
J. J. SOPKA, Ph.D. (Harvard) Mathematician, Johns Hopkins University.
ARTHUR STEGER, M.A. (California) Instr., University of New Mexico.
R. A. STOKES, M.A. (Mississippi) Instr., University of Mississippi.
N. S. THOMPSON, M.A. (Columbia) Instr., Manhattan College.
R. G. THOMPSON, M.A. (Nebraska) Instr., University of Colorado.
R. H. THOMPSON, Student, M.I.T.
C. J. TREMBLAY, M.A. (Southern California) Asst. Prof., Bard College.
J. R. VAN ANDEL, M.A. (Michigan) Asso. Research Engineer, Burroughs Corp., Philadelphia, Pa.
MARCELLE M. WALKER, M.S. (Virginia S.C.) Instr., Virginia State College.
J. W. WELLAND, U. S. Air Force.
C. R. WHITE, M.A. (Howard) Prof., State A & M College, Orangeburg, S. C.
F. H. M. WILLIAMS, M.A. (Pennsylvania) Prof., Drexel Institute of Technology.
C. S. WOLFE, M.S. (Arizona) Asst. Prof., Shepherd College, Shepherdstown, W. Va.
W. B. WOOLF, B.A. (Pomona) Grad. Student, Claremont Graduate School.

THE APRIL MEETING OF THE KANSAS SECTION

The thirty-eighth annual meeting of the Kansas Section of the Mathematical Association of America was held at Washburn University, Topeka, Kansas, on April 11, 1953. Professor P. M. Young, Chairman of the Section, presided at both the morning and afternoon sessions.

One hundred thirty persons were present including the following forty-one members of the Association:

R. W. Babcock, Wealthy Babcock, Florence L. Black, Jeneva J. Brewer, E. L. Dubowsky, Paul Eberhart, W. C. Foreman, L. E. Fuller, Albert Furman, W. H. Garrett, F. C. German, Laura Z. Greene, Georgia M. Haswell, Sabrina M. Hecht, C. V. Holmes, H. V. Huneke, W. C. Janes, J. L. Kelley, L. E. Laird, C. F. Lewis, Anna Marm, Margaret E. Martinson, Thirza A. Mossman, Agnes E. Nibarger, S. T. Parker, P. S. Pretz, G. B. Price, C. B. Read, L. M. Reagan, R. G. Sanger, Sister M. Nicholas, F. B. Sloat, G. W. Smith, R. G. Smith, E. C. Stopher, E. B. Stouffer, Wilmont Toalson, C. B. Tucker, A. E. White, Ferna E. Wrestler, P. M. Young.

At the business session the following officers were elected: Chairman, Professor W. C. Foreman, Baker University; Vice-Chairman, Professor E. C. Stopher, Fort Hays Kansas State College; Secretary-Treasurer, Professor Laura Z. Greene, Washburn University.

The program consisted of the following papers:

1. *Topology for the undergraduate*, by Professor J. L. Kelley, University of Kansas.

By using elementary notions on the homotopy of paths, correct and simple proofs of the basic results of complex function theory are possible. It is not necessary to assume or prove the Jordan curve theorem. As an example of technique, the fundamental theorem of algebra may be proved as follows: Suppose $p(z)=\sum_{k=0}^{n}a_kz^{n-k}$ is a polynomial without roots and that $a_0\neq 0$. For $r\geqq 0$ let $\phi(r)$ be the integral of $p'(z)/p(z)$ along the path re^{it}, for $0\leqq t\leqq 2\pi$. Letting $q(z)=zp'(z)/p(z)$, $\phi(r)=i\int_0^{2\pi}q(re^{it})dt$ and $\phi'(r)=i\int_0^{2\pi}q'(re^{it})e^{it}dt=q(re^{it})i/r]_{t=0}^{2\pi}=0$. Hence ϕ is constant, and since $\phi(0)=0$, ϕ is identically zero. However, as $r\to\infty$, $\phi(r)$ approaches $i\int_0^{2\pi}ndt=2\pi in$, which is a contradiction.

2. *Some problems in the teaching of advanced calculus*, by Professor R. G. Smith, Kansas State Teachers College.

This paper considers two problems in the teaching of advanced calculus: Euler's theorem and the definite integral.

1) If $e^{i\theta}$ is defined to be the limiting value of $\zeta=(1+i\theta/n)^n$, it is a simple exercise to show that mod ζ approaches 1 and arg ζ approaches θ as n increases without limit. This method of proof has an advantage over most methods in that it is associated with a simple geometrical representation in the plane of complex numbers.

2) Although the definite integral is defined in most texts in terms of the limiting value of an infinite series—few authors give any examples or exercises to illustrate the evaluation of the integral as so defined. Several examples with equal or unequal subintervals were here presented, each resulting in a well known series.

3. *Solutions of systems of linear congruences*, by Professor L. E. Fuller, Kansas State College.

Systems of linear congruences with integral coefficients modulo a power of a prime are considered. These congruences are assumed to be already in a form where the coefficient matrix is a modification of the ordinary triangular form for linear equations. In this form, the determinant of the coefficient matrix is the product of the diagonals. Using the augmented coefficient matrix, it is shown that if the constants are multiples of the diagonal elements of their rows, then the number of solutions is exactly equal to the product of the diagonal elements where any zero is replaced by the modulus.

4. *A survey of the certification requirements in mathematics for elementary and high school teachers*, by Sister Mary Nicholas, Marymount College.

In this paper, the deficiency in elementary mathematics of college freshmen was pointed out. The fact that no mathematics is required for the certification of elementary and high school teachers in some states led to the study of the certification requirements of the forty-eight states, Hawaii, and Puerto Rico. It was found that mathematics is required for elementary and high school certificates in only thirteen of these. Colleges may help this situation by offering courses for teachers and advising those who plan to teach to enroll in these courses.

LAURA Z. GREENE, *Secretary*

THE NOVEMBER MEETING OF THE PHILADELPHIA SECTION

The annual meeting of the Philadelphia Section of the Mathematical Association of America was held on November 28, 1953 at Drexel Institute of Technology, Philadelphia, Pennsylvania. The chairman for this meeting was Professor C. A. Nelson, New Jersey College for Women.

Sixty-five were in attendance including the following forty-four members of the Association:

R. J. Bickel, James Elmer Davis, F. L. Dennis, E. K. Dorff, E. D. Glenney, D. S. Greenstein, H. M. Gurk, V. H. Haag, Katherine E. Hazard, J. R. Holzinger, R. F. Jackson, P. B. Johnson, C. E. Kerr, R. W. Klopfenstein, H. W. Kuhn, V. V. Latshaw, W. S. Lawton, Marguerite Lehr, K. L. Loewen, Harold Luxenberg, A. W. Mall, Helen M. Marston, B. H. McCandless, Edith A. McDougle, S. S. McNeary, C. A. Nelson, C. O. Oakley, J. C. Oxtoby, C. F. Pinzka, M. A. Rader, G. E. Raynor, B. E. Rhoades, Pincus Schub, C. B. Sensenig, C. A. Shook, E. P. Starke, Alexander Tartler, G. L. Thompson, A. W. Tucker, R. M. Walter, G. C. Webber, D. W. Western, Albert Wilansky, H. M. Zerbe.

A short business meeting was held following the luncheon at which the following officers were elected for the ensuing year: Chairman, Professor Alexander Tartler, Drexel Institute of Technology; Secretary, Professor G. C. Webber, University of Delaware. The following members of the Program Committee were elected: Chairman, Professor H. W. Kuhn, Bryn Mawr College; Professor F. L. Dennis, Ursinus College; Dr. Harold Luxenberg, Remington Rand, Philadelphia. It was voted that the executive committee of the Section should consist of the chairman, the secretary, the program chairman and the sectional governor.

The program for this meeting consisted of the following papers:

1. *Linear equations and inequalities: solvability versus inconsistency*, by Professor H. W. Kuhn, Bryn Mawr College.

Basic to all questions of the existence of solutions for systems S of linear equations in unknowns $x_1, \cdots, x_n$,is the elementary result: the process of elimination either yields a solution or exhibits an inconsistent equation $0x_1 + \cdots + 0x_n = d$ (with $d \neq 0$) as a linear combination of the equations of S. This same result holds with natural modifications for systems in which inequalities (strict and ordinary) may appear. Its applications embrace many special cases including the existence of positive solutions to linear equations, equality theorems for linear programming, and the fundamental theorem of matrix game theory.

2. *Logical development of knot theory*, by Professor R. H. Fox, Princeton University, introduced by the Secretary.

The question "What is the mathematical definition of a knot?" and "When are two knots called equivalent?" may be answered in various ways. Examples, illustrating the difference between some of the possible answers, were considered, and those which seemed to approximate the pre-mathematical intuition most closely were adopted.

The principal means of distinguishing between inequivalent knots are (1) the fundamental group G of the residual set (the so-called group of the knot) and (2) the coverings of 3-space branched over the knot. Construction of an orientable surface spanning the knot was sketched, and thereby certain conjugate classes, the meridian class and the longitude class, of G were selected. Two knots are equivalent only if their groups are isomorphic, and their preferred conjugate classes correspond under the isomorphism. Calculation of the Alexander polynomial of G by means of the free calculus was sketched, and the overhand knot, the figure eight knot and the trivial knot were thereby shown to be equivalent.

3. *Mathematical techniques used in economics theory*, by Professor Jan Tinbergen, Haverford College, introduced by the Secretary.

As far as economics is a quantitative science, mathematical treatment makes sense. It started in 1838 (Cournot) but was stagnant for quite some time. An important upswing began in 1930 (The Econometric Society).

Statistical economic systems are characterised by a large number of variables, related to each other by a certain number of balance, technical and institutional equations which are the side conditions under which the economic subjects are maximizing their "utility functions" (Wabras, Pareto). Wald gave an existence proof for a solution under specified conditions.

Dynamics trys to explain fluctuations in economic variables by a combination of exogenous and endogenous factors. In order to interpret fluctuations entirely endogenously one may make use of differential equations, difference equations or mixed equations. A well-known attempt is the

so-called acceleration principle, which applies to stocks: the rate of increase in stocks, or the investment in stocks move parallel to the rate of increase in sales. In order to obtain second-order or higher differentials in the equation (a prerequisite for periodic solutions) however, one has to consider a succession of production processes (Chait). Fixed lags between causes and effects lead to difference equations. Various types of mixed equations are used (Fisch, Kalecki).

Recent tendencies are to work with large numbers of linear equations and inequalities as boundary conditions (Koopmans, Leontief). Sometimes abstract logistics have been used (Arrow). Where uncertainty comes in, these techniques are combined with probability theory (Morgenstern and Von Neumann).

4. *A new approach to freshman mathematics*, by Professor C. O. Oakley, Haverford College.

This paper consisted of a report on the Haverford College freshman course developed over the past seven years. The subject matter includes logic, groups, the number system, fields, functions, analytic geometry, calculus of polynomials, probability and statistics. Partial results indicate that students (a) even at the freshman level, handle abstract mathematics just as readily as they do the concrete, (b) find immense pleasure in studying modern concepts, (c) can attain a reasonable degree of accomplishment even with poor high school preparation, (d) are better prepared for further work in mathematics than they usually are following the conventional sequence of trigonometry, algebra and analytic geometry.

G. C. WEBBER, *Secretary*

THE DECEMBER MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The fall meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America was held at The George Washington University, Washington, D. C., on December 5, 1953. Professor D. C. Lewis, Jr., Chairman of the Section, presided at the morning and afternoon sessions.

There were one hundred five persons in attendance, including the following seventy-nine members of the Association:

J. C. Abbott, M. I. Aissen, D. F. Atkins, R P. Bailey, T. J. Benac, R. O. Blummer, Jr., T. A. Botts, J. W. Brace, J. H. Braun, B. H. Buikstra, W. E. Byrne, J. F. Canu, B. R. Cato, Jr., H. J. Cheston, Jr., B. H. Chovitz, C. E. Clark, G. R. Clements, G. F. Cramer, A. R. DiDonato, J. A. Duerksen, P. J. Federico, Gloria C. Ford, C. H. Frick, Michael Goldberg, R. A. Good, E. S. Grable, J. J. Greever, III, D. W. Hall, M. Gweneth Humphreys, Louise S. Hunter, W. R. Hydeman, J. E. Ikenberry, S. B. Jackson, R. D. Johnson, Jr., F. E. Johnston, L. M. Kells, F. B. Key, H. L. Kinsolving, C. F. Koehler, Karl Kozarsky, A. E. Landry, Victor Lewicke, D. C. Lewis, Jr., D. B. Lloyd, D. B. Lowdenslager, Ella Marth, M. H. Martin, E. S. Mayer, Carol V. McCamman, J. M. McLynn, E. J. McShane, Joseph Milkman, George Millman, R. W. Moller, T. W. Moore, C. H. Murphy, Jr., W. R. Murray, W. H. Norris, P. L. Oglesby, M. W. Oliphant, O. J. Ramler, R. W. Rector, J. N. Rice, J. W. Sawyer, Veryl G. Schult, Paul Shapiro, W. F. Shenton, C. H. Sisam, Sister Gabrielle Marie, Sister Mary Cordia, W. S. Soar, S. P. Spaulding, W. J. Strange, C. T. Taam, J. H. Taylor, Feodor Theilheimer, J. A. Tierney, P. M. Whitman, D. M. Young, Jr.

Professor W. H. Norris, Chairman of the High School Contest Committee, reported that the number of schools replying to the initial letter sent out by his committee indicated the possibility of having a contest in May, 1954. At the request of the committee the Section voted: (1) That the Chairman of the Section appoint a board of members at the collegiate level which shall make the

final selection of the winning papers; and (2) That the members of the Section approach their respective schools and companies in regard to contributions for prizes. The Chairman pointed out that this and future projects of the Section will be aided by including more high school teachers as members of the Association. An announcement was made regarding the program of having visiting lecturers appear before undergraduate students within the Section. Professors G. R. Clements, United States Naval Academy, B. Z. Linfield, University of Virginia, and O. J. Ramler, Catholic University of America, were appointed to nominate officers for the academic year 1954–55.

The following papers were presented:

1. *A natural approach to the fundamental theorem of the integral calculus*, by Professor J. P. Hoyt, United States Naval Academy, introduced by the Chairman.

By means of a concept from the calculus of finite differences, the evaluation of the limit of a sum as the difference of two "anti-differentials" is made intelligible to the first year calculus student.

2. *The Ballistic Research Laboratories' Geda*, by Mr. C. H. Murphy, Jr., Ballistic Research Laboratories, Aberdeen Proving Ground.

The Ballistic Research Laboratories have recently obtained an analogue computer installation almost completely composed of Geda (Goodyear Electronic Differential Analyzer) units. The installation itself as well as the way in which an analogue computer performs basic mathematical operations was described and a simple problem was solved.

3. *A remark on a result of Leighton*, by Professor Choy-tak Taam, The Catholic University of America.

If r, p and $(rp)'$ are continuous and if $rp>0$, $(rp)'\geqq 0$ on I: $a\leqq x<\infty$, then every solution of $(ry')'+py=0$ is bounded on I (W. Leighton, *Proc. Nat. Acad. of Sci.*, vol. 35, 1949, pp. 190–191). It was shown that the condition "$(rp)'$ is continuous and $(rp)'\geqq 0$ on I" may be replaced by "rp is non-decreasing on I." Further generalization was indicated.

4. *Cyclically-ordered sets and separation theorems*, by Dr. M. I. Aissen, Radiation Laboratory, The Johns Hopkins University.

The well-known separation theorems about zeros of orthogonal polynomials were improved and generalized to the case of functions satisfying certain Kth order linear difference equations.

5. *Analysis and the undergraduate*, by Professors D. W. Hall and G. L. Spencer, II, University of Maryland, presented by Professor Hall. (By invitation.)

The undergraduate advanced calculus course should be an introduction to rigorous analysis. Students entering this course are normally trained only in techniques and have little or no idea of logic or the nature of proofs. Time must be spent in remedying these deficiencies if there is to be any hope of the students understanding the fundamental theorems of analysis. One way in which these problems might be met was discussed.

C. H. Frick, *Secretary*

EMPLOYMENT OPPORTUNITIES

Memphis State College, Memphis, Tenn. Assistant professor. Prefer man with Ph.D. and some experience.

Mathematics Research, Inc., State College, Pa. Senior Applied Mathematician.

The MONTHLY is devoting this space to paid announcements of employment opportunities for mathematicians. The text of such announcements should be in want-ad form and must be in the hands of the editor (C. B. Allendoerfer, Mathematics Department, University of Washington, Seattle 5, Wash.) before the first day of the month preceding the issue in which the notice is to appear. Announcements should indicate the academic rank or similar description of the opening, but should not mention a specific salary. Blind ads are permissible which direct replies to a specific box number in care of the Mathematical Association of America, Buffalo 14, N. Y. In order to conserve space and achieve uniformity, the privilege is reserved to rearrange advertisements. Advertisers will be billed at the rate of $1.50 per line. Rates for display advertising may be obtained from the Advertising Manager.

CALENDAR OF FUTURE MEETINGS

Thirty-fifth Summer Meeting, University of Wyoming, Laramie, Wyoming, August 30–31, 1954.

Thirty-eighth Annual Meeting, University of Pittsburgh, Pittsburgh, Pennsylvania, December 30, 1954.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Marshall College, Huntington, West Virginia, May 1, 1954.
ILLINOIS, Knox College, Galesburg, May 14–15, 1954.
INDIANA, Rose Polytechnic Institute, Terre Haute, May 1, 1954.
IOWA, Iowa State College, Ames, April 30–May 1, 1954.
KANSAS
KENTUCKY, University of Kentucky, Lexington, May 8, 1954.
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA University of Maryland, College Park, May 1, 1954.
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA, Hamline University, St. Paul, May 8, 1954.
MISSOURI, University of Missouri, Columbia, May 7, 1954.
NEBRASKA, Omaha, April 24, 1954.
NORTHERN CALIFORNIA
OHIO, Ohio State University, Columbus, April 17, 1954.
OKLAHOMA, Oklahoma City University, October, 1954.
PACIFIC NORTHWEST, Reed College, Portland, Oregon, June 18, 1954.
PHILADELPHIA, Princeton University, Princeton, New Jersey, November 27, 1954.
ROCKY MOUNTAIN, Colorado Agricultural and Mechanical College, Fort Collins, April 30-May 1, 1954.
SOUTHEASTERN
SOUTHERN CALIFORNIA
SOUTHWESTERN, Arizona State College, Tempe, April 16–17, 1954.
TEXAS, Texas Technological College, Lubbock, April 23–24, 1954.
UPPER NEW YORK STATE, College for Teachers at Albany, May 1, 1954.
WISCONSIN, State Teachers College, Eau Claire, May 8, 1954.

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

FRESHMAN MATHEMATICS AS AN INTEGRAL PART OF WESTERN CULTURE

MORRIS KLINE, New York University

1. Introduction. I would like to re-open the very hackneyed question of what to teach the liberal arts student. My excuse for doing this is that the answer which has been given for generations is equally hackneyed and, what is more to the point, a totally unsatisfactory one.

Let me state at the outset that by the liberal arts student I mean the one who does not intend to use mathematics in some profession or career, and is taking mathematics because the subject is supposed to contribute to a liberal education. Though much might be said about the proper freshman courses for students who intend to continue their mathematical training beyond this first year or who may take just one year of mathematics but will have to apply it in specialized physics, chemistry, and biology courses (*e.g.*, pre-medical students), the needs of these latter two groups of students will not be discussed here. The recommendations to be made in this article concerning the liberal arts students are based on the assumption that this group can be segregated from the others. This segregation can be effected even in small colleges without creating any serious administrative problems.

As defined above, the liberal arts students constitute only one group of freshmen. However, by far the greatest percentage of freshmen belongs to this group. Also, since the segregation that is presupposed is in accordance with interests rather than ability, the liberal arts group will contain some of the most worthwhile students. For these reasons, then, this group must be given the utmost consideration. Moreover, since many of these students will become leaders in our society they will determine the fate of mathematics in some areas. Hence there are selfish reasons too for being concerned about the knowledge and impressions of mathematics which these students will acquire.

2. Objections to the traditional courses. What have we been feeding the liberal arts students? The almost universal diet has been college algebra and trigonometry. I believe that these courses are a complete waste of time. What educational values are there really in exponents, radicals, logarithms, Horner's method, partial fractions, binomial theorem, the trigonometric identities, and the law of tangents, just to mention some of the conventional topics?

Let us, in fact, examine this material in terms of the values commonly claimed as warranting its inclusion in the curriculum. One hears most often that students learn to reason by studying this material. Leaving aside the question of transfer of training, a moot point at best, I maintain that one cannot teach reasoning with this material because the material does not permit it. Every one knows that it is impossible to give freshmen a satisfactory definition of an irrational number. Hence we not only beg many questions of logic in teaching operations with irrational numbers but we compound confusion by using irrational

numbers as exponents in the subject of logarithms. Of course similar difficulties are ignored in the use of irrational numbers as trigonometric ratios associated with angles. In college algebra we use many properties of continuous functions without proof, relying upon a graph to convince the student of the correctness of these properties. For example, we use the fact that if the polynomial $f(x)$ has opposite signs for $x=a$ and $x=b$ then $f(x)=0$ has a root for a value of x between a and b. And we use the fact that $f(x)=0$ has at least one root. Even where we present proofs very few students can follow some of them. For example, how many students follow the proof of the binomial theorem for positive integral exponents, let alone fractions? When we teach partial fractions we generally omit the proofs because so much time would be required to present them correctly. Likewise when we teach the addition formulas and other identities of trigonometry we do not present proofs for all values of the variables because these proofs would take an unconscionable amount of time. To sum up, in our presentations of college algebra and trigonometry we mix proofs, intuitive arguments, graphical evidence, and assertions accepted without proof. Nevertheless, we expect the student to learn all the material. How he can come through with an appreciation of and respect for proof as exemplified by this one year of mathematics is not at all clear to me. I would say that we have confused him.

A second argument for mathematics is its beauty. There is beauty in mathematics. However, the question of what is beautiful must be answered subjectively. I contend that if I were to select portions of mathematics that are beautiful, they would not come from college algebra and trigonometry. As a matter of fact we know that these subjects are taught not because they are beautiful but because they are the next step towards higher mathematics after the high school subjects. And even if portions, at least, of college algebra and trigonometry were beautiful, it would be most difficult to sell mathematical beauty to students who are psychologically indisposed. To sell this beauty is like trying to give away ten dollar gold pieces. People are too suspicious even to examine one.

A third argument presented for the usual material is that everybody makes some use of mathematics and hence must learn some of it. But who of us has used college algebra and trigonometry outside of his professional work? There is actually nothing in these subjects which is ever used by an educated layman. Of course there are significant applications of college algebra and trigonometry but not only do these have no bearing on daily life but they occur in the context of advanced physics and mathematics so that the student cannot see where these techniques are of use even to scientists and engineers.

The truth about college algebra and trigonometry is that these subjects comprise nothing but a series of dry, boring, unmotivated, disconnected, and, to the student, unimportant techniques. The subjects are taught as techniques and the students are expected to master and reproduce them in parrot-like fashion. The entire body of material is of no value to the non-specialist and no argument for it, such as mental discipline, will ever make such material palatable

or pregnant with significance. The only argument in favor of this material is that it is readily forgotten.

There is unfortunately too much evidence to support the above evaluation of college algebra and trigonometry as courses for the liberal arts students. Those of us who have taught college freshmen know how little they get from the material. We know that we have failed to reach them and that all we succeed in doing is to intensify their dislike of mathematics. We know that they leave our courses grateful only for the fact that the year is over and vowing never again to become involved with mathematics.

Indirectly and unintentionally many of us have supplied evidence that the above evaluation is correct. A recent advertisement for one of the new college algebra texts described it as a real innovation, "a humanized approach to mathematical concepts." Algebra was supposedly presented as a language, as a logical science, and as a branch of human endeavor. It was also described as a collection of techniques and as a collection of puzzles. I could readily appreciate these last two values but was curious as to how the others might be incorporated. I therefore made haste to examine this "novel" algebra text and found exactly two and one-half pages out of more than 400 devoted to these values of algebra. There was in fact exactly one paragraph on algebra as a language, one on algebra as a logical science, and one on algebra as a branch of human endeavor. The rest of the book was the same as any one of the $4097\frac{1}{2}$ college algebras which have appeared since 1900. The point of this illustration is not that the author failed to present the values of algebra claimed in the advertisement, though, of course, he did fail. Rather, here is an author who sees the need to present these values of mathematics, who tries to do so through the medium of college algebra, and who succeeds only in showing that the material does not lend itself to the exposition of these values.

Other authors, presumably equally dissatisfied with the meaningless and poverty stricken material of college algebra and trigonometry, add historical notes. After a long chapter on the theory of equations they tell us that Descartes was born in 1596, that he died in 1650, and that he was a philosopher. Such feeble, ridiculous, and pediculous efforts to incorporate human values are more to be pitied than scorned.

By persisting in the teaching of college algebra and trigonometry to the liberal arts student, college teachers of mathematics have been committing a heinous crime against mathematics and humanity. We have been guilty of teaching brick laying instead of architecture, and color mixing instead of painting. Such teaching has discouraged latent interest in mathematics and has embittered many young people towards mathematics, and in many cases, towards all learning. More than that, it has cut the ground from underneath the subject so that it is gradually losing its place in the liberal arts curriculum. Apropos of the teaching of the conventional material is the remark Mephistopheles makes to Faust in Goethe's drama:

Is it life, I ask, is it even prudence,
To bore thyself and bore the students?

To this quotation I would add that even the devil speaks the truth sometimes.

3. Some recent innovations in freshman mathematics courses. In recent years serious attempts have been made to improve the standard offering and I should like to discuss some of these briefly. A reasonably common alternative to college algebra and trigonometry is the mathematical analysis or general mathematics course which selects material from college algebra, trigonometry, co-ordinate geometry, and the calculus. Though the attempt to give the student a somewhat broader picture of elementary mathematics can be commended, the contents of the texts that have been published cannot be, for in effect all that has been accomplished is to select techniques from four fields instead of two. Since the basic ideas underlying the fields from which the techniques are drawn are now more numerous and in the case of the calculus, more difficult, the student is left even more at sea as to what is accomplished by all these techniques. Rigor is hardly even envisaged as an objective in these courses.

Some teachers, who are quite willing to break sharply with traditional material, attempt to teach the basic and broad concepts of modern mathematics and hence concentrate on such topics as functionality, transformation and invariance, and groups, rings, and fields. These topics are indeed basic ones in mathematics and they do constitute some of the most beautiful themes. However they are far too sophisticated and abstract to mean anything to freshmen and their beauty makes no appeal to the uninitiated. Moreover, one could hardly say that these topics present a rounded view of the multifaceted role of mathematics in our civilization.

Other teachers, highly conscious of the absence of rigor in conventional college algebra and trigonometry courses, have attempted to supply it by teaching the foundations of the real and complex number systems. This material is over the heads of freshmen. Real numbers are so familiar to them that the problem of defining them and rigorously establishing their properties does not strike them as significant. Such courses must devote a great deal of time convincing students that they ought to learn what the entire mathematical world did not miss for thousands of years. Moreover, aside from the difficulties in a really rigorous approach to the irrational number, the material is not representative either of pure or applied mathematics.

To my mind some better efforts to meet the problem presented by the liberal arts student were made by Dresden of Swarthmore in his *Invitation to Mathematics* and—if I may so speak of a contribution in which I was personally involved—by Cooley, Gans, Kline, and Wahlert of New York University in their *Introduction to Mathematics*. These two books are by no means alike. Yet both indicated a willingness to break away from conventional patterns and to abandon techniques in favor of ideas. They differ in that Dresden's

book is devoted to a wide range of ideas of pure mathematics and to the presentation of rigorous proofs, whereas the second one is more descriptive, includes some applications of mathematics, and points out the significance of some of the creations in our culture. Because the style and level of Dresden's book were difficult for freshmen it did not meet wide approval. The *Introduction to Mathematics* did receive a very encouraging response as measured in terms of the number of adoptions and did stimulate other authors to write similar books. (One author liked the idea of this book so much that he took over the entire pattern and used many sections verbatim, acknowledging, of course, the inspiration of Euclid, Newton, and Einstein.) Both of these books were a step in the right direction. However, to my mind both still compromised too much with conventional patterns and failed to present adequately the significance of mathematics in our civilization.

4. The philosophy underlying the cultural approach to mathematics. Having criticized past efforts so strongly I know that the time has come to state what I think should be done. This I propose to do next. First, however, I should like to state the philosophy which guides what I propose. Actually the conviction that a particular course seems right comes first and one invents the philosophy afterwards to rationalize the conviction. Nevertheless, a wise philosophy argues eloquently for the course which abides by it.

My philosophy contains three principles. The first of these states that *knowledge is a whole* and that mathematics is a part of that whole. However the whole is not the sum of its parts. The present procedure is to teach mathematics as a subject unto itself and somehow expect the student who takes only one year of the subject to see its importance and significance for the general body of knowledge. This is like giving the student incomplete pieces of a very complicated jig-saw puzzle and expecting him to put the puzzle together. It follows from this principle that mathematics must be taught in the context of human knowledge and culture.

The principle I have stated is not at all unfamiliar or unrecognized by teachers. But either because they themselves are ignorant of the true place of mathematics in our civilization and culture or because they become too steeped in their own specialty they ignore the broad picture and expect the student to supply it.

My second principle is that *mathematics must contribute to the objectives of a liberal arts program*. Though it is difficult to state these objectives precisely one might say that they are to acclimate the student to the civilization and culture in which he lives, to increase his appreciation of what that culture contains, and to prepare him for life in his cultural environment. It follows from this principle as well as the first one that a mathematics course for liberal arts students must relate the subject to other branches of our culture.

Many teachers agree with this principle but conform to it in strange ways. They teach how to mix x pounds of coffee at 20¢ per pound with y pounds of

tea at 50¢ per pound to make 100 pounds of some unspeakable mixture. In very modern textbooks this application has been made more realistic. Coffee is one dollar per pound and tea, two dollars.

My third principle is a negative one but it seems to be necessary to include it. *Don't compromise with your objectives.* Choose material and a presentation which directly fulfill the purposes of your course. Don't let tradition dictate subject matter. Don't drag the important ideas in by the tail while actually stressing conventional techniques.

5. Sketch of the proposed course. I believe the above principles and objectives can be adhered to in a satisfactory freshman liberal arts course and I propose to sketch the outlines of such a course. Briefly described, the object of the course is to present mathematics as a major constituent of modern culture. The course treats significant ideas of mathematics and the influences of these creations on other branches of our culture.

The material is arranged in historical order. The purpose is not to present the history of mathematics. Rather, the historical order is roughly the logical order. Moreover, by following the historical order it is possible to present the circumstances which led to the creation of a mathematical idea and to show the influence it exerted. Modern civilization and culture are an accumulation, an amalgamation, and a fusion of contributions from many earlier civilizations and cultures. The historical order of events permits us to break apart the whole complex of modern ideas insofar as they relate to our subject and to examine the contributions one by one.

I shall illustrate how some topics are treated. The material begins with some facts about mathematics in Egypt and Babylonia. The primary object of this topic is to present the origins of mathematics and its uses in relatively primitive civilizations. At the same time one can emphasize the empirical nature of pre-Greek mathematics in order to prepare the student for the change which takes place in the classical Greek period. In this chapter familiar facts about number are reviewed and accepted on the same basis as the ancient peoples themselves used, namely, experience.

We then proceed to the classical Greek period (600 B.C.–300 B.C.) and emphasize the change in the nature of mathematical activity introduced by the Greeks. Abstraction, deductive proof from explicitly stated axioms, and the emphasis on geometry as opposed to algebra are the salient changes and these can be related to the nature of classical Greek thought and society. In this same period I review a few facts about Euclidean geometry and then show that the outstanding characteristics of this creation are precisely those of Greek philosophy and Greek art. In addition, the significance of Euclidean geometry for later ages as a model of rigorous reasoning is pointed out.

The next topic is the mathematics of the Alexandrian Greek period, which lasted from about 300 B.C. to about 600 A.D. The mathematics of this period

favored science and measurement. Trigonometry arose at this time as a step towards quantitative astronomy and so I teach the trigonometry of right triangles and the use of trigonometry in the measurement of the sizes and distances of the heavenly bodies. Other applications of the trigonometry of the right triangle such as navigation and the analysis of forces into components can be included.

The final topic in the Greek period centers about the Greek doctrines that nature is designed in accordance with mathematical laws and that man can fathom that design through the mastery of mathematics. The importance of Euclidean geometry and of the supreme Greek achievement in astronomy, Ptolemaic theory, as evidence for the rationality of nature, is discussed. This material also shows that mathematics made possible the first truly great astronomical theory and the first great scientific synthesis.

The transition from the Greek age to the Renaissance is made by giving a brief historical account of the state of affairs in medieval Europe, the role of mathematics in this period, and the importance for subsequent developments of the Catholic doctrine that nature is rationally designed by God and intelligible to man. It is also relevant to emphasize at this stage that a lack of interest in life in the physical world is detrimental to the growth of mathematics.

The historical movement known as the Renaissance is then sketched. The flow of Greek works to Europe gave the Europeans the opportunity to take up where the Greeks left off. The revived interest in the natural world suggested a host of problems. And the fusion of the Catholic doctrine that nature is the handiwork of God and therefore worthy of study with the Greek doctrine of the mathematical design of nature inspired the mathematicians and scientists to take up their chosen tasks with religious zeal.

Among developments of the Renaissance the first one treated is the creation and introduction of the heliocentric theory by Copernicus and Kepler. This particular creation involves no new mathematical ideas but it does give one a chance to show how, for purely mathematical reasons and despite weighty scientific and religious counter-arguments, the courses of astronomy, science, philosophy, and even religion were altered.

A second great development of the Renaissance originates with the work of the painters. These men sought to depict nature realistically and hence faced the problem of reproducing three-dimensional scenes on canvas. By applying Euclidean geometry to this problem the Renaissance artists developed the science of perspective. A few principles of this science are readily taught and the effect of this development on painting can be demonstrated by displaying and contrasting late medieval and Renaissance paintings. In the consideration of projection and section, the basic idea underlying their science of perspective, the painters raised questions which the mathematicians took over and answered by the creation now known as projective geometry. A few theorems of this

geometry can be taught partly to enlarge the student's knowledge and partly to show that a vast and vital branch of mathematics was inspired by the art of painting.

The usual material of co-ordinate geometry is the next topic. However it appears to me to be very much worth while to lead into this subject by showing how Descartes, lost in the intellectual storms of the seventeenth century, turned to mathematics to find a new approach to truth. Descartes isolated the principles of mathematical method and applied his "universal mathematics" to philosophy. He then applied the method to the study of curves and came up with a much needed method of handling all curves. The potentialities in co-ordinate geometry for the study of new curves and higher dimensional geometry can now be discussed.

Like many other courses this one emphasizes the notion of functionality. The introduction of this concept is motivated by first discussing Galileo's new view of the role of science, namely, to describe natural phenomena quantitatively. Such descriptions are best made in terms of relationships among variables; hence the importance of the function concept. However instead of using artificial and disconnected examples of functions one can use the laws of motion and gravitation. Moreover, to show that functional relations expressed as formulas can be extremely helpful to scientists one can derive new laws of motion from Newton's laws and the law of gravitation. Derivations employing only elementary algebra and yet yielding significant conclusions can be given. It is then possible to show how successful these laws and their implications were in describing motions on the Earth and in the heavens. The flight of projectiles and the motions of the planets around the sun are encompassed. So far-reaching was the range of these laws that they were called universal laws. The role of the calculus in deriving universal laws, at least insofar as the concept of instantaneous rate of change is concerned, can be included in this general topic.

The remarkable evidence provided by the universal laws of motion for a mathematically designed and lawful universe was acclaimed by all European intellectuals and inspired a rationalistic movement. Imbued with the conviction that Reason, personified by mathematics, could solve all of man's problems, the great minds of the eighteenth century undertook a sweeping reorganization of science, philosophy, religion, ethics, literature, and the social sciences. It is possible to explain how science became more dependent upon mathematics, that doctrines such as materialism and determinism were built upon mathematical and scientific foundations, that the source and nature of truths were re-examined, that new religious movements culminating in Deism resulted, and that poetry was deprecated in favor of prose. All these topics can be presented in concrete terms. In a few cases they can be assigned as outside reading.

Returning to mathematics proper one could treat next the trigonometric functions and apply them to the mathematical analysis of musical sounds. Using sound waves as a concrete analogy one could also include a somewhat loose and largely qualitative account of electromagnetic phenomena. Though the mathematical theory of these phenomena proved to be invaluable for the design of the

radio, the telephone, television, and other modern miracles of science, its value in organizing and interpreting a whole class of seemingly diverse natural phenomena is stressed. Further, since the physical nature of all electromagnetic phenomena and of radio waves in particular is completely unknown, it is possible to illustrate the meaning of the prevalent philosophical doctrine that in the last analysis our best scientific knowledge reduces to mathematical formulas. These are the nature of the physical world.

A major objective of the course is to introduce the fundamentals of statistics. However, it seems highly desirable to explain why statistical methods were sought and emphasized. Hence one might point out first that the rationalistic spirit of the Age of Reason had infused the social scientists with the desire to discover the universal laws of their domains. But the attempts to deduce these laws of man and society from *a priori* principles failed to produce realistic sciences. The social scientists then sought and created new mathematical techniques for the derivation of laws by working from statistical data. The way is now open to treat the elements of statistics and probability and to show how significant laws were derived with these techniques.

From the statistical techniques there resulted statistically likely conclusions which, nevertheless, seemed to apply as infallibly as did the mathematically deduced, necessary laws of the Newtonian era. A new philosophy, the statistical view of nature, arose to challenge the philosophy of determinism and has had grave implications for philosophy, religion, and science. These implications should be discussed.

We are now close to modern times and, in view of the rapid development of mathematics in this period, some selection of topics is forced upon us. Among many possibilities the notion of transfinite numbers and the value of this concept in providing a sound analysis of length, time, and motion seem to warrant inclusion. Even more important in this period is the remarkable creation of non-Euclidean geometry. Several modern texts now include this subject but why they avoid pointing out the implications of this creation is a mystery to me. In intellectual spheres no creation of modern times has been more revolutionary for, in effect, non-Euclidean geometry has taught us that there are no truths. Moreover, this creation caused a marked revision of our understanding of the nature of mathematics.

Without expecting to do any more than pander to popular interest one could show how non-Euclidean geometry is applied in the theory of relativity. The presentation should be, of course, purely qualitative and highly simplified.

The entire presentation closes with a discussion of the twentieth century's understanding of the nature of mathematics. The essential features, mathematics as a method of approach to knowledge and mathematics as an art, should be stressed.

It is perhaps needless to point out that many a good intention has gone awry in the execution. The handling of the above material must produce a coherent, rounded, sober presentation of elementary mathematics. There must be proper balance between the large ideas and concrete illustrations. There

must be precise assertions and proofs. However, there is no need to prove all the conclusions. Moreover, the proofs must be distinguished from intuitively grounded and loose arguments. Finally, the treatment must be on a level suitable for freshmen.

6. Criticisms of the proposed course and rebuttal. The above course does seem to be a radical departure from conventional ones and in discussions with colleagues I have encountered criticisms which I should like to consider for a moment. One such criticism is that the course would amount to just a lot of vague talk, an ill-defined mass of material. One answer to this criticism could be that no course is vaguer as to the nature and role of mathematics than college algebra and trigonometry. However, more to the point is it that each topic has definite mathematical content and establishes definite relationships of mathematics to our culture. The critic who believes that the kind of course I have outlined would be vague evidently doesn't believe that mathematics has any ideas or significance and hence that there would be little to say about it. A course which emphasizes concepts and their influences can be made substantial by insisting upon a good understanding of what is taught.

Again, I have been told that some of the topics are too sophisticated to be grasped by freshmen. This criticism is justified in part. However, each course must open up new fields and new concepts which will be pursued and more fully understood through work in subsequent courses. That is, each course must break fresh ground to some extent. In later courses some of these ideas will be reconsidered and the student will progress farther in these courses because he already has some notion of what is being discussed. The mathematics course I have sketched would break ground for science and philosophy courses in particular. More than that, many colleges are now experimenting with broad humanities courses in history, literature, physical science, and the social sciences. Each of these approaches our culture through its own avenues. The mathematics course described above would fit excellently into such a program and reinforce the other courses.

Some teachers would argue that the course I have outlined is mathematically thin. Where, for example, is the quadratic formula? Of course these critics are begging the question. What is mathematics? Evidently to these people it is a series of techniques. It does not include the meaning, purpose, and significance of the technical material. These critics merely reflect their own failure to see mathematics broadly. Better one idea well understood as to meaning, purpose, and significance than a thousand techniques however well mastered. A virtuoso on the violin is not a musician.

Moreover, concepts are more difficult to grasp than mechanical procedures which call for only parrot-like responses. Any child could learn that if $y=x^2$ then $dy/dx=2x$. However, how many calculus students can describe the concept represented by dy/dx and how many can state what is accomplished by the calculus?

Then there are "practical" objections. Some teachers point out that a student can't go on from a course such as I have described to more advanced mathematics. My answer is this. If a student who has indicated at the outset that he does not intend to pursue mathematics or use it in later life changes his mind at the end of the course, then he should be willing to take a technical course such as college algebra and trigonometry before going on to the higher courses. In any case the 99% who won't go on should not be sacrificed to the one per cent who might.

I know that some chairmen object to innovations in freshman mathematics because it is supposed to be hard for the student to transfer credit from one college to another. Hence they permit innovations as long as the first semester contains college algebra and the second, trigonometry. Other chairmen object to my course because it cannot be taught by the graduate students who, in some places called universities, teach the bulk of the freshmen. Of course these chairmen are putting the cart before the horse or are just rationalizing their unwillingness to consider new ideas.

There is a very real practical difficulty and that is to find college teachers of mathematics who will undertake to present the cultural aspects of mathematics. For various reasons which I propose to discuss at greater length at another time, mathematics teachers pay least attention to their most critical teaching problem, namely, the education of the liberal arts student. However, the task of obtaining the proper teachers is merely the task of recognizing and re-orienting ourselves to the problem posed by the liberal arts student. It is not the task of finding persons with extraordinary talents.

The best answer to all dubious objections is experimentation. This we are now doing at New York University. The material I have sketched was tried in one section of about 30 students during the academic year 1951–1952. Two sections were taught this material during the year 1952–1953. Two more sections of this course are presently being used for this experiment. The general cultural material on which the course draws has been gathered and published under the title of *Mathematics in Western Culture* (Oxford University Press, N. Y., 1953).

It is too early to assess fully the results of the experiment. One positive accomplishment can be noted. The students understand what is being tackled in each topic and participate fully in the class work. They feel that this material is their meat, so to speak, and partake of it. Since the course makes contact with painting, philosophy, literature, the social sciences, and other fields, even the most disinterested student is drawn into participation at some time. The interest displayed contrasts sharply with the reticence, resistance, and helplessness one encounters in teaching the traditional material.

7. The major objective of the proposed course. What should be the prime accomplishment of the proposed course? Essentially it should teach an appreciation of the role of mathematics in Western culture. Appreciation rather

than skill has long been recognized as an objective in literature, music, and art. It seems to me equally justifiable as an objective in mathematics, especially in view of the facts that interest in mathematics must be aroused and that the subject is more difficult to grasp.

If we are successful in teaching appreciation of mathematics we shall replace the present dislike and complete rejection of the subject on the part of students who have suffered through college algebra and trigonometry by respect and possibly liking for the subject. We may even succeed in inspiring them to maintain some contact with mathematics in later life and thereby secure for them one of the noblest of intellectual interests. We know that this interest will be well rewarded.

COUPON COLLECTING FOR UNEQUAL PROBABILITIES*

HERMANN VON SCHELLING, U. S. Submarine Base, New London, Connecticut

1. Introduction. Let us assume that a firm encloses in each package of a certain product one coupon which is chosen randomly from a stock of k different items. A customer who collects these coupons may gather some of them several times before he completes a whole set. The problem arises to determine the probability of obtaining all of the k coupons in n packages. An obvious generalization is what will be the probability that the lth different coupon is found in the nth package.

Since 1938 this question was treated by several authors [1], [2], [3] who always assumed that each coupon has the same probability $1/k$. The method was applied by M. G. Kendall and Babington Smith [4] for checking the randomness of their random sampling numbers.

In the general case the ith coupon has the probability p_i with $p_1+p_2+\cdots+p_k=1$. The solution of the general problem was published by the writer [5] in a German journal of a limited circulation as early as 1934. This paper remained practically unknown; the authors mentioned above did not refer to it. Besides, the formulas were given without proofs. Since the problem seems to be of a continuous interest, a short demonstration of the general case might be justified.

2. The general distribution. Let us assume k kinds of events which occur with the probabilities

$$p_1, p_2, \cdots, p_k; \qquad p_1 + p_2 + \cdots + p_k = 1. \tag{1}$$

---

* Opinions or conclusions contained in this paper are those of the author. They are not to be construed as necessarily reflecting the views or endorsement of the Navy Department.

Only one event can be observed at any one time. We are using the following notation:
At the n_1th observation the last of the missing events is noted,
at the n_2th observation the second last,
at the n_3th observation the third last,
. . . .
at the $n_{(k-1)}$th observation the $(k-1)$th last, *i.e.*, the second event is noted.

In order to illustrate this notation, let us throw a die twelve times with the following results:

$$3\ 5\ )\ 3\ 2\ )\ 5\ 4\ )\ 1\ )\ 2\ 2\ 3\ 1\ 6\ ).$$

The number 6 appears as the last one, on the 12th place, $n_1=12$;
the number 1 appears as the second last, on the 7th place, $n_2=7$;
the number 4 appears as the third last, on the 6th place, $n_3=6$;
the number 2 appears as the fourth last, on the 4th place, $n_4=4$;
the number 5 appears as the fifth last, on the 2nd place, $n_5=2$.

A symbol from the theory of symmetrical functions is to be used for getting compact formulas. The character

$$S_k f(p_1, p_2, \cdots, p_m)$$

means that the function f is to be formed for all $\binom{k}{m}$ combinations of the k indices indicated in the subscript of S and that all these terms are to be added. For instance

$$S_4 \frac{1}{p_1+p_2} = \frac{1}{p_1+p_2} + \frac{1}{p_2+p_3} + \frac{1}{p_3+p_4} + \frac{1}{p_4+p_1} + \frac{1}{p_1+p_3} + \frac{1}{p_2+p_4} \cdot$$

This understood, the probability $w(n_m)$ that the mth last event is observed at the time n_m, reads

$$(2)\quad \begin{cases} w(n_m) = S_k\left\{\binom{m-1}{m-1}(p_1+p_2+\cdots+p_m)(p_{m+1}+p_{m+2}+\cdots+p_k)^{n_m-1}\right. \\ \qquad -\binom{m}{m-1}(p_1+p_2+\cdots+p_{m+1})(p_{m+2}+p_{m+3}+\cdots+p_k)^{n_m-1} \\ \qquad +\binom{m+1}{m-1}(p_1+p_2+\cdots+p_{m+2})(p_{m+3}+p_{m+4}+\cdots+p_k)^{n_m-1} \\ \qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \qquad \left.+(-1)^{k-m+1}(p_1+p_2+\cdots+p_{k-1})(p_k)^{n_m-1}\right\} \\ \qquad \text{with } k-m+1 \leqq n_m < \infty. \end{cases}$$

Proof: The requested probability $w(n_m)$ is the sum of terms

$$p_{i_1}^{l_1} p_{i_2}^{l_2} \cdots p_{i_{k-m}}^{l_{k-m}} p_{i_{k-m+1}} \text{ with } l_1 + l_2 + \cdots + l_{k-m} = n_m - 1.$$

All exponents l_j are positive. These terms occur in the sum

$$S_k (p_1 + p_2 + \cdots + p_m)(p_{m+1} + p_{m+2} + \cdots + p_k)^{n_m - 1}.$$

However, this sum contains in addition terms of the same form for which one or more of the exponents l_j vanish. Let us consider the following expression in which the coefficients c_j are constant, *i.e.*, independent of the probabilities p_i:

$$\begin{aligned} S_k \{ & c_1 (p_1 + p_2 + \cdots + p_m)(p_{m+1} + p_{m+2} + \cdots + p_k)^{n_m - 1} \\ & - c_2 (p_1 + p_2 + \cdots + p_{m+1})(p_{m+2} + p_{m+3} + \cdots + p_k)^{n_m - 1} \\ & + c_3 (p_1 + p_2 + \cdots + p_{m+2})(p_{m+3} + p_{m+4} + \cdots + p_k)^{n_m - 1} \\ & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ & + (-1)^{k-m+1} c_{k-m} (p_1 + p_2 + \cdots + p_{k-1})(p_k)^{n_m - 1} \}. \end{aligned}$$

The coefficients c_j are to be determined so that all terms are cancelled which contain less than $(k-m+1)$ different probabilities p_i whereas all terms with exactly $(k-m+1)$ different probabilities p_i occur just once. These conditions yield the following system of $(k-m)$ linear equations. The first refers to the terms with only two probabilities, the second to the terms with three probabilities and so on.

$$(3) \quad \left\{ \begin{array}{ll} \binom{k-2}{m-1} c_1 - \binom{k-2}{m} c_2 + \cdots + (-1)^{k-m+1} \binom{k-2}{k-2} c_{k-m} & = 0 \\ \binom{k-3}{m-1} c_1 - \binom{k-3}{m} c_2 + \cdots + (-1)^{k-m} \binom{k-3}{k-3} c_{k-m-1} & = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ \binom{m}{m-1} c_1 - \binom{m}{m} c_2 & = 0 \\ \binom{m-1}{m-1} c_1 & = 1 \end{array} \right.$$

The solution of this triangular system is found to be

$$(4) \qquad c_j = \binom{m-2+j}{m-1}; \qquad j = 1, 2, \cdots, k-m.$$

In order to establish this result, we just have to solve the equations step by step, starting with the last one. Formula (4) completes the proof of the basic equation (2).

The probability $w(n_m)$ is a sum of geometrical progressions. Therefore it is

easy to find the cumulative distribution $W(n_m)$. We get

$$W(n_m)=S_k\left\{\binom{m-1}{m-1}(p_{m+1}+\cdots+p_k)^{k-m}-\binom{m}{m-1}(p_{m+2}+\cdots+p_k)^{k-m}\right.$$
$$\left.+\cdots+(-1)^{k-m+1}\binom{k-2}{m-1}(p_k)^{k-m}\right\}$$
$$-S_k\left\{\binom{m-1}{m-1}(p_{m+1}+\cdots+p_k)^{n_m}-\binom{m}{m-1}(p_{m+2}+\cdots+p_k)^{n_m}\right.$$
$$\left.+\cdots+(-1)^{k-m+1}\binom{k-2}{m-1}(p_k)^{n_m}\right\}.$$

If n_m goes to infinity, $W(n_m)$ converges to one and the second sum at the right side vanishes. Therefore the first constant sum must be unity. Indeed it can be proved that this sum equals $(p_1+p_2+\cdots+p_k)^{k-m}=1$. Therefore

$$\begin{aligned}W(n_m) = 1 - S_k\Bigg\{&\binom{m-1}{m-1}(p_{m+1}+\cdots+p_k)^{n_m}\\ &-\binom{m}{m-1}(p_{m+2}+\cdots+p_k)^{n_m}+\cdots\\ &+(-1)^{k-m+1}\binom{k-2}{m-1}(p_k)^{n_m}\Bigg\}.\end{aligned}\tag{5}$$

Let us note that $W(n_m)$ is simpler than $w(n_m)$. This is a decisive advantage of the "sequential approach." It is inherent to this method, as I have confirmed again and again with various distributions.

The arithmetic mean becomes

$$\begin{aligned}E(n_m) = S_k\Bigg\{&\binom{m-1}{m-1}\frac{1}{p_1+p_2+\cdots+p_m}\\ &-\binom{m}{m-1}\frac{1}{p_1+p_2+\cdots+p_{m+1}}+\cdots\\ &(-1)^{k-m}\binom{k-1}{m-1}\frac{1}{p_1+p_2+\cdots+p_k}\Bigg\}.\end{aligned}\tag{6}$$

This formula is not prohibitive anymore, if we confine ourselves to small values of k and m. The second moment may be obtained from

$$E\binom{n_m+1}{2} = S_k\left\{\binom{m-1}{m-1}\frac{1}{(p_1+p_2+\cdots+p_m)^2}\right.$$
$$(7) \qquad -\binom{m}{m-1}\frac{1}{(p_1+p_2+\cdots+p_{m+1})^2}+\cdots$$
$$\left.+(-1)^{k-m}\binom{k-1}{m-1}\frac{1}{(p_1+p_2+\cdots+p_k)^2}\right\}.$$

Of course, the variance follows from

$$(8) \qquad \operatorname{Var}(n_m) = 2\left[E\binom{n_m+1}{2} - \binom{E(n_m)+1}{2}\right].$$

These formulas cover the general case completely.

3. The special case of equal probabilities. Since the case $p_1=p_2=\cdots=p_k=1/k$ was treated by several authors, it is sufficient to list the final formulas. From equation (2) follows

$$(9) \qquad w(n_m) = \binom{k-1}{m-1}\frac{1}{k^{n_m-1}}\sum_{j=0}^{k-m}(-1)^j\binom{k-m}{j}(k-m-j)^{n_m-1}.$$

By using the so-called "differences of zero" we get

$$(10) \qquad w(n_m) = \frac{(k-1)(k-2)\cdots m}{k^{n_m-1}}\left(\frac{\Delta^{k-m}0^{n_m-1}}{(k-m)!}\right).$$

Fisher and Yates [6] tabulated the function $(\Delta^n 0^s)/r!$. Therefore the probabilities $w(n_m)$ can be evaluated easily up to $n_m=26$, the range of Fisher and Yates' tables.

The cumulative distribution $W(n_m)$ becomes according to equation (5)

$$(11) \quad W(n_m) = 1 - \binom{k-1}{m-1}\frac{1}{k^{n_m-1}}\sum_{j=0}^{k-m}(-1)^j\frac{1}{m+j}\binom{k-m}{j}(k-m-j)^{n_m}.$$

Simple formulas result for the arithmetic mean and the variance, namely the well known equations

$$(12) \qquad E(n_m) = k\left(\frac{1}{k}+\frac{1}{k-1}+\cdots+\frac{1}{m}\right)$$

and

$$(13) \qquad \operatorname{Var}(n_m) = k^2\left(\frac{1}{k^2}+\frac{1}{(k-1)^2}+\cdots+\frac{1}{m^2}\right) - k\left(\frac{1}{k}+\frac{1}{k-1}+\cdots+\frac{1}{m}\right).$$

All these relationships agree with the results gained independently by the authors who were mentioned in the introduction.

References

1. F. G. Maunsell, A problem in cartophily. Math. Gazette, vol. 22, 1938, p. 328; vol. 23, 1939, p. 77.

2. H. J. Goodwin, On cartophily and motor cars. Math. Gazette, vol. 33, 1949, pp. 169–171.

3. W. Feller, Probability Theory and Its Applications, Vol. I. John Wiley & Sons, Inc., New York 1950. See pages 175 and 181 and index references to coupon collecting.

4. M. G. Kendall and B. Babington Smith, Randomness and random sampling numbers. Journal of the Royal Statistical Society, vol. 101, 1938, pp. 147–166.

5. H. von Schelling, Auf der Spur des Zufalls. Deutsches Statistisches Zentralblatt, vol. 26, 1934, pp. 137–146.

6. R. A. Fisher and F. Yates, Statistical Tables for Biological, Agricultural and Medical Research. Oliver & Boyd, London and Edinburgh, 1948. See Table XXII.

BARBILIAN GEOMETRY AND THE POINCARÉ MODEL

P. J. KELLY, University of California, Santa Barbara

1. Introduction. In a very brief paper in *Casopis Matematiky a Fysiky* (1934–1935), D. Barbilian defined a class of metric spaces (which we will call Barbilian spaces) and stated some of their properties. The paper is an outline of results and contains no proofs, but one of the stated properties is that a Barbilian space has unique geodesic connection of each pair of its points when and only when it coincides with the Poincaré model of hyperbolic geometry. From this point of view, general Barbilian spaces are not geometrically fruitful. However, it seems to the author that the Barbilian approach to the Poincaré model has certain advantages of simplicity and generality. The object of the present paper is to illustrate these in a short development.

2. Barbilian geometry. A metric space is a set of points such that to each pair of its points, a, b, there is assigned a real number $d(a, b)$ with the properties

(1) $d(a, b) \geqq 0$,
(2) $d(a, b) = 0$ if and only if $a = b$,
(3) $d(a, b) = d(b, a)$,
(4) $d(a, b) + d(b, c) \geqq d(a, c)$.

Now, in the euclidean plane, let K be a set of points forming the interior of a

region bounded by a simple, closed curve J (Fig. 1). There is, then, a euclidean distance between each pair, a, b, in K and we denote this distance by ab. The problem we wish to consider is that of assigning some new distance law, or metric, to the points of K.

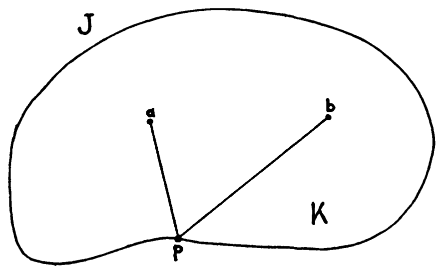


FIG. 1

Let a and b be particular points of K and let p be any point on J. As p varies on J, the distances pa and pb vary continuously and each assumes a maximum and a minimum value at some position of p. Any continuous function of pa and pb also has this property and so either the maximum or the minimum of the function associates a particular function value with the pair, a, b. What we seek is a function such that the value associated with a and b also satisfies the properties (1)–(4), that is, is a metric function.

To be specific, let the maximum be selected. The simplest functions of pa and pb we could consider are the basic arithmetic ones, $pa+pb$, $pa-pb$, $pa \cdot pb$, and pa/pb. That is, we inspect as possible distance functions

$$\max_{p \in J} (pa + pb), \qquad \max_{p \in J} (pa - pb),$$

$$\max_{p \in J} (pa \cdot pb), \qquad \max_{p \in J} (pa/pb).$$

All of these satisfy property (1). However, for $a=b$ the sum and product cases do not yield zero distance and can be discarded. The difference function is constantly zero for $a=b$ and so has zero as a maximum. It does not have the symmetry property (3), but this is easily corrected by taking

$$d(a, b) = \max_{p \in J} (pa - pb) + \max_{q \in J} (qb - qa).†$$

It is not difficult to show that this function has the four basic properties and is a metric. However, under this metric, like the euclidean one, K is a bounded space, and there is a more interesting possibility.

† A slight generaliztion is obtained by using $kd(a, b)$ where k is a positive constant for all points a, b.

For $a=b$,

$$\max_{p\epsilon J}(pa/pb) = \max_{p\epsilon J}(pa/pa) = 1.$$

Thus, the quotient maximum is one when $a=b$. Because J is a simple closed curve, there are always points of J on each side of the perpendicular bisector of the segment from a to b. Thus for $a\neq b$, the quotient maximum is greater than one. These facts suggest the use of log $[\max_{p\epsilon J}(pa/pb)]$, which has properties (1) and (2), and can be modified, as before, to have property (3) by defining

$$d(a, b) = \log\left[\max_{p\epsilon J}(pa/pb)\right] + \log\left[\max_{q\epsilon J}(qb/qa)\right].$$

This is the metrization of K which Barbilian defined and which leads to an infinite metric space for any choice of J.

To investigate this metric, it will be convenient to define

$$r(a, b) = \left[\max_{p\epsilon J} pa/pb\right]$$

$$r(b, a) = \left[\max_{q\epsilon J}(qb/qa)\right]$$

so that

$$d(a, b) = \log\left[r(a, b)\cdot r(b, a)\right].$$

If a, b, and c are three points of K, we have,

(5) $$r(a, b) \geqq pa/pb \qquad \text{for all } p\epsilon J$$

and

(6) $$r(b, c) \geqq pb/pc \qquad \text{for all } p\epsilon J,$$

hence

(7) $$r(a, b)\cdot r(b, c) \geqq pa/pc \qquad \text{for all } p\epsilon J$$

which implies

(8) $$r(a, b)\cdot r(b, c) \geqq r(a, c).$$

By a similar argument,

(9) $$r(b, a)\cdot r(c, b) \geqq r(c, a).$$

Combining (8) and (9), we have

(10) $$r(a, b)\cdot r(b, a)\cdot r(b, c)\cdot r(c, b) \geqq r(a, c)\cdot r(c, a),$$

and, upon taking the logarithm of both sides, this yields

(11) $$d(a, b) + d(b, c) \geqq d(a, c),$$

which establishes d as a metric.

In this last argument, it can be observed that the equality will hold in (11) if and only if the equality holds in both (8) and (9). Now let p^* be a maximizing point of J for $r(a, c)$. Then

$$r(a, c) = \frac{p^*a}{p^*c} = \frac{p^*a}{p^*b} \cdot \frac{p^*b}{p^*c} \tag{12}$$

shows that the equality in (8) holds if and only if p^* is also a maximizing point for $r(a, b)$ and $r(b, c)$. Similarly, the equality in (9) holds if and only if there is a common maximizing point q^* on J for $r(b, a)$, $r(c, b)$ and $r(c, a)$. That is, we have a result stated by Barbilian:

THEOREM: *Three points of K are collinear in the Barbilian sense if and only if their three distances can be defined by a common pair of points on J.*

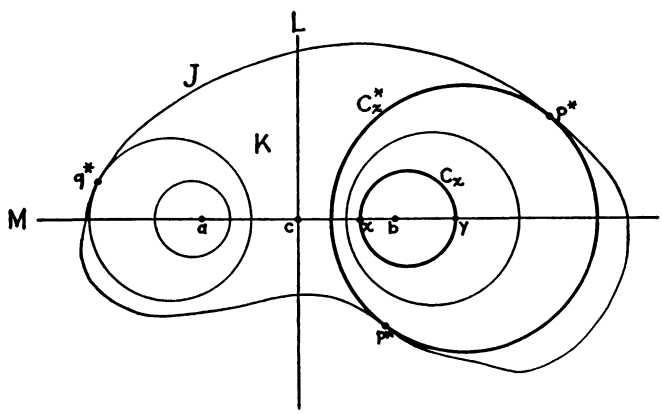


FIG. 2

For a given pair of points, a, b, in K, maximizing points on J, p^* and q^* for $r(a, b)$ and $r(b, a)$ respectively are determined very simply. Let c be the midpoint of the open, euclidean segment from a to b, namely $S^*(a, b)$ (Fig. 2). Take L to be the euclidean, perpendicular bisector of $S^*(a, b)$. If x is any point of $S^*(c, b)$, and y is the harmonic conjugate of x with respect to a, b, then the circle C_x, with the closed segment $S(x, y)$ as a diameter, is the locus of all points t such that $ta/tb = xa/xb = \alpha$. As x proceeds from c toward b, the circles C_x are decreasingly nested and α increases from 1 monotonically without bound. Because b is not on J there exists a circle C_{x_1} lying wholly in K. But all the circles C_x, for x on $S^*(c, b)$, fill (except for b) the half-plane defined by L and to which b belongs. It follows that as x proceeds from x_1 toward c there is a circle C_x^* of first contact with J. If p^* is any point on both C_x^* and J then $p^*a/p^*b = r(a, b)$. For if $C_{x_2} \subset C_x^* \subset C_{x_3}$, then C_{x_2} contains no points of J, while C_{x_3} corresponds to a smaller value of α than does C_x^*. By a similar argument, for x on $S^*(a, c)$, there

is a second circle, centered on M, separating a and b harmonically, and tangent to J at some point q^*. Then $q^*b/q^*a = r(b, a)$.

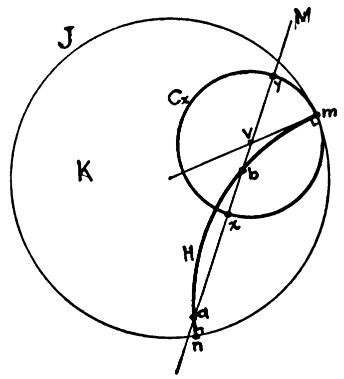


FIG. 3

Consider now the case where J is a circle and K is its interior (Fig. 3). If the line M through two distinct points a and b in K does not pass through the center of J, there is a unique circle H which passes through a and b and its orthogonal to J. Let m and n denote the intersection points of J and H, with b on the smaller arc $\widehat{am}$ of H. The radius of J to point m cuts M at a point v. Let C_x be the circle with center v and radius vm. Because C_x is tangent to J it is orthogonal to H and so separates a and b harmonically. Therefore m maximizes ta/tb for t on J. Similarly n maximizes tb/ta for t on J. Thus,

$$d(a, b) = \log\left[(ma/mb)\cdot(nb/na)\right],$$

which is also the Poincaré definition for distance. The points m and n are, by the same argument, the maximizing points for all pairs a and b on H, hence H is a geodesic. Because of the theorem established previously, H is the unique geodesic connecting a and b. When a and b are collinear with the center of J, the diameter of J containing a and b is simply a limiting position for H and the same results hold.

In this development, the cross ratio form $(pa/pb)(qb/qa)$ for distance appears in a natural way. Since this form is invariant under inversion, it is easily shown that inverting with respect to a point outside J sends K to an isometric space, K'. The whole exploratory approach suggests that there may be other alternatives and it is natural to consider the possibility of replacing the maximizing points on J by the intersection points of J with the line through a and b. This leads, of course, to a Hilbert geometry, if J is convex, and gives the Klein model of hyperbolic geometry when J is an ellipse.†

† The comparison of Barbilian, Hilbert, and Poincaré spaces was pointed out by L. M. Blumenthal, *Distance Geometries*, 1938, pp. 27–28.

3. A generalization. Let a and b be two distinct points of K with a to the left of L, the perpendicular bisector of the segment from a to b. The ratio pa/pb, $p \neq b$, is less than, equal to, or greater than one according as p is to the left, on, or to the right of L. The condition that K be interior to a closed curve J simply insures that $r(a, b)$ and $r(b, a)$ are determined by points to the right and left of L respectively, and hence that $\log [r(a, b)r(b, a)]$ is a positive number. But this can be accomplished in a more general way.

Suppose J is any closed set which is disjoint from a set K to be metrized, and take a, b and L as before. To the previous definitions of $r(a, b)$ and $r(b, a)$, add the condition that if J has no point to the left of L then $r(b, a)=1$ and if J has no point to the right of L then $r(a, b)=1$. Since $\lim_{ta\to\infty} ta/tb = \lim_{ta\to\infty} tb/ta = 1$, this amounts to adjoining the line at infinity to J. With one exceptional case, namely when J is L or a subset of L, the product $r(a, b)\cdot r(b, a)$ is always greater than one for distinct points.

The new metric can be defined directly by taking

$$r(a, b) = \max \left\{1, \max_{p\in J} pa/pb\right\}$$

$$r(b, a) = \max \left\{1, \max_{q\in J} qb/qa\right\}$$

$$d'(a, b) = \log \left[r(a, b)\cdot r(b, a)\right].$$

The validity of $d'(a, b)$ as a metric can be established, save for the exceptional case, by much the same arguments as before, and when J, K define an ordinary Barbilian space then d and d' are the same. One advantage of the more general definition concerns the inversion of an ordinary Barbilian space with respect to a point of J. The image sets, J' and K', define an isometric space with respect to d' but not with respect to d. For example, when J is a circle with interior K, then inversion with respect to a point of J yields, under d', the familiar half-plane representation of hyperbolic space.

4. An example. It is easy to exhibit some particular geodesics in an ordinary Barbilian space. For if C is any circle, containing no point exterior to J, and touching J at m and n, then the circular arc in K orthogonal to C at m and n is a Barbilian geodesic.

To students who have seen the difference in character between euclidean and hyperbolic parallels, it is usually interesting to examine a space in which both types occur, and to see that this is achieved at the expense of unique geodesic connection. As an example, let K be the set of all points interior to either of two, equal-sized, orthogonal circles C_1 and C_2, which intersect at points m and n (Fig. 4). Then J consists of the larger arcs $\overset{\frown}{mn}$ of C_1 and C_2. The points interior to both circles form the "lens" and the smaller arcs $\overset{\frown}{mn}$ will be called the "lens arcs." If o_1 and o_2 are the centers of C_1 and C_2 respectively, then every point of $S(o_1, o_2)$ is the center of a circle C which passes through m and n and which possesses no point exterior to J. The circular arc which is orthogonal to C

at m and n, and which lies inside the lens, is therefore a Barbilian line. In particular, the segment $S^*(m, n)$ is also a straight line in the Barbilian sense.

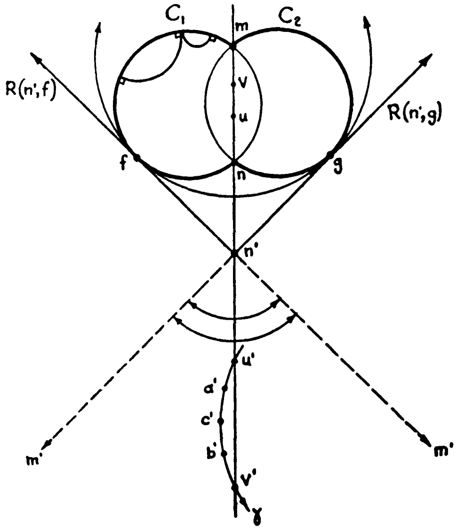


FIG. 4

Consider now an inversion with respect to a circle which is centered at m and which is tangent to C_1 and C_2 at f and g respectively. The transformation sends J into $R(n', g)$ (the ray through g which emanates from n') and into the perpendicular ray $R(n', f)$. These rays separate the plane into two regions and K' is the non-convex portion. The former family of circular arcs through m and n goes into the pencil of rays through n' which lies within the angle which is vertical to $\angle fn'g$. Let u and v be any two points of $S^*(m, n)$ and take γ to be a curve through u' and v'. If at all points of $\widehat{u'v'}$ on γ the curvature of γ is sufficiently small, it is apparent that for any pair of points a' and b' on $\widehat{u'v'}$ the perpendicular bisector of $S(a', b')$ will not intersect J'. If a' is nearer to n' than is b', then $d(a', b') = \log (n'b'/n'a')$. For any point c' of γ between a' and b', we then have $d(a', b') = d(a', c') + d(c', b')$, since $n'b'/n'a' = (n'c'/n'a')(n'b'/n'c')$. Thus the arc $\widehat{u'v'}$ of γ is a straight line segment in the Barbilian sense. There are therefore infinitely many straight line connections of u' and v' (including $S(u', v')$). Since the mapping is an isometry, there are therefore infinitely many straight lines connections of u and v.

The circular arcs which are orthogonal to both lens arcs, and which lie in the lens, invert into the arcs of concentric circles which are centered at n'. These are easily shown to be euclidean parallels. However, they are only Barbilian segments. The lens arcs themselves are examples of complete lines which are parallel in the euclidean sense. To see this, let o be a fixed point of the lens

arc which lies on C_1 and let x be a variable point on the lens arc which lies on C_2 (Fig. 5). Take p to be the intersection of J and the circle H_x which passes through o and x and which is orthogonal to C_2. Let q be the intersection of J with the circle M_x which passes through o and x and which is orthogonal to C_1. Then the Barbilian distance from o to x is given by

$$d(o,\, x) = \log\left(\frac{px}{po}\,\frac{qo}{qx}\right).$$

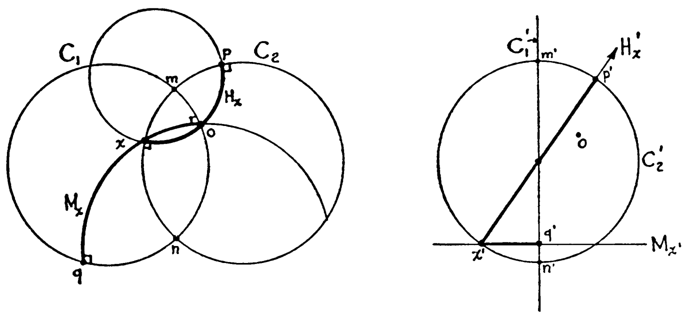


FIG. 5

If primes indicate the image points under an inversion centered at o, then from $px/po = x'p'/x'o$ and $qo/qx = x'o/x'q'$ we also have

$$d(o,\, x) = \log\,(x'p'/x'q').$$

Under the inversion, C_2 is transformed to a circle C_2'. Since C_1, H_x, and M_x pass through o, they invert into straight lines C_1', H_x', and M_x'. Because inversion preserves angles, it follows that C_1' is orthogonal to C_2' and therefore it passes through the center of C_2'. For the same reason, H_x' passes through the center of C_2', while M_x' is perpendicular to C_1'. The lines H_x' and M_x' intersect at x' on an arc $\widehat{m'n'}$ of C_2'. The point p' is the other intersection of H_x' and C_2', while C_1' and M_x' intersect at q'. As x varies, $x'p'$ is always the diameter length of C_2', $x'q'$ is always half the length of a chord perpendicular to the fixed diameter C_1'. The ratio $x'p'/x'q'$ is therefore a minimum when $x'q'$ is a maximum. This occurs when $x'q'$ is the radius length of C_2', that is when $d(o, x) = \log 2$. Since this result is independent of the position of o on the lens arc, the two lens arcs are equidistant straight lines in the Barbilian geometry at the constant distance log 2, and are therefore parallels in the euclidean sense. On the other hand, any two circles which are orthogonal to J at a common point of J (see Fig. 4) are Barbilian parallels which behave as do hyperbolic parallels.

In the construction just given, it is clear that the minimum for $d(o, x)$ occurs when H_x' and M_x' coincide, that is when H_x' is perpendicular to both C_1' and C_2'. In the original figure, then, the circle through o which is orthogonal to

both C_1 and C_2 carries the Barbilian perpendicular to both lens arcs. This, with the previous remarks, shows that the two conjugate systems of circles, defined by the base points m and n, form a gridwork of lines within the lens. It is also clear that inversion with respect to any circle which is orthogonal to both C_1 and C_2 is a motion of the space.

MATHEMATICS IN THE SECONDARY SCHOOLS FOR THE EXCEPTIONAL STUDENT‡

H. W. BRINKMANN, Swarthmore College

1. Introduction. The work to be described in this paper was done under the auspices of the School and College Study of Admission with Advanced Standing. This Study, which was initiated by President Gordon Chalmers of Kenyon College, was undertaken by twelve colleges* and twenty-seven secondary schools. Financial support for the Study was given by The Fund for the Advancement of Education, and Dr. William H. Cornog, President of the Central High School, Philadelphia, Pennsylvania, served as its Executive Director. The fundamental principle behind the study is that certain bright students are wasting time in the schools as set up at present, and that they could profitably spend some time in anticipating the work they would normally do in their first year of college. It was proposed that the twelve colleges should reach agreement on the content of the central subjects of the college freshman year, with a view to granting advanced credit in these subjects to those students who qualified in them. It was inevitable that mathematics be one of these subjects.† A committee was appointed to deal with each of these subjects and to define the subject matter concerned. This was done in consultation with representatives of the various institutions, and many lengthy meetings were held by each committee throughout the year to discuss the problems raised by the study. The committee for mathematics consisted of the following: Julius Hlavaty (Bronx High School of Science, New York City); Elsie Parker Johnson (Oak Park and River Forest High School, Oak Park, Ill.); Charles Mergendahl (Newton High School, Newtonville, Mass.); George B. Thomas (Massachusetts Institute of Technology);

‡ Presented to the Mathematical Association of America on December 31, 1953.

* Bowdoin College, Brown University, Carleton College, Haverford College, Kenyon College, Massachusetts Institute of Technology, Middlebury College, Oberlin College, Swarthmore College, Wabash College, Wesleyan University, Williams College.

† The other subjects were: English Composition, Literature, Physics, Chemistry, Biology, Latin, French, German, Spanish, and History.

Elbridge P. Vance (Oberlin College); Volney H. Wells (Williams College), with the writer (Swarthmore College) as chairman. The reports of all the committees are available at the office of Dr. Cornog, the Executive Director of the Study.

It should be mentioned here that the twelve colleges have all adopted the proposed plan to grant advanced credit for the work as outlined by the committees. Similar committees are now at work constructing examinations, which will be used to test the candidates for such credit. Furthermore, seven specially selected schools* are testing out these ideas by preparing candidates in several of the subjects mentioned. Many of the other cooperating schools are also giving courses of the sort proposed by our committees.

The problem before our committee was to define the work in mathematics which would be acceptable by our twelve institutions for college credit in place of a first-year college course. On the basis of consultation with the departments of mathematics of the twelve institutions, our committee decided that such a first-year course should consist essentially of a substantial course in calculus with applications, along with the analytic geometry that is needed for such a course. Our committee did not want to recommend that advanced credit be given for courses in college algebra, solid geometry, and the like, even though such courses are occasionally given in the freshman year at some colleges. When we attempted to integrate our freshman course into the High School program we found that it was necessary to give consideration to the whole program in mathematics in the secondary schools. We accordingly worked out a detailed program in mathematics for the last three years of secondary school, which culminates in a course that will be acceptable for advanced standing in college. When we got through with this we found that such a program is preferable in many ways to the standard program in mathematics as it is now pursued in most secondary schools, although we started out by devising it for the exceptional student.

2. The problem. It has been realized for many years that the mathematics curriculum in the secondary schools was in for a drastic revision. For example, the section on mathematics in *General education in school and college* (reprinted in this MONTHLY, vol. 60, 1953, pages 380–383) deals with this topic, and certain general proposals for revising the curriculum are made there. It will be seen that the program planned by our committee agrees in many ways with these proposals; in addition, we have given detailed suggestions as to how such a program can be carried out.

The main ideas behind our plan are: to break down the standard compartmentalized program for these years, to introduce certain new subject matter, and at the same time to suggest the elimination of certain traditional items. As

---

* Bronx High School of Science (New York City); Central High School (Philadelphia, Pa.); Evanston Township High School (Evanston, Ill.); Germantown Friends School (Philadelphia, Pa.); Horace Mann School (New York City); Newton High School (Newtonville, Mass.); St. Louis Country Day School (St. Louis, Mo.).

a background for this proposed program we assumed a course of instruction carrying the student through the ninth grade; he would thus be acquainted with the number system, the vocabulary and ideas of elementary, intuitive geometry, the beginnings of the use of algebraic symbolism, and the ideas of graphical representation. Our program for the three years of senior High School would then be the following:

(1) A course in 10th year mathematics stressing deductive thinking, but dealing with various types of mathematical subject matter.

(2) A year's course consisting of a continuation of algebra, of analytic geometry, and of trigonometry.

(3) A year's course made up of certain advanced topics in algebra and analytic geometry, with a substantial introduction to calculus and its applications.

3. Tenth Year Mathematics. The work in this year has traditionally dealt with deductive geometry. Our proposal would continue to stress the deductive method, but would apply it not merely to geometrical subject matter, but to material from algebra and other subjects as well. To achieve this it is not necessary that all of the geometric content of the course be organized into one logical sequence; the role of undefined terms, definitions, assumptions, theorems, can be taught by exhibiting several instances of short groups of propositions. Each such group would illustrate the meaning of a deductive system. Furthermore, it is desirable that non-geometric material—for example, from algebra and other subjects—be organized in this fashion. Our report gives some detailed examples along these lines. In addition to the geometric content of the year's work—and we would urge that the simpler concepts of three dimensional geometry be considered along with their plane counterparts—it is essential that the study and development of the algebra begun in the 9th grade be continued here. This is best done by introducing the subject of analytic geometry at this time and carrying it to a point where the student is able to prove simple theorems by algebraic methods. The equations of simple curves (circle, parabola) can also be introduced. The student will thus continue his use of algebra and at the same time increase his geometric insight. In this connection the study of trigonometry should also be begun.

4. Eleventh Year Mathematics. The work of this year should serve as an introduction to analysis. Since the students taking mathematics at this level are generally those who are planning to go to college, the material should be of a college preparatory character. The subject matter is a continuation of work begun at earlier levels and is essentially algebra, analytic geometry, and trigonometry. Among the subjects to be treated in algebra are the theory of polynomials—including the remainder theorem; systems of linear equations—including determinants; complex numbers; logarithms. The work in analytic geometry should carry the student through the study of linear geometry and the elements of conic sections. The trigonometry to be studied in this year should be largely analytic trigonometry, the work being centered around a study

of the trigonometric functions as functions, with a minimum of work done in geometric trigonometry and computation. Moreover, the application of trigonometric ideas to complex numbers should certainly be included here.

5. Twelfth Year Mathematics. The primary objective of the work in this year is to give a substantial introduction to differential and integral calculus, with enough applications to bring out the meaning and to illustrate the fundamental importance of this subject. The work to be done in the two previous years was planned so as to lead up to this subject, and the necessary prerequisite topics from algebra, analytic geometry, and trigonometry were included. It is expected that skills in these subjects will be further developed when they are employed in the calculus, and that those topics which were not covered during previous years will be studied now. In particular, the analytic geometry previously studied would be used here and would be enriched by applications of the calculus.

6. Remarks. It will be seen that there is nothing very unconventional about the subject matter suggested in our program. In fact, it is quite definitely built around the development and applications of analysis. It seems to the writer that this is the way it should be, especially when one keeps in mind the variety of students for whom this work is planned. A student entering college prepared with such a program will have many advantages. He will in fact be ready to take the normal sophomore course in mathematics. Thus, if he becomes a mathematician he will be able to accelerate his progress in analysis and will, because of this fact, be able to take additional work in other fields, such as higher algebra, advanced geometry, statistics, mechanics, and so forth, and thus round out his mathematical background. If the student is preparing to study science or engineering, he will be greatly helped by having had a course in calculus before he enters college; his first course in physics, for example, can then be a real meaningful introduction to the subject. Finally, there will be certain students who will not need to study mathematics in college, because the course here outlined will give them the kind of preparation that is sufficient for their needs. Such students will thus be free to take work in college which will perhaps be more to their advantage educationally.

As stated at the beginning, the program here outlined was originally proposed by our committee for those exceptional students who wish to present a certain amount of mathematics for advanced credit. It seems to us, however, that the ideas behind it could be equally well applied to secondary school mathematics in general. Thus it may be that work along these lines will become the standard program in secondary school mathematics. Or, it may be desirable for the less able student to stretch out over three years the work that we have laid out for the tenth and eleventh years. As a third possibility, a student may wish to take the tenth and eleventh year courses proposed in our program, and not take the twelfth year course; courses in advanced algebra, statistics, solid

geometry, and so forth could be made available for such a student in his twelfth year. Such students would then take the normal freshman college course when they enter college. There are already many schools in this country in which work of this type in mathematics is being done. It is our hope that this will be continued and that other schools will follow.

ALTERNATIVE SOLUTION TO THE EHRENFEST PROBLEM*

F. G. HESS, University of British Columbia

1. Introduction. Explicit expressions for the probabilities connected with the so-called Ehrenfest model (see Section 2 below) have been obtained by M. Kac [1], who has applied the usual method of dealing with problems involving discrete Markov chains. It is the purpose of this article to show that the solution to the problem can be obtained in a simpler way if the problem is formulated in terms of a direct product representation.† Such a formulation should be useful for calculating probabilities connected with discrete Markov chains of similar complexity.

2. The Ehrenfest model. In order to illustrate certain features of statistical mechanics, P. and T. Ehrenfest [2] have considered the following simple model.

$2R$ labelled balls are placed in 2 boxes. We have $2R$ labelled tickets, a one-to-one correspondence existing between the tickets and balls. A ticket is drawn at random. The ball represented by this ticket is removed from the box it is in and placed in the other box. The ticket is replaced in the pack which is then shuffled. The process is repeated.

One of the questions which arose in connection with this model is—what is the probability, $P(n|m; s)$, that there are $R+m$ balls in box 1 after s draws if there are $R+n$ balls in box 1 initially? This is the problem treated here.

3. Procedure. A brief outline of the procedure for calculating P follows. A set of orthonormal column vectors, ξ_i, is found such that each ξ_i represents a possible state of the system. (We shall define a state explicitly below.) A matrix operator H is then found such that when it operates on a state vector, ξ_i, it forms the sum of all those state vectors which can result from ξ_i in a single draw, *i.e.*,

---

* I wish to thank Professor W. Opechowski for his interest in this problem. I should also like to express my indebtedness to the National Research Council of Canada for a Studentship.

† The referee has kindly brought to my attention the existence of a paper by A. J. F. Siegert [4], in which essentially the same method is used to obtain the solution to the Ehrenfest problem. However, the present paper differs considerably from that of Siegert's in the explicit formulation of the method. It may be mentioned that Siegert has investigated other problems for which the same method of solution applies.

$$H\xi_i = \sum_j p^1_{ji}\xi_j, \tag{1}$$

where p^1_{ji} is the probability that the state ξ_j is obtained from the state ξ_i in a single draw. From (1) it can be shown that [3],

$$H^s\xi_i = \sum_j p^s_{ji}\xi_j, \tag{2}$$

where

$$p^s_{ji} = \xi'_j H^s \xi_i \tag{3}$$

is the probability that the state ξ_j is obtained after s draws if the system is initially in the state ξ_i; ξ'_j is the transpose of ξ_j. To evaluate p^s_{ji} we must know the eigenvalues and eigenvectors of H and the expansion of ξ_i in terms of these eigenvectors.

Let us now define explicitly a state of the system. Following Kac, we could say that a state is defined if we know the number of balls in box 1. If we let $\xi_m(-R \leqq m \leqq R)$ be a column vector having the number 1 in its $R+m$th row and zeros elsewhere, then ξ_m would represent the state having $R+m$ balls in box 1. By calculating all p^1_{ji} from their definition we then obtain H from (1) as a $(2R+1)$ by $(2R+1)$ matrix. We must then find the eigenvalues and eigenvectors of H in order to calculate $p^s_{ji} = P(i|j;\ s)$. Essentially, this is the procedure carried out by Kac.

However, in this paper we shall introduce a more complicated definition of a state which simplifies the solution of the problem. We shall say that a state is defined if we know which ball is in which box. Thus, we have $\binom{2R}{R+m}$ different states possible having $R+m$ balls in box 1. As a result, let the column vector ξ_{mk} denote the kth state of the possible $\binom{2R}{R+m}$ states which have $R+m$ balls in box 1; k is just an index distinguishing between the different $\binom{2R}{R+m}$ states. Replacing i and j in (1), (2), and (3) by mk and nl respectively, we obtain,

$$p^s_{mk,nl} = \xi'_{mk} H^s \xi_{nl} \qquad (\xi'_{mk}\xi_{nl} = \delta_{mn}\delta_{kl}). \tag{4}$$

Now $\sum_k p^s_{mk,nl}$ is the probability that there are $R+m$ balls in box 1 after s draws if the system is initially in one of the $\binom{2R}{R+n}$ states ξ_{nl} having $R+n$ balls in box 1. This probability must be independent of l. Hence

$$P(n \mid m;\ s) = \binom{2R}{R+n}^{-1} \sum_{kl} p^s_{mk,nl}. \tag{5}$$

If we let

$$X(a) = \sum_m a^{R+m} \sum_k \xi_{mk}, \tag{6}$$

where a is some parameter, then $\binom{2R}{R+n} P(n|m; s)$ is the coefficient of $a^{R+m}b^{R+n}$ in the expansion of

$$g(a, b) = X'(a)H^s X(b). \tag{7}$$

In the next section, we shall express ξ_{mk}, $X(a)$, and H explicitly. For our formulation of the problem it is a fairly simple matter to find the eigenvalues and eigenvectors of H. In addition $X(a)$ is easily expressed in terms of the eigenvectors of H.

4. Solution. Suppose we had only 1 ball. Let α, β defined by

$$\alpha = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \beta = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{8}$$

represent the fact that the ball is in box 1, box 2 respectively. If

$$S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \left(S^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I \right), \tag{9}$$

then, for the case of only 1 ball, S is the operator H because

$$S\alpha = \beta, \qquad S\beta = \alpha. \tag{10}$$

For the case of $2R$ balls, we let ϵ_i be α or β depending on whether the ith ball is in box 1 or box 2. Then, a state ξ_{mk} of the system can be represented by the direct product of the matrices ϵ_i, *i.e.*,

$$\xi_{mk} = \epsilon_1 \times \epsilon_2 \times \cdots \times \epsilon_{2R}, \tag{11}$$

where $R+m$ of the ϵ_i's are α's and the rest β's. The ξ_{mk}'s are orthonormal.

(It may be noted here that our formulation of the problem is easily extended to include similar problems involving more than 2 boxes.)

If we define

$$[a\alpha + \beta]^{2R} \equiv [a\alpha + \beta] \times [a\alpha + \beta] \times \cdots \times [a\alpha + \beta], \tag{12}$$

then

$$[a\alpha + \beta]^{2R} = \sum_m a^{R+m} \sum_k \xi_{mk},$$

and so, from (6),

$$X(a) = [a\alpha + \beta]^{2R}. \tag{13}$$

H is given by

$$H = \frac{1}{2R} \sum_{i=1}^{2R} S_i$$

where S_i is the direct product of $2R$ matrices, one of which is S (9), located in the ith position, and all the rest of which are the unit matrix, I. Each S_i, operating on some ξ_{mk}, changes the ith ball from the box it is in to the other box as a result of (10). Thus H, operating on ξ_{mk}, produces a sum of $2R$ new states, each one obtained from the state ξ_{mk} by moving one of the balls. $1/2R$ is the probability that a particular ball interchanges boxes in a draw.

Hence,

$$H\xi_{mk} = \sum_{nl} p^1_{nl,mk}\xi_{nl}, \tag{14}$$

each $p^1_{nl,mk}$ being $1/2R$ or zero depending on whether or not ξ_{nl} can be obtained from ξ_{mk} by moving just one ball.

We now evaluate $g(a, b)$ in (7). We first find the eigenvalues and eigenvectors of H. We note that μ and ν defined by

$$\mu = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}, \qquad \nu = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} \tag{15}$$

are eigenvectors of S to the eigenvalues 1 and -1 respectively. Furthermore, it is easily shown that each orthonormal vector η_{mk} defined by

$$[c\mu + \nu]^{2R} = \sum_m c^{R+m} \sum_k \eta_{mk} \tag{16}$$

is an eigenvector of H to the eigenvalue

$$(R + m)1 + (R - m)(-1) = 2m, \tag{17}$$

i.e.,

$$H\eta_{mk} = 2m\eta_{mk}. \tag{18}$$

Now, from (8) and (15), we have

$$\alpha = \frac{1}{\sqrt{2}}(\mu + \nu), \qquad \beta = \frac{1}{\sqrt{2}}(\mu - \nu). \tag{19}$$

Therefore, from (13), (19) and (16), we obtain

$$\begin{aligned} X(a) &= \frac{1}{2^R}[(a + 1)\mu + (a - 1)\nu]^{2R} \\ &= \frac{1}{2^R}\sum_m (a + 1)^{R+m}(a - 1)^{R-m}\sum_k \eta_{mk}. \end{aligned} \tag{20}$$

Substituting (20) and (18) into (7) and using the fact that the η_{mk}'s are orthonormal, we obtain

$$(21)\quad g(a, b)=\frac{1}{2^{2R}} \sum_{j=-R}^{R}(a+1)^{R+j}(a-1)^{R-j}(b+1)^{R+j}(b-1)^{R-j}\left(\frac{2j}{2R}\right)^{s}\binom{2R}{R+j}.$$

Defining K_l^j by

$$(22)\qquad (a+1)^{R+j}(a-1)^{R-j} = \sum_{l=0}^{2R} K_l^j a^l,$$

we obtain finally,

$$(23)\qquad \binom{2R}{R+n}P(n\mid m; s) = \frac{1}{2^{2R}}\sum_{j=-R}^{R}\left(\frac{j}{R}\right)^{s}\binom{2R}{R+j}K_{R+m}^{j}K_{R+n}^{j}.$$

From (23), we see that $P(n|m; s)$ has the symmetry property

$$(24)\qquad \binom{2R}{R+n}P(n\mid m; s) = \binom{2R}{R+m}P(m\mid n; s),$$

a result which follows quite easily from the physical nature of the problem.

Kac's result, in terms of our notation, is given by

$$(25)\qquad P(n\mid m; s) = \frac{1}{2^{2R}}\sum_{j=-R}^{R}\left(\frac{j}{R}\right)^{s}K_{R+m}^{j}K_{R+j}^{n}.$$

This result is the same as ours since

$$(26)\qquad \binom{2R}{R+j}K_{R+n}^{j} = \binom{2R}{R+n}K_{R+j}^{n},$$

a relation proved as follows:

From(22) we have

$$(27)\qquad \sum_{nj}\binom{2R}{R+j}K_{R+n}^{j}a^{R+n}b^{R+j} = \{b(a+1)+a-1\}^{2R}.$$

Also

$$(28)\qquad \sum_{nj}\binom{2R}{R+n}K_{R+j}^{n}a^{R+n}b^{R+j} = \{a(b+1)+b-1\}^{2R}.$$

Since the right-hand sides of (27) and (28) are the same, the relation (26) immediately follows.

References

1. M. Kac, Random walk and the theory of Brownian motion, This MONTHLY, vol. 54, 1947, pp. 369–391.

2. P. and T. Ehrenfest, Über zwei bekannte Einwände gegen das Boltzmannsche H-Theorem,

Phys. Zeit., vol. 8, 1907, pp. 311–314.
3. W. Feller, An Introduction to Probability Theory and Its Applications, vol. 1, Ch. 15, 16 (1950), Wiley.
4. A. J. F. Siegert, On the approach to statistical equilibrium, Phys. Rev., vol. 76, 1949, pp. 1708–1714.

MATHEMATICAL NOTES

EDITED BY F. A. FICKEN, University of Tennessee,

Material for this department should be sent to F. A. Ficken, University of Tennessee, Knoxville 16, Tenn.

PASCAL HEXAGONS ASSOCIATED WITH A TRIANGLE*

VICTOR THÉBAULT, Tennie, Sarthe, France

The object of this note is to point out the use that can be made of a little-known theorem concerning a Pascal hexagon inscribed in a conic when the conic is taken as the nine point circle of a triangle.

THEOREM. *If A, a, B, b, C, c are six arbitrary points on a conic (C), then the Pascal lines of the inscribed hexagons $AaBbCc$, $AbBcCa$, $AcBaCb$ are concurrent.*

In the inscribed hexagon $AaBbCc$, the pairs of opposite sides Aa and bC, Bb and cA intersect in D, D' on the corresponding Pascal line; in the inscribed hexagon $AbBcCa$, the pairs of opposite sides Bc and aA, Ca and bB intersect in E, E'; and in the inscribed hexagon $AcBaCb$, the pairs of opposite sides Cb and cB, Ac and aC intersect in F, F'. In order to show that the Pascal lines DD', EE', FF' are concurrent in a point S, it suffices to show that triangles DEF and $D'E'F'$ are homological, or, in other words, that the corresponding sides DE and $D'E'$, EF and $E'F'$, FD and $F'D'$ intersect in three collinear points M, N, P. But this is easily seen to be the case, since these three points belong to the Pascal line of the inscribed hexagon $AcBbCa$.

We also observe that triangles aCD and BcD', whose corresponding vertices lie on the concurrent lines aB, Cc, DD', are homological.

In a triangle $T \equiv ABC$, let A_1, B_1, C_1 be the midpoints of the sides BC, CA, AB, let A', B', C' be the feet of the corresponding altitudes, and let H be the orthocenter and K the symmedian point. Let N and P be the points of intersection, with lines AB and CA, of parallels to CA and AB through an arbitrary point M on line BC, and let X, Y, Z be the second points of intersection of circle $(C) \equiv MNP$ with the lines BC, CA, AB.

* Translated from the French by Howard Eves.

COROLLARY I. *The points of intersection A, D, E of the opposite sides of the hexagon $MXYPZN$ inscribed in circle (C) are collinear on the symmedian AK of triangle T, and the Pascal lines of the inscribed hexagons $MPYNZX$ and $MNYXZP$ are concurrent on AK.*

It suffices to show that the Pascal line ADE of the inscribed hexagon $MXYPZN$ coincides with the symmedian AK, for the Pascal lines of the other two hexagons are concurrent on the preceding one by our opening theorem. Now it is clear that the line ADE remains fixed as M varies along BC, for triangle DPZ remains homothetic to itself inasmuch as trapezoid $MPZN$ is isosceles and the angles at D, P, Z of the triangle are equal to the angles at C, A, B of T. It follows that the distances of D from sides CA and AB are proportional to these sides, and the proof is complete.

COROLLARY II. *The points A, $a_1 \equiv (A_1B_1, C'A')$, $a_1' \equiv (C_1A_1, A'B')$ are collinear on the symmedian AK of triangle T; the points B, $(B_1C', A'B')$, $(B'C_1, A_1B_1)$ are collinear on a line (D); the points C, $(B'C_1, C'A')$, (B_1C', C_1A_1) are collinear on a line (D'); the lines (D), (D') intersect on AK.*

This proposition is a particular case of the preceding one, the circle (C) being the nine point circle of triangle T. The symmedian AK coincides with the Pascal line of the inscribed hexagon $B'B_1A_1C_1C'A'$, and (D), (D') coincide with the Pascal lines of the hexagons $B'C_1A_1A'C'B_1$, $B'A'A_1B_1C'C_1$, obtained by permuting the vertices of the first hexagon according to the statement of the opening theorem.

Alternatively, one can show that AK passes through a_1 and a_1' by observing that these points bisect the segments CC_2, BB_2 of parallels to $B'C'$ drawn through C, B and limited by AB, AC. In fact, the parallel to AB drawn through C cuts $C'A'$ and $C'B'$ at the vertices V, W of an isosceles triangle $C'VW$ of altitude $C'C$. Consequently, $C'A'$ cuts CC_2 in its midpoint, which is situated on A_1B_1 and is thus located at a_1. A similar conclusion follows for a_1' on BB_2, whence the points A, K, a_1, a_1' are collinear.

The point of intersection C_2' of $A'B'$ and CC_2 is collinear with points H and B_2 on a symmedian of triangle HBC.

COROLLARY III. *The sets of points*

$$\begin{array}{lll} (C_1C', B'B_1), & (C'A_1, B_1A'), & (A_1B', A'C_1), \\ (A_1A', C'C_1), & (A'B_1, C_1B'), & (B_1C', B'A_1), \\ (B_1B', A'A_1), & (B'C_1, A_1C'), & (C_1A', C'B_1), \end{array}$$

determine three concurrent cevians (a), (b), (c) of triangle T.

The lines (a), (b), (c) coincide with the Pascal lines of the inscribed hexagons $C_1C'A_1B'B_1A'$, $C_1B'A_1A'B_1C'$, $C_1A'A_1C'B_1B'$, and are concurrent by virtue of the opening theorem. The point of concurrency has $(a^2 \cos A, b^2 \cos B, c^2 \cos C)$ for trilinear coordinates with respect to the fundamental triangle T.

COROLLARY IV. *The Pascal lines of the inscribed hexagons $C'C_1A_1A'B_1B'$, $C'A'A_1B'B_1C_1$, $C_1C'A'A_1B_1B'$ form a triangle homothetic to the fundamental triangle T.*

For the opposite sides B_1C_1 and A_1A', C_1A_1 and B_1B', A_1B_1 and C_1C' of these hexagons are parallel.

With the six points A_1, B_1, C_1, A', B', C' one can form 60 hexagons inscribed in the nine point circle of triangle T. The consideration of the 60 corresponding Pascal lines, which are not necessarily all distinct,* and the points which they determine by their intersections in triples according to the statement of the opening theorem of this note, would constitute a subject for study.

* Ch. Bioche, *Bulletin de la Société Mathématique de France*, t. 58, p. 27, (1930).

A NOTE ON NORMAL MATRICES

W. V. PARKER, Alabama Polytechnic Institute

A matrix A is said to be normal if $AA^*=A^*A$, where A^* denotes the transpose conjugate of A [1]. The purpose of this note is to extend a theorem of an earlier paper [2] concerning such matrices. That theorem may be stated as follows:

THEOREM 1. *If $A=(a_{ij})$ is an $n\times n$ normal matrix and the characteristic roots of A are $a_{11}, a_{22}, \cdots, a_{nn}$ then A is a diagonal matrix.*

In this note we establish

THEOREM 2. *If $A=(a_{ij})$ is an $n\times n$ normal matrix which can be partitioned into $A=(A_{\alpha\beta})$, $\alpha, \beta=1, 2, \cdots, k$, such that the characteristic function of A is the product of the characteristic functions of the square matrices $A_{\alpha\alpha}$, $\alpha=1, 2, \cdots, k$, then $A=\text{diag. }\{A_{11}, A_{22}, \cdots, A_{kk}\}$.*

We first prove this for A an Hermitian matrix.

LEMMA 1. *Let $A=(A_{\alpha\beta})$, $\alpha, \beta=1, 2, \cdots, k$, be an $n\times n$ Hermitian matrix and let $D=\text{diag. }\{A_{11}, A_{22}, \cdots, A_{kk}\}$. If A and D have the same characteristic function, then $A=D$.*

The coefficient of x^{n-2} in the characteristic function of an $n\times n$ matrix is the sum of all second order principal minors of the matrix. Denote these coefficients for matrices A and D by S and S' respectively. Then $S-S'=\sum a_{ij}\bar{a}_{ij}$ where the sum is taken over all i, j such that $i<j$ and no $A_{\alpha\alpha}$ contains both the ith row and the jth row of A. Since $S=S'$, $a_{ij}=0$ for every element of every matrix $A_{\alpha\beta}$, $\alpha\neq\beta$. This completes the proof of Lemma 1.

LEMMA 2. *Let $A=(A_{\alpha\beta})$, $\alpha, \beta=1, 2, \cdots, k$, be an $n\times n$ normal matrix and let $D=\text{diag. }\{A_{11}, A_{22}, \cdots, A_{kk}\}$. If A and D have the same characteristic function, D is also normal.*

There exist unitary matrices U_α such that $U_\alpha A_{\alpha\alpha} U_\alpha^*$, $\alpha=1, 2, \cdots, k$, is triangular. Let $U=\text{diag.}\ \{U_1, U_2, \cdots, U_k\}$. Then UAU^* is a normal matrix having its characteristic roots as diagonal elements. It follows from Theorem 1 that UAU^* is a diagonal matrix and consequently $U_\alpha A_{\alpha\alpha} A_\alpha^*$ is a diagonal matrix and $A_{\alpha\alpha}$ is normal. A direct sum of normal matrices is a normal matrix and hence D is normal.

We now return to the proof of Theorem 2. We may write $A=G+iH$, where G and H are Hermitian. Also, write $G=(G_{\alpha\beta})$ and $H=(H_{\alpha\beta})$ where $G_{\alpha\beta} =\frac{1}{2}(A_{\alpha\beta}+A_{\alpha\beta}^*)$ and $H_{\alpha\beta}=(A_{\alpha\beta}+A_{\beta\alpha}^*)/2i$. Since A is normal there exists a unitary matrix V such that $VAV^*=\text{diag.}\ \{L_1, L_2, \cdots, L_k\}$ where L_α is a diagonal matrix having as diagonal elements the characteristic roots of $A_{\alpha\alpha}$. From Lemma 2, we know that $A_{\alpha\alpha}$ is normal and hence there exist unitary matrices U_α such that $U_\alpha A_{\alpha\alpha} U_\alpha^*=L_\alpha$, $\alpha=1, 2, \cdots, k$. If $U=\text{diag.}\ \{U_1, U_2, \cdots, U_k\}$ and $W=U^*V$, then $WAW^*=\text{diag.}\ \{A_{11}, A_{22}, \cdots, A_{kk}\}$; consequently $WGW^* =\text{diag.}\ \{G_{11}, G_{22}, \cdots, G_{kk}\}$, and $WHW^*=\text{diag.}\ \{H_{11}, H_{22}, \cdots, H_{kk}\}$. It then follows from Lemma 1 that $G=\text{diag.}\ \{G_{11}, G_{22}, \cdots, G_{kk}\}$ and $H =\text{diag.}\ \{H_{11}, H_{22}, \cdots, H_{kk}\}$. Thus we have $A=\text{diag.}\ \{A_{11}, A_{22}, \cdots, A_{kk}\}$.

References

1. B. E. Mitchell, Normal and diagonalizable matrices, this MONTHLY, vol. 60, 1953, pp. 94–96.

2. W. V. Parker, Characteristic roots and the field of values of a matrix, Duke Math. Jour., vol. 15, 1948, pp. 439–442.

A NOTE ON A GENERALIZATION OF THE CAUCHY-MACLAURIN INTEGRAL TEST

T. A. NEWTON, University of Nebraska

1. De la Vallee-Poussin presented the following theorem as an exercise [8, p. 414]. (We shall use the notation $\Delta M_n=M_{n+1}-M_n$.)

THEOREM 1. *If* $0<M_n\leqq M_{n+1}$, $n=1, 2, 3, \cdots$, $M_n\to\infty$ *with* n, *and if* $F(x)$ *is positive monotone decreasing, then* $\sum_{n=1}^{\infty} F(M_{n+1})\Delta M_n$ *converges with* $\int^\infty F(x)dx$, *and* $\sum_{n=1}^{\infty} F(M_n)\ \Delta M_n$ *diverges with* $\int^\infty F(x)dx$.

When added conditions are placed on $\{M_n\}$, the following theorem results.

THEOREM 1a. *If, in Theorem* 1, *either* $\Delta M_n=O(1)$ *or* $M_{n+1}=O(M_n)$, *then* $\sum_{n=1}^{\infty} F(M_{n+1})\Delta M_n$ *and* $\sum_{n=1}^{\infty} F(M_n)\Delta M_n$ *converge and diverge together.*

These results have been known for some time. But since the author has been unable to find explicit proof of both theorems in the more readily available literature, they are proved in Section 2.

It is seen that these results contain the Cauchy-Maclaurin integral test and Cauchy's condensation test as special cases. By setting $F(x)=1/x^p$, the Abel-Dini theorem is easily proved using these results [6]. In turn, the Abel-

Dini theorem is closely related to Kummer's very general criteria [4, p. 310]. Extensive discussions concerning convergence and divergence criteria based upon Theorem 1 and 1a are given by Rajagopal in [5] and [7].

In Sections 3 and 4 of this note, we shall show that aside from serving as a basis for various convergence and divergence criteria, Theorems 1 and 1a give us a means for examining the behavior of the general terms of certain convergent series.

2. *Proof of Theorem* 1. Since $F(x)$ is monotone decreasing and $\{M_n\}$ is monotone increasing, it follows that $F(M_v) \geqq F(x) \geqq F(M_{v+1})$ for $M_v \leqq x \leqq M_{v+1}$, $v=1, 2, \cdots$. Consequently,

$$F(M_v)\Delta M_v \geqq \int_{M_v}^{M_v+1} F(x)dx \geqq F(M_{v+1})\Delta M_v$$

for $v=1, 2, 3, \cdots$. Summing for $v=1, 2, \cdots n$, it follows that

$$\sum_{v=1}^{n} F(M_v)\Delta M_v \geqq \int_{M_1}^{M_n+1} F(x)dx \geqq \sum_{v=1}^{n} F(M_{v+1})\Delta M_v$$

for $n>0$. Letting n become infinite, we obtain the desired result.

Proof of Theorem 1a. Considering first that $\Delta M_n = O(1)$, determine G such that $\Delta M_n \leqq G$ for all n. It follows that

$$0 \leqq \left\{ \sum_{v=1}^{n} F(M_v)\Delta M_v - \sum_{v=1}^{n} F(M_{v+1})\Delta M_v \right\} \leqq G \sum_{v=1}^{n} \{F(M_v) - F(M_{v+1})\}$$

or

$$0 \leqq \left\{ \sum_{v=1}^{n} F(M_v)\Delta M_v - \sum_{v=1}^{n} F(M_{v+1})\Delta M_v \right\} \leqq GF(M_1)$$

for $n>0$. Letting n become infinite, it follows that since the terms of both series are positive then the series must either both diverge or both converge.

For the case in which $M_{n+1} = O(M_n)$, we first note that if $\{M_n\}$ satisfies the hypothesis of Theorem 1, then so does $\{KM_n\}$ for any $K>0$. It then follows from Theorem 1 that the divergence of $\int^\infty F(x)dx$ implies the divergence of both $\sum_{n=1}^\infty F(M_n)\Delta M_n$ and $\sum_{n=1}^\infty F(KM_n)\Delta(KM_n)$. But since $M_{n+1} = O(M_n)$, we can choose K such that $M_{n+1} \leqq KM_n$ for $n>0$, and it follows from the monotone nature of $F(x)$ that $K^{-1}F(KM_n)\Delta(KM_n) \leqq F(M_{n+1})\Delta M_n$ for $n>0$. It follows by comparison that $\sum_{n=1}^\infty F(M_{n+1})\Delta M_n$ diverges with $\int^\infty F(x)dx$. That the series $\sum_{n=1}^\infty F(M_n)\Delta M_n$ and $\sum_{n=1}^\infty F(M_{n+1})\Delta M_n$ converge with $\int^\infty F(x)dx$ follows in a similar manner from the inequality $F(M_n)\Delta M_n \leqq KF(K^{-1}M_{n+1})\Delta(K^{-1}M_n)$ for $n>0$.

3. For the remainder of this paper, $\sum c_n$ (or $\sum c(n)$) and $\sum d_n$ (or $\sum d(n)$) denote respectively arbitrary convergent and divergent series of positive terms,

and $\{M_n\}$ denotes an arbitrary sequence satisfying the hypothesis of Theorem 1. The following facts have been pointed out in classical literature.

(3.1) If $\sum c_n$ is monotone, then $nc_n \to 0$, and if $p_n \to +\infty$ then there exists a monotone $\sum c_n'$ such that $\overline{\lim}\, np_nc_n' = +\infty$.

(3.2) Given any $\sum c_n$, there always exists a monotone $\sum d_n$ such that $d_n \to 0$ and $\underline{\lim}\, d_n/c_n = 0$.

(3.3) Given any $\sum d_n$ where $d_n \to 0$, there always exists a $\sum c_n$ such that $\overline{\lim}\, c_n/d_n = +\infty$.

These results are discussed by Knopp [**4**, pp. 300–305]. More recently, (3.1) was extended by R. W. Hamming in [**2**]. His results are included in the following theorem.

THEOREM 2. *If $\sum c(n)$ is monotone, M_{n+1}/M_n bounded from* 1 *and* ∞, *and $\sum c(M_n)\Delta M_n$ monotone decreasing, then $n(\log n)c(n) \to 0$.*

Proof. Since $M_{n+1} = O(M_n)$, it follows from Theorems 1 and 1a that $\sum c(M_n)\Delta M_n$ is convergent. The remainder of the proof is quite similar to that used by Hamming in [**2**], where he considers the particular case $M_n = 2^n$, and will not be presented here.

Furthermore, Theorem 2 is the best such statement in that if $p_n \to +\infty$, then there exists a $\sum c(n)$ and $\{M_n\}$ satisfying the conditions of Theorem 2 such that $np_n (\log n)c(n)$ does not approach zero. A theorem by Hamming in [**2**] suffices as a proof of this assertion in that he proved that if we let $M_n = 2^n$, then there exists a monotone $\sum c(n)$ such that $\sum 2^n c(2^n)$ is also monotone and yet $np_n (\log n)c(n)$ does not approach zero.

4. With respect to statements (3.2) and (3.3), Hamming in [**3**] and Aryeh Dvoretzky in [**1**] have proved some theorems which show, in a certain sense, the rarity of the occurrence $c_n \geqq d_n$ for arbitrary monotone $\sum c_n$ and $\sum d_n$. Theorem 1 of Dvoretzky implies Hamming's results, while the following theorem implies Theorem 1 of Dvoretzky. This theorem is only a slight extension of Dvoretzky's theorem, but is thought to be of interest since its proof is based upon Theorems 1 and 1a above.

THEOREM 3. *If $\sum c_n$ and $\sum d_n$ are monotone, and for arbitrary $\{M_n\}$, either $\Delta M_n = O(1)$ or $M_{n+1} = O(M_n)$, then there exist infinitely many n such that for all v with $M_n \leqq v \leqq M_{n+1}$, we have $c_v < d_v$.*

Proof. Assume that $c(x)$ and $d(x)$ are monotone decreasing functions such that $c(n) = c_n$, and $d(n) = d_n$. Further assume that there exists an n_0 such that for all $n > n_0$, $c(M_n) \geqq d(M_{n+1})$, or $c(M_n)\Delta M_n \geqq d(M_{n+1})\Delta M_n$ where $\{M_n\}$ satisfies the hypothesis of the theorem. But it follows from Theorems 1 and 1a that $\sum d(M_{n+1})\Delta M_n$ diverges with $\sum d_n$. Consequently the latter inequality implies the divergence of $\sum c(M_n)\Delta M_n$. Again, considering Theorems 1 and 1a, we see that the divergence of $\sum c(M_n)\Delta M_n$ implies the divergence of $\sum c_n$,

thus arriving at a contradiction. Consequently there exist infinitely many n such that $c(M_n) < d(M_{n+1})$. However, due to the monotone nature of the functions $c(x)$ and $d(x)$, if the latter inequality is true for n, then for all v with $M_n \leqq v \leqq M_{n+1}$, $c(v) \leqq c(M_n) < d(M_{n+1}) \leqq d(v)$.

References

1. Aryeh Dvoretzky, On monotone series, Amer. Jour. Math., vol. 70, 1948, pp. 167–173.
2. R. W. Hamming, Convergent monotone series, this MONTHLY, vol. 52, 1945, pp. 70–72.
3. R. W. Hamming, Monotone series, Amer. Jour. Math., vol. 68, 1946, pp. 133–136.
4. Konrad Knopp, Theory and Application of Infinite Series. Translated from second German Edition, London, 1946.
5. C. T. Rajagopal, Remarks on some generalizations of Cauchy's condensation and integral tests, this MONTHLY, vol. 46, 1939, pp. 338–341.
6. C. F. Rajagopal, The Abel-Dini and allied theorems, this MONTHLY, vol. 51, 1944, pp. 566–570.
7. C. T. Rajagopal, Convergence theorems for series of positive terms, Jour. Indian Math. Soc., vol. 3, 1938, pp. 118–125.
8. De la Vallee-Poussin, Cours d'Analyse Infinitesimale, vol. 1, First American Edition, New York, 1946.

TRISECTION APPROXIMATION

FREE JAMISON, San Jose State College, California

On January 13, 1953, C. R. Lindberg† proposed a method for the approximate trisection of the angle. His method with one slight refinement follows. Using V, the vertex of the angle as a center, let a circle intersect the sides at A and B (Fig. 1). Draw BV and extend to C on the circle. Through D, the midpoint of $\widehat{AB}$, draw CD and extend to E so that $DE=BC$. Draw EV which makes $\angle BVE$ approximately $1/3\angle BVA$. The error can be found by consulting a tables of sines, for in $\triangle EVD$, $\sin V = 2 \sin E$, and $\angle V + \angle E = 1/4\angle BVA$. By solving, the error for a given positive angle $X < \pi$ is

$$X/12 - \arctan\left[(\sin X/4)/(2+\cos X/4)\right].$$

If $Y = X/4$, the error is

$$\begin{aligned}\frac{Y}{3} - \arctan\frac{\sin Y}{2+\cos Y} &< \frac{Y}{3} - \arctan\frac{Y - Y^3/6}{3 - Y^2/2 + Y^4/24}\\ &< \frac{Y}{3} - \left[\frac{Y}{3} - \frac{Y^5}{3(72 - 12Y^2 + Y^4)}\right.\\ &\quad \left. + 1/3\left(\frac{Y}{3}\right)^3\right] < \frac{Y^5}{194} + \frac{Y^3}{81} < \frac{X^3}{4000}.\end{aligned}$$

† C. R. Lindberg: Method of trisecting an angle, Unpublished paper, San Jose State College.

In an extension of Lindberg's method (Fig. 2), D is located so that $\widehat{BD} = (3/8)\widehat{BA}$ and C is diametrically opposite F where $\widehat{BF} = (1/4)\widehat{BA}$. In this extension the error for a given positive angle $X < \pi$ is

$$X/48 - \arctan\left[(\sin X/16)/(2+\cos X/16)\right] < X^3/320{,}000.$$

A comparison of the accuracy and simplicity of the five approximations described by Yates‡ with those based on Lindberg's method is given in the following table. The column headed "N" is the total of the number of arcs, number of settings of compasses to a definite length, and the number of lines required including an extension of a side where necessary.

Method	N	Approximate Maximum Errors	
		Acute Angles	Obtuse Angles
Von Cusa—Snellius	15	3° 26′	*
Dürer	>20	18″	31′
Karajordanoff	16	2′ 12″	*
Kopf-Perron	14	15″	*
D'Ocagne	8	22′	*
Lindberg	9	3′	22′
Lindberg's Extended	12	3″	20″

* Not applicable or else errors are more than 3°.

Topel§ made some improvement on the method of Kopf-Perron applicable to angles less than 45°, which reduced the error to less than 0.5″. However, Lindberg's method extended requires a smaller N than Topel's and gives an approximate maximum error of 0.3″ for angles less than 45°.

The small errors by Lindberg's method are not surprising when it is considered that small angles are almost proportional to their sines; and a triangle with two small angles whose opposite sides are in the ratio 2:1 will have its smallest angle approximately equal to one-third of the exterior angle at the third vertex. It may be of interest that Lindberg arrived at his method unaware

‡ Yates, R. C.: The Trisection Problem, West Point, N. Y.,

§ Topel, Bernard J.: Concerning a remark of Canon Lemâitre about Kopf's trisection of the angle, Reports of a Mathematical Colloquium, Second Series, Issue 1, Notre Dame, 1939, pp. 49–52.

that $DE = 2VD$; the refinement mentioned heretofore concerned the method of locating but not the position of E.

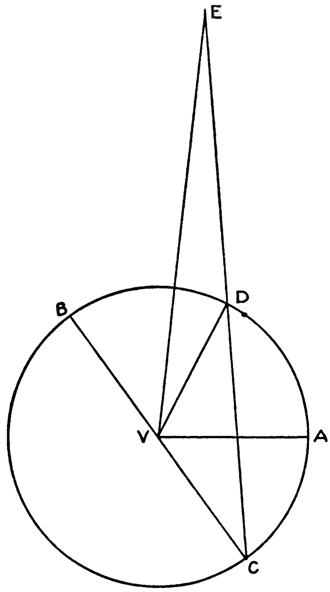


FIG. 1

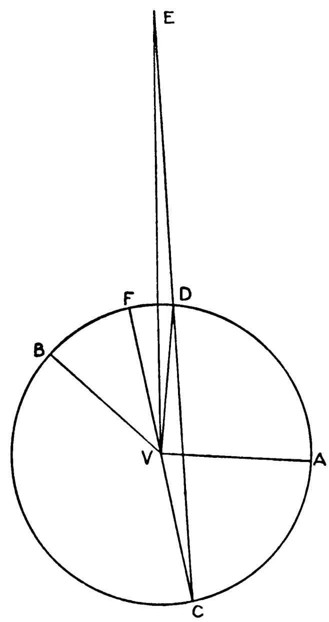


FIG. 2

DIVISORS OF ZERO IN POLYNOMIAL RINGS

W. R. SCOTT, The University of Kansas

The purpose of this note is to give a brief proof of the following theorem, proved by Alexandra Forsythe in a note with the above title, this MONTHLY, vol. 50, 1943, pp. 7–8. (*Cf.* also, N. H. McCoy, *Rings and Ideals*, p. 34, Th. 4.)

THEOREM. *Let R be a commutative ring, and let $R[x]$ be the ring of polynomials over R. If f in $R[x]$ is a divisor of zero, then there is a c in R such that $c \neq 0$, $cf = 0$.*

Proof. Deny the theorem, and let g be a non-zero polynomial of smallest degree such that $fg = 0$. Let

$$f = a_0 + a_1 x + \cdots + a_m x^m,$$
$$g = b_0 + b_1 x + \cdots + b_n x^n,$$

where $b_n \neq 0$ and $n \geqq 1$. Since $b_n f \neq 0$, $a_i b_n \neq 0$ for some i, and therefore $a_i g \neq 0$. Let r be the largest integer such that $a_r g \neq 0$. Then

$$fg = (a_0 + \cdots + a_r x^r)(b_0 + \cdots + b_n x^n) = 0.$$

Hence $a_r b_n = 0$ and deg $(a_r g) < n$. However, $(a_r g) f = 0$, which is a contradiction.

CLASSROOM NOTES

EDITED BY G. B. THOMAS, Massachusetts Institute of Technology

All material for this department should be sent to G. B. Thomas, Department of Mathematics, Massachusetts Institute of Technology, Cambridge 39, Mass.

NOTE ON HERO'S FORMULA

J. P. BALLANTINE, University of Washington

In the development of the formulas for the solution of a triangle, given three sides, by logarithms, the usual derivation of the formula for r or K (the area) is rather long, starting from the law of cosines. Following is a much briefer derivation.

The formulas, $\tan \frac{1}{2}A = r/(s-a)$, $\tan \frac{1}{2}B = r/(s-b)$, and $\tan \frac{1}{2}C = r/(s-c)$ can be read easily from the diagram. Then

$$\tfrac{1}{2}A + \tfrac{1}{2}B = 90° - \tfrac{1}{2}C.$$

Take the tangent of both members:

$$\frac{\tan \frac{1}{2}A + \tan \frac{1}{2}B}{1 - \tan \frac{1}{2}A \tan \frac{1}{2}B} = \frac{1}{\tan \frac{1}{2}C}.$$

Clear:

$$\tan \tfrac{1}{2}A \tan \tfrac{1}{2}C + \tan \tfrac{1}{2}B \tan \tfrac{1}{2}C = 1 - \tan \tfrac{1}{2}A \tan \tfrac{1}{2}B,$$

$$\tan \tfrac{1}{2}B \tan \tfrac{1}{2}C + \tan \tfrac{1}{2}A \tan \tfrac{1}{2}C + \tan \tfrac{1}{2}A \tan \tfrac{1}{2}B = 1.$$

Now substitute the values of the various tangents:

$$\frac{r^2}{(s-b)(s-c)} + \frac{r^2}{(s-a)(s-c)} + \frac{r^2}{(s-a)(s-b)} = 1,$$

$$r^2(s - a + s - b + s - c) = (s-a)(s-b)(s-c),$$

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

TRIGONOMETRY FROM DIFFERENTIAL EQUATIONS

D. E. RICHMOND, Williams College

1. Introduction. This note shows how analytic trigonometry may be developed in an elementary manner (with no use of infinite series) from the differential equation

$$\text{(I)} \qquad \frac{d^2y}{dt^2} + y = 0$$

to which one is naturally led through the study of simple harmonic motion.

First write (I) as

$$\frac{dx}{dt} = -y \tag{1}$$

by setting

$$\frac{dy}{dt} = x. \tag{2}$$

Multiplying (1) by x and (2) by y and adding,

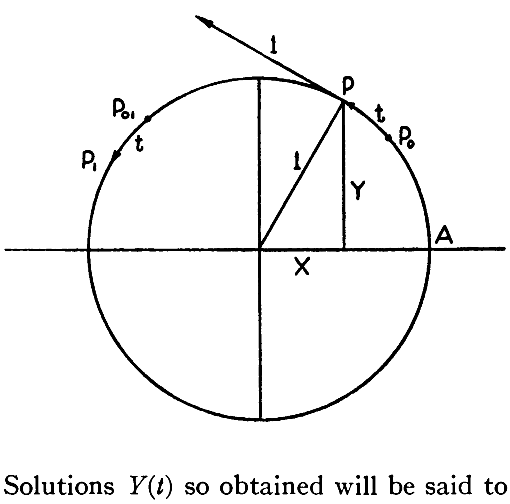
$$x\frac{dx}{dt} + y\frac{dy}{dt} = 0.$$

Hence $x^2+y^2=r^2$ where r is a constant. For given t, x and y are the coordinates of a point on a circle, omitting the trivial case $r=0$.

The radius of this circle becomes 1 if we set $x=rX$, $y=rY$. Then

$$\frac{dX}{dt} = -Y \tag{3}$$

$$\frac{dY}{dt} = X \tag{4}$$



where $X^2+Y^2=1$. Solutions $Y(t)$ so obtained will be said to be *normalized.*

Clearly (3) and (4) give the components of a motion with the uniform velocity 1 in a counterclockwise direction along the unit circle C. If arc length s is measured along C in this direction, $ds/dt=1$ and $s=t+c$. In fact, $s=t$ if s is measured from P_0, the position at $t=0$.

Two normalized solutions of (I), $Y(t)$ and $Y_1(t)$, can differ only in the positions of their initial points, P_0 and P_{01} respectively. But then $Y_1(t)=Y(t+a)$

where a is the arc length from P_0 to P_{01}. Moreover, given one normalized solution of (I), $Y(t)$, then all functions $Y(t+a)$ with constant a are normalized solutions of (I). The general solution of (I) is therefore

$$y = rY(t + a)$$

with arbitrary constants r and a.

Let us define $\sin t$ to be that solution $Y(t)$ for which $Y=0$ and $X(=dY/dt)=1$ at $t=0$, that is, that solution for which $P_0=A$ (see Figure). The corresponding $X(t)=dY(t)/dt$ will be defined to be $\cos t$. From (3) and (4)

$$\frac{d \sin t}{dt} = \cos t,$$

$$\frac{d \cos t}{dt} = -\sin t.$$

It follows at once that $\cos t$ (as well as $\sin t$) is a solution of (I).

It is geometrically obvious that $\sin(t+2\pi)=\sin t$ and $\cos(t+2\pi)=\cos t$ for all t, and also that

$$\sin(-t) = -\sin t, \qquad \cos(-t) = \cos t.$$

The radian (or circular) measure of angles becomes extremely natural.

2. The addition formulas. To derive the addition formulas, we verify by differentiation that

$$c_1 \sin t + c_2 \cos t$$

is a solution of (I) for arbitrary constants c_1 and c_2. Then for some r and some a,

$$c_1 \sin t + c_2 \cos t = r \sin(t + a). \tag{5}$$

Differentiating,

$$c_1 \cos t - c_2 \sin t = r \cos(t + a). \tag{6}$$

Substituting $t=0$ in (5) and (6),

$$c_1 = r \cos a, \qquad c_2 = r \sin a.$$

Inserting these values in (5) and (6) and cancelling the factor r,

$$\sin(t + a) = \sin t \cos a + \cos t \sin a, \tag{7}$$

$$\cos(t + a) = \cos t \cos a - \sin t \sin a. \tag{8}$$

The remaining formulas of analytic trigonometry follow without difficulty.

3. Calculation of sin *t* and cos *t*. If it is desired to *calculate* $\sin t$ and $\cos t$, it suffices to consider the case $t>0$ and integrate both members of $\cos t \leqq 1$ from 0 to t, obtaining $\sin t<t$. Continuing,

$$1 - \cos t < \frac{t^2}{2}$$

or

$$\cos t > 1 - \frac{t^2}{2};$$

$$\sin t > t - \frac{t^3}{3!};$$

$$\cos t < 1 - \frac{t^2}{2!} + \frac{t^4}{4!};$$

$$\cdots\cdots\cdots .$$

The theorem used here is the intuitively obvious one that if $f(t)$ and $g(t)$ are two different integrable functions such that $f(t) \leqq g(t)$, $(t>0)$, then

$$\int_0^t f(t)dt \leqq \int_0^t g(t)dt, \qquad (t > 0).$$

It is hoped that this note will nourish the suspicion that the conventional semester course in trigonometry involves considerable educational waste.

A GEOMETRIC DETERMINATION OF THE NATURE OF THE ROOTS OF THE CUBIC, BIQUADRATIC, AND QUINTIC EQUATIONS

M. S. KLAMKIN, Polytechnic Institute of Brooklyn

The cubic equation,

(1) $$ax^3 + 3bx^2 + 3cx + d = 0, \qquad a > 0,$$

can be reduced by the substitution, $x = y/m - b/a$, to the canonical form:

(2) $$y^3 + 3py + 2 = 0.$$

Here,

(3) $$m^3[2b^3 - 3abc + da^2] = 2a^3$$
$$m^2[ac - b^2] = pa^2.$$

We will assume that $2b^3 - 3abc + da^2 \neq 0$, and that $ac - b^2 \neq 0$, for otherwise $x = -b/a$ is a root or the cubic can be solved by completing the cube.

Let us now consider the graphs of

(4) $$z = y^2 + 2/y, \quad \text{and} \quad z = -3p.$$

The first curve is asymptotic to $y=0$ and to the parabola $z=y^2$. Also, the curve has a minimum point at $(1, 3)$ and an inflection point at $(-2^{1/3}, 0)$. The second curve is a straight line parallel to the y-axis. The intersections of these two

curves represent the real roots of equation (2). The possible intersections for different values of p are shown in Figure 1.

Consequently, if $-p>1$, there are three real and unequal roots. If $-p=1$, there are three real roots, two of which are equal. And, finally, if $-p<1$, there is only one real root, and the other two are complex.

The nature of the roots is also determined by the sign of the discriminant. However, the two criteria are equivalent, since here

$$\Delta = 4(1 + p^3).$$

This method not only determines the nature of the roots but affords a convenient way of obtaining the real roots of a cubic, since the curve of Figure 1

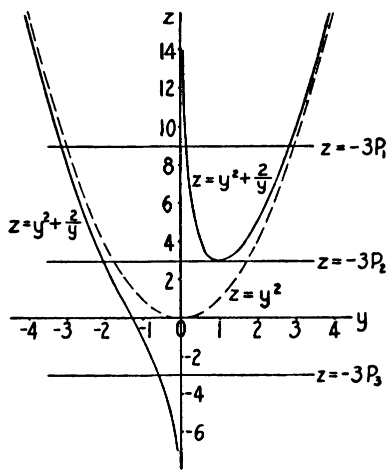


FIG. 1

is the same for all cubics. It should be noted that for a certain range of p, there is no need to resort to a graphical solution of the equation $y^3+3py+2=0$, since the roots are tabulated in Jahnke, Emde, *Tables of Functions*.

The biquadratic equation,

$$ax^4 + 4bx^3 + 6cx^2 + 4dx + e = 0, \qquad a > 0, \tag{5}$$

can be reduced by the substitution, $x=ry-b/a$, to the form

$$y^4 + py^2 + my + n = 0, \tag{6}$$

where $p=1$, -1, or 0, according to whether $ac-b^2>0$, <0, or $=0$, respectively.

Let us now consider the graphs of

$$z = y^4 + py^2, \quad \text{and} \quad -z = my + n. \tag{7}$$

Some possible intersections are shown in Figures 2 and 3. The figure for $p=0$ is similar to that for $p=1$. Thus, when $ac-b^2 \geqq 0$, there are either two or zero real

roots. Moreover, if $n \leqq 0$, there are two real roots. When $ac-b^2<0$, there are either four, two, or zero real roots. If in addition, $n<-\frac{1}{12}$, $-\frac{1}{12}<n<\frac{1}{4}$, or $n>\frac{1}{4}$, then there are two, four or two, or two or zero, real roots, respectively. For a more complete description of the roots, we would have to consider the case where the curves of (7) are tangent to each other. This can be done, but not readily. However, by plotting the curves of (7) we can determine all the real roots of (6). The three quartic curves $z=y^4+py^2$ can be drawn once to serve for all quartics, while the straight line $-z=my+n$ is easily drawn.

The quintic equation,

$$ax^5 + 5bx^4 + 10cx^3 + 10dx^2 + 5ex + f = 0, \qquad a > 0, \tag{8}$$

can be reduced by the substitution, $x=sy-b/a$, to the form

$$y^5 + py^3 + my^2 + ny + r = 0, \tag{9}$$

where $p=1$, -1, or 0, according to whether $ac-b^2>0$, <0, or $=0$, respectively.

Let us now consider the graphs of

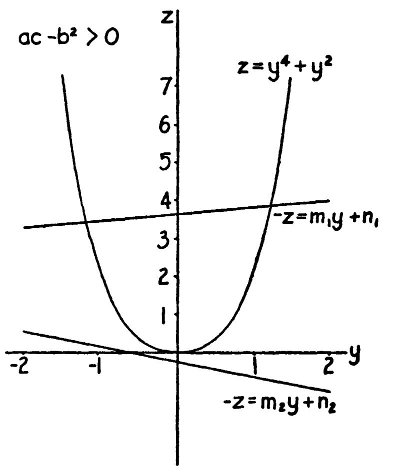


FIG. 2

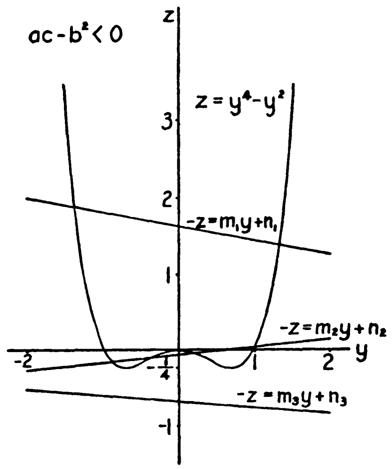


FIG. 3

$$z = y^5 + py^3, \quad \text{and} \quad -z = my^2 + ny + r. \tag{10}$$

The figures are not drawn here, but can easily be constructed by the reader. From them we find that when $ac-b^2 \geqq 0$, there are either three or one real roots, depending on where the vertex of the parabola of (10) is located, and on the sign of m. When $ac-b^2<0$, there are either five, three, or one real roots. Again, by plotting the curves of (10) (the only variable one being the parabola) we can obtain the real roots of (9).

The usefulness of the above method for solving quintic equations would be greatly increased if there existed a simple device for drawing parabolas.

PRODUCT OF DETERMINANTS BY INDUCTION

C. M. FULTON, University of California, Davis

In this paper we prove the multiplication theorem of determinants by mathematical induction. We use only properties of determinants which are usually encountered in college algebra texts.

Subscripts take the values $1, \cdots, n$. If a_{ij} is the element of a determinant of order n in the ith row and jth column, its cofactor is denoted by A_{ij} and the value of the determinant by A.

Let a_{ij} and b_{ij} be the elements of two determinants. Consider a third determinant whose elements are defined by the relation

$$c_{ik} = a_{i1}b_{1k} + a_{i2}b_{2k} + \cdots + a_{in}b_{nk}.$$

We want to show that $C=AB$.

We need not comment on the validity of the theorem for $n=1$. Our induction proof will be carried out for the special case in which

$$a_{21} = a_{31} = \cdots = a_{n1} = 0.$$

To save space we suggest that the reader write out the three determinants. In the particular case under consideration, for $i \neq 1$,

$$c_{ik} = a_{i2}b_{2k} + \cdots + a_{in}b_{nk}.$$

The sum on the right side has $n-1$ terms only. We now direct our attention to the cofactors C_{1j}. If our theorem is true for determinants of order $n-1$, these cofactors can be written as products as follows

$$C_{11} = A_{11}B_{11},\ C_{12} = A_{11}B_{12}, \cdots,\ C_{1n} = A_{11}B_{1n}.$$

We then see from the well-known properties of cofactors that

$$\begin{aligned} C &= C_{11}c_{11} + C_{12}c_{12} + \cdots + C_{1n}c_{1n} \\ &= A_{11}B_{11}(a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1n}b_{n1}) \\ &\quad + A_{11}B_{12}(a_{11}b_{12} + a_{12}b_{22} + \cdots + a_{1n}b_{n2}) + \cdots \\ &\quad + A_{11}B_{1n}(a_{11}b_{1n} + a_{12}b_{2n} + \cdots + a_{1n}b_{nn}) \\ &= A_{11}a_{11}(B_{11}b_{11} + B_{12}b_{12} + \cdots + B_{1n}b_{1n}) \\ &\quad + A_{11}a_{12}(B_{11}b_{21} + B_{12}b_{22} + \cdots + B_{1n}b_{2n}) + \cdots \\ &\quad + A_{11}a_{1n}(B_{11}b_{n1} + B_{12}b_{n2} + \cdots + B_{1n}b_{nn}) = AB. \end{aligned}$$

To complete the proof of our theorem we reduce the general case to the special case that has been treated. With regard to the elements in the first column of the determinant of the a's we distinguish between two cases. Either all of them are zero or at least one is not zero. The first case has been covered. In the second case we may assume without loss of generality that $a_{11} \neq 0$. We then add to the elements of the ith row $(i \neq 1)$ the corresponding elements of

the first row multiplied by the same number r_i. This leaves A unchanged. But at the same time C remains unchanged since

$$(a_{i1} + r_i a_{11})b_{1k} + (a_{i2} + r_i a_{12})b_{2k} + \cdots + (a_{in} + r_i a_{1n})b_{nk} = c_{ik} + r_i c_{1k}.$$

We can now establish the situation of our special case by choosing r_i in such a way that $a_{i1}+r_i a_{11}=0$, $(i\neq 1)$.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, State University of New York

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Harpur College, Endicott, New York. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1116. *Proposed by C. W. Trigg, Los Angeles City College*

We define a pandiagonal heterosquare as a square array of the first n^2 positive integers, so arranged that no two of the rows, columns, and diagonals (broken as well as straight) have the same sum. Is there any n for which these $4n$ sums are consecutive numbers?

E 1117. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Construct a right triangle in which the legs and the altitude on the hypotenuse can be taken as the sides of another right triangle.

E 1118. *Proposed by Joel Brenner, State College of Washington*

Let us say a permutation is of type T if it is a product of mutually permutable transpositions. Is every permutation p a product of two permutations of type T?

E 1119. *Proposed by L. C. Graue, Sacramento State College*

Consider two families of circles, one tangent at the origin to the x-axis and the other tangent at the point (1, 1) to a line of slope m. Find the locus of the points of tangency of the two families.

E 1120. *Proposed by W. B. Carver, Cornell University*

(a) A student working a numerical problem gets the result

$$59\sqrt{90 - 14\sqrt{7}} + 4\sqrt{4555 + 1721\sqrt{7}},$$

but he finds the answer in the book, which happens to be correct, is

$$145\sqrt{26 + 2\sqrt{7}}.$$

Show that the student's result is correct.

(b) Find all sets of integers k, R, S, T, A, B, C, D, E, F which satisfy the equation

$$R\sqrt{A + B\sqrt{k}} + S\sqrt{C + D\sqrt{k}} + T\sqrt{E + F\sqrt{k}} = 0,$$

with $k>0$ and having no square factors, $A+B\sqrt{k}>0$, $C+D\sqrt{k}>0$, $E+F\sqrt{k}>0$, and the radicals meaning in all cases the positive square root.

SOLUTIONS

Time Soliloquy

E 1086 [1953, 626]. *Proposed by T. A. Bickerstaff, University of Mississippi*

"That was a good lunch; now for a good cigar and then I must catch the one o'clock train. Let's see—my watch says exactly nine o'clock but that can't be right. It's still running and well wound. Now I remember I wound it and set it just this morning by the radio. Maybe I carelessly set the hands in reverse position. If so, exactly what time is it?"

Solution by C. F. Pinzka, Educational Testing Service, Princeton, N. J. Let x and y $(0\leqq x<y<12)$ represent the positions of the hands when they were set in reverse position. If both the correct and the reverse positions are to be meaningful, we must have $12x-y\equiv 12y-x\equiv 0 \pmod{12}$, or $13(y-x)\equiv 0 \pmod{12}$. The error due to reversing the hands is $y-x=(12/13)k$, k being an integer. Only $k=4$ leads to a time that is reasonable, this time being $12{:}41\frac{7}{13}$.

Also solved by Leon Bankoff, A. P. Boblétt, Julian Braun, W. B. Carver, Monte Dernham, S. H. Eisman, L. R. Ford, A. H. Freitag and Herta Freitag (jointly), Vern Hoggatt, R. T. Hood, R. W. Huff, A. R. Hyde, R. Klopfenstein, Sam Kravitz, L. V. Mead, E. F. Myers, C. S. Oglivy, Azriel Rosenfeld, C. Swanson, Daniel Weiner, R. H. Wilson, Jr., and the proposer.

A Special Case of Stern's Formula

E 1087 [1953, 626]. *Proposed by P. A. Piza, San Juan, Puerto Rico*

Let a be an arbitrary positive integer and let

$$S_m = \sum_{j=1}^{a} j^m.$$

We have

$$a(a+1)^2 = S_1 + 3S_2, \qquad a^2(a+1)^3 = S_2 + 2S_3 + 5S_4,$$
$$a^3(a+1)^4 = S_3 + 5S_4 + 3S_5 + 7S_6, \cdots .$$

Determine coefficients c_k such that

$$a^n(a+1)^{n+1} = \sum_{k=n}^{2n} c_k S_k.$$

Solution by F. R. Olson, Duke University. We write

$$x^n(x+1)^{n+1} = \sum_{s=0}^{n+1} \binom{n+1}{s} x^{2n+1-s} \tag{1}$$

and

$$(x-1)^n x^{n+1} = \sum_{s=0}^{n} \binom{n}{s} (-1)^s x^{2n+1-s}. \tag{2}$$

Subtraction of (2) from (1) and summation over x from $x=1$ to $x=a$ gives

$$a^n(a+1)^{n+1} = \sum_{s=0}^{n} \left[\binom{n+1}{s} + (-1)^{n-s} \binom{n}{s-1} \right] S_{n+s}, \tag{3}$$

where we define $\binom{s}{-1}=0$. This method may be applied to handle the more general case $a^n(a+1)^m$.

These results are also obtained in a different manner in Nielson, *Traité Élémentaire des Nombres de Bernoulli* (1933), p. 305.

Also solved by W. E. Briggs, Leonard Carlitz, P. L. Chessin, F. J. Duarte, S. H. Eisman, N. J. Fine, A. R. Hyde, Bernard Jacobson, C. W. Karns, M. S. Klamkin, Kovina Milosevich, M. J. Pascual, C. F. Pinzka, R. C. Read, D. C. Russell, E. P. Starke, Chih-yi Wang, and the proposer. Late solutions by Julian Braun and C. D. Olds.

Probability of Winning a Game

E 1088 [1953, 627]. *Proposed by M. J. Pascual, Siena College*

A and B agree to play n games and he who goes first has a chance of $a/(a+b)$ of winning that game. If the winner of any game goes first in the following game, then find (1) the probability of A winning the nth game if he goes first in the opening game, (2) the expected value of A's winnings in n games if each game is worth d dollars.

Solution by D. S. Greenstein, University of Pennsylvania. Let $P=a/(a+b)$, and let P_n denote the probability that A win the nth game. Then

$$P'_{n+1} = P_nP + (1-P_n)(1-P) = (2P-1)P_n + (1-P), \qquad P_1 = 1.$$

These conditions are uniquely satisfied by $P_n = [1+(2P-1)^n]/2$. A's expected winning in the nth game is

$$P_nd - (1-P_n)d = (2P_n - 1)d = d(2P-1)^n.$$

His expected winning for the first n games, therefore, is given by

$$d\sum_{k=1}^{n}(2P-1)^k = d(2P-1)[1-(2P-1)^n]/2(1-P).$$

Also solved by P. H. Arnold, J. W. Baldwin, Julian Braun, S. H. Eisman, N. J. Fine, L. A. Fulk, R. E. Greenwood, J. M. Howell, A. R. Hyde, M. S. Klamkin, A. E. Livingston, C. F. Pinzka, Azriel Rosenfeld, William Small, and the proposer.

Greenwood and Klamkin pointed out that this problem appears in a more general context in Uspensky, *Introduction to Mathematical Probability*, p. 75.

Generalization of a Real Function Theorem

E 1089 [1953, 627]. *Proposed by W. R. Utz, University of Missouri*

Show that if $f(x)$ is a real function bounded below on $I=[a, b]$ and $g(x)$ is real and either (i) monotone increasing on I or (ii) continuous with $g(b)>g(x)$ for $a\leqq x\leqq b$, then given $\epsilon>0$ there is a constant $\lambda(\epsilon)>0$ such that $f(x)-\lambda g(x)$ cannot attain $\inf_{x\epsilon I}[f(x)-\lambda g(x)]$ on $[a, b-\epsilon]$. (The special case $g(x)=x$ is used by E. Baiada, *Ann. Scuola Norm. Super. Pisa*, vol. 15 (1950), p. 111.)

Solution by L. E. Ward, Jr., University of Nevada. Let $M(\epsilon)$ denote the g.l.b. of $f(x)$ for $a\leqq x\leqq b-\epsilon$, and let $g(x_0)$ be the maximum of $g(x)$ for $a\leqq x\leqq b-\epsilon$. We are assured of the existence of x_0 if g is continuous, and, if g is monotone increasing, then $x_0=b-\epsilon$. Choose $\lambda(\epsilon)$ such that

$$\lambda(\epsilon) > \max\{[M(\epsilon)-f(b)]/[g(x_0)-g(b)],\ 0\}.$$

Since $g(x_0)-g(b)<0$, and since $f(x)\geqq M(\epsilon)$ and $g(x)\leqq g(x_0)$ for $x\leqq b-\epsilon$, we have

$$f(x)-\lambda g(x) > f(b)-\lambda g(b), \qquad x\leqq b-\epsilon.$$

Therefore, for $x\leqq b-\epsilon$, it follows that $f(x)-\lambda g(x)$ cannot attain $\inf_{x\epsilon I}[f(x)-\lambda g(x)]$.

Also solved by R. C. Read and the proposer.

An Interesting Construction Problem

E 1090 [1953, 627]. *Proposed by B. M. Stewart, Michigan State College*

From one vertex of a triangle lines are to be drawn dividing the triangle into a set S of n triangles having equal inscribed circles.

(1) Show that in general the set S may be constructed by ruler and compass if and only if $n=2^s$.

(2) Show that the $n-k+1$ triangles formed by taking sets of k adjacent triangles of the set S have equal inscribed circles ($k=2, 3, \cdots, n-1$).

(3) Find a neat construction when $n=2$. (*Note.* From (2) it follows that repeated application of (3) will solve the problem when $n=2^s$.)

Solution by the Proposer. Given triangle ABC with the ordered points $C_0=A$, $C_1, \cdots, C_{n-1}, C_n=B$ on the side AB, let r_{ik} be the radius of the in-

scribed circle of triangle CC_iC_{i+k}, $i=0, 1, \cdots, n-k$. Let h be the length of the altitude drawn from C and let $\alpha=\tan A/2$, $\beta=\tan B/2$. We can show that the conditions $r_{01}=r_{11}= \cdots =r_{n-1,1}$ imply

$$r_{ik} = [1 - (\alpha\beta)^{k/n}]h/2. \tag{1}$$

On the one hand, when $k=1$, formula (1) shows by the Galois conditions for constructibility of the new quantity $(\alpha\beta)^{1/n}$ from the known quantities α and β, that (a) holds.

On the other hand, for any k, formula (1) is independent of i, which establishes (b).

To prove (1) we note that for any triangle such as ABC we may express the area in two ways involving h, r, α, β, and $\gamma=\tan C/2$, as follows:

$$(r/\alpha + r/\beta)h/2 = (r/\alpha + r/\beta + r/\gamma)r.$$

Since $\gamma=(1-\alpha\beta)/(\alpha+\beta)$, we obtain

$$r = (1 - \alpha\beta)h/2, \qquad (h - 2r)/h = \alpha\beta. \tag{2}$$

If we set $\alpha_i=\tan C_{i+1}C_iC/2$ and $\beta_i=\tan CC_iC_{i-1}/2$, then

$$\alpha_i\beta_i = 1, \tag{3}$$

because the external and internal angle bisectors are perpendicular. Applying (2) to the triangle CC_iC_{i+1} we have $(h-2r_{i1})/h=\alpha_i\beta_{i+1}$. Thus the conditions $r_{01}=r_{11}= \cdots =r_{n-1,1}$ hold if and only if

$$\alpha\beta_1 = \alpha_1\beta_2 = \alpha_2\beta_3 = \cdots = \alpha_{n-1}\beta.$$

Using (3) we readily check that this system of equations is solved by

$$\alpha_i^n = \alpha^{n-i}/\beta^i, \qquad i = 0, 1, \cdots, n-1.$$

For

$$(\alpha_i\beta_{i+k})^n = (\alpha^{n-i}/\beta^i)(\beta^{i+k}/\alpha^{n-i-k}) = (\alpha\beta)^k.$$

When $k=1$, this shows that the system of equations is solved; and for any k, this shows starting from (2) that

$$r_{ik} = (1 - \alpha_i\beta_{i+k})h/2$$

is equivalent to (1).

For part (c) we may let I be the incenter for triangle ABC and draw BQ parallel to IC to meet AC at Q. Then the circle through A, B, Q is the locus of points subtending an angle $C/2$ from AB. Extend IC to meet this circle at C'. From C draw CX parallel to $C'A$ to meet AI at X; from C draw CY parallel to $C'B$ to meet BI at Y Since triangle XYC is homothetic to triangle ABC', it follows that X, Y are the centers of the desired equal inscribed circles, and their common tangent drawn from C will locate the desired point C_1 on the side AB.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4588. *Proposed by M. R. Spiegel, Rensselaer Polytechnic Institute, Troy, N. Y.*

Let $n>1$. Evaluate

$$\int_0^\infty (\sqrt[n]{x^n+1} - x)dx.$$

4589. *Proposed by Casper Goffman, Wayne University*

It is well known that the interval (0, 1) is the union of a set of the first category and a set of measure zero. Generalize this result to arbitrary separable metric spaces.

4590. *Proposed by Paul Erdös, University of Notre Dame*

Fermat's conjecture that all numbers of the form

$$F_n = 2^{2^n} + 1$$

are prime was proved wrong by Euler. Show, however, that $\sum 1/d \to 0$ as $n\to\infty$, where d ranges over all the divisors of F_n except 1.

4591. *Proposed by D. J. Newman, Republic Aviation Corporation, Farmingdale, N. Y.*

Let $f(x)$ be analytic. For $a=1$, the equation $f'(ax)+f(x)=0$ $(-\infty<x<\infty)$ has the property that its solution, $f(x)=e^{-x}$, approaches zero as $x\to\infty$. Is this true for any real $a>1$?

4592. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn, N. Y.*

Find the sum

$$\sum_{r=1}^{\infty} \frac{(-1)^{r+1}\log r}{r}.$$

SOLUTIONS

Postulates for a Group

4504 [1952, 554]. *Proposed by Olga Taussky, National Bureau of Standards, Washington, D. C.*

Prove that a set S which is closed under an associative composition law which satisfies the following three axioms is a group:

1. There exists an idempotent e such that $e^2=e$.
2. Every element has at least one left inverse with respect to e.
3. Every element has at most one right inverse with respect to e.

Note by D. W. Sasser, Branford, Connecticut. In the recent solution [1954, 55] the following statement is quoted as a "well known theorem": a semigroup with a right identity and left inverses with respect to it is a group. This is false as can be seen by the following example. Let S be any set with more than one element and define $ab=a$ for any two elements a, b in S. S is clearly a semigroup. Moreover, any element serves as a right identity and is in fact the left inverse of every element. But S is not a group since the identity is not unique—and for other reasons as well.

The statement is correct—and is easily proved—if one assumes that the right identity is unique. It is also true that: a semigroup with a right (or left) identity and with right (or left) inverses with respect to it is a group.

The proof as printed is correct when the alternate finish is followed.

Correction also noted by G. B. Preston.

Density of the Set of Sums of Two Squarefull Numbers

4525 [1953, 123]. *Proposed by Paul Erdös, University of Notre Dame*

Let $u_1<u_2< \cdots$ be the sequence of integers all of whose prime factors have exponents exceeding one. Prove that the density of integers of the form u_i+u_j is zero (*i.e.* the number of integers $\leqq x$ of the given form is $<\epsilon x$ for any ϵ if x is large enough).

Solution by D. J. Newman, Republic Aviation Corporation, Farmingdale, N. Y. Borrowing from number theory the result that, for any fixed positive integers a, b, the set of numbers ax^2+by^2 has zero density, we are able to prove the general result:

If $\sum a_m^{-1/2}$, $\sum b_n^{-1/2}$ both converge then the sets of all integers of the form $a_mx^2+b_ny^2$ is of zero density.

From this the problem follows, as all squarefull numbers are of the form m^3x^2 and we need only apply the above result to the case where $a_m=m^3$, $b_n=n^3$.

Proof of result: We need only consider x, $y>0$. First note that the number of integers $\leqq N$ which are of the form Ax^2+By^2 is at most

$$\sqrt{\frac{N}{A}}\cdot\sqrt{\frac{N}{B}}=\frac{N}{\sqrt{AB}}\cdot$$

Now split the integers $a_m x^2 + b_n y^2$ into three sets

$$S_1:\ m, n \leqq M; \qquad S_2:\ m > M; \qquad S_3:\ n > M.$$

Clearly S_1 has zero density (it is made up of a finite number of sets of the form ax^2+by^2). S_2 contains, among the first N integers, at most

$$\sum_{m,n;m>M} \frac{N}{\sqrt{a_m b_n}} = N \sum b_n^{-1/2} \sum_{m>M} a_m^{-1/2},$$

so that the upper density of S_2 is $\sum b_n^{-1/2} \sum a_m^{-1/2}$ with $m > M$. Likewise $\sum a_m^{-1/2} \sum b_n^{-1/2}$, with $n > M$, is the upper density of S_3. By letting $M \to \infty$ we see that S_2 and S_3 are of density zero, and so is our whole set.

Also solved by K. Prachar.

k-chromatic Graphs without Triangles

4526 [1953, 123, 336]. *Proposed by Peter Ungar, New York University*

Show that for any $k > 1$ there exist k-chromatic graphs which contain no three mutually connected nodes. (If it requires k colors to color the nodes so that no two nodes of the same color are connected by an edge, the graph is k-chromatic.)

Solution by Blanche Descartes, Toronto, Ontario. A stronger result can be proved. It can be shown that for any $k > 1$ there exists a k-chromatic graph which has no circuit of less than 6 edges.

The case $k = 2$ is trivial. For the other cases we define a sequence of graphs $G_3, G_4, G_5, \cdots$. The graph G_3 is a circuit of just 7 edges. (Any larger odd number would do.) When G_i is defined, with m_i nodes say, we construct G_{i+1} as follows. We take

$$\binom{im_i - i + 1}{m_i}$$

disjoint copies of G_i. We adjoin $im_i - i + 1$ extra nodes, which we call "central." We set up a 1-1 correspondence between the copies of G_i and the sets of m_i central nodes. We join each copy of G_i to the members of the corresponding set of central nodes by m_i new edges of which no two have a common end. The resulting graph is G_{i+1}.

This construction ensures that no graph G_i has a circuit of less than 6 edges.

Clearly G_3 is 3-chromatic. If $i > 3$ and G_{i+1} has a coloring C in i or fewer colors then some m_i of the central nodes of G_{i+1} must have the same color in C. The corresponding copy of G_i must be colored in $i-1$ or fewer colors. We may thus show inductively that G_k cannot be colored in less than k colors. This does not prove that G_k is k-chromatic but if it is not we can obtain a k-chromatic graph from it by deleting some vertices and their incident edges. Thus for all $k > 1$ there exists a k-chromatic graph having no circuit of less than 6 edges.

The writer discussed a problem equivalent to the special case $k=4$ in *Eureka* (March, 1948).

Also solved by J. B. Kelly and the Proposer.

Continuous Solution of a Functional Equation

4527 [1953, 123]. *Proposed by D. J. Newman, Republic Aviation Corporation, Farmingdale, N. Y.*

Is there a function $f(x)$ continuous in the closed interval (0, 1) and such that in this interval $f(x)+f(x^2)=x$?

I. *Solution by a reader whose name does not appear.** If the defining relationship is used iteratively the form of $f(x)$ is seen to be

$$\begin{aligned} f(x) &= x - f(x^2) = x - x^2 + f(x^4) = \cdots \\ &= x - x^2 + x^4 - \cdots + (-1)^k x^{2^k} + (-1)^{k+1} f(x^{2^{k+1}}). \end{aligned}$$

The continuity of $f(x)$ at $x=0$ implies that $f(0)=0$ and therefore the last term above approaches zero as $k\to\infty$, so that

$$f(x) = \sum_{k=0}^{\infty} (-1)^k x^{2^k}$$

gives the unique solution on the open interval $0<x<1$ which is also continuous at $x=0$. If the desired $f(x)$ exists, it must be true also that $f(1)=\frac{1}{2}$.

Suppose there exists a point x_0, $0<x_0<1$, at which $f(x_0)=\frac{1}{2}+\Delta$, $\Delta>0$. Let $x_n=(x_{n-1})^{1/2}$. Then $x_n>x_{n-1}$ and $x_n\to 1$ as $n\to\infty$. From the defining relationship $f(x_n)=x_n-f(x_{n-1})$. In particular $f(x_1)=x_1-f(x_0)=x_1-(\frac{1}{2}+\Delta)<\frac{1}{2}-\Delta$, and $f(x_2)=x_2-f(x_1)=\frac{1}{2}+\Delta+(x_2-x_1)$. Since $x_2>x_1$, $f(x_2)>f(x_0)$. By induction, the points x_{2n} form a sequence approaching 1 as a limit such that

$$f(x_{2n+2}) > f(x_{2n}) > \cdots > f(x_2) > \tfrac{1}{2} + \Delta,$$

for all $n>0$. Similarly $f(x_{2n+1})<f(x_{2n-1})<\cdots<\frac{1}{2}-\Delta$. Thus $\lim f(x)$ as $x\to 1$ cannot exist.

Actually the value of $f(.995)$ by direct calculation exceeds 0.5008 which proves the impossibility of the proposed function.

II. *Solution by Norman Greenspan, Polytechnic Institute of Brooklyn, N. Y.* In G. H. Hardy, *Divergent Series*, p. 77, there is discussed the functional equation

$$f(x) + f(x^a) = x, \qquad a > 1,$$

in connection with the two distinct continuous solutions:

$$F(x) = \sum_{n=0}^{\infty} (-1)^n x^{a^n} \qquad 0 \leqq x < 1,$$

* Each month many contributions are received unsigned—eight for the single problem 4527. Wherever possible from attached notes or return addresses on envelopes, the editors add the names. We urge each author to make sure his name appears on every sheet submitted.

$$\Phi(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(1+a^n)}\left(\log \frac{1}{x}\right)^n \qquad 0 < x \leqq 1.$$

Hardy shows that $F(x)$ is not continuous at $x=1$ nor is $\Phi(x)$ continuous at $x=0$. Furthermore $F(x)$ and $\Phi(x)$ are the only functions of x that satisfy $f(x)+f(x^a)=x$ and are continuous for $0 \leqq x < 1$ and $0 < x \leqq 1$ respectively. Thus there exists no function of the type proposed.

Also solved by A. A. Bennett, P. M. Cohn, A. E. Currier, D. S. Greenstein and S. G. Kneale, Oliver Gross, P. G. Kirmser, Walter Knödel, H. W. Oliver, R. F. Reeves, E. M. Wright, and the Proposer.

Editorial Note. From the fact that $F(1)$ assumes the form $1-1+1-\cdots$, several readers assumed that $F(x) \to \frac{1}{2}$ as $x \to 1$. For further clarification of this point consult Levinson, *Gap and Density Theorems*, 1940, p. 186, for the theorem of Hardy-Littlewood; also Hardy, *Divergent Series*, p. 173, for the high-indices theorem.

Sequences with One Term Relatively Prime to All Others

4528 [1953, 192, 423]. *Proposed by Paul Erdös, University of Notre Dame*

Prove that the primes have the following property: If a sequence of consecutive integers $n, n+1, \cdots, n+k$ contains a prime p, then at least one of these integers is relatively prime to all others. Prove also that infinitely many integers which are not primes have the same property. (That sequences exist in which no member has this property has been shown by Sivasankaranarayana, Pillai and Szkeres.)

Partial Solution by Stanislaw Leja, Buffalo, N. Y. Let $n+i$ be the greatest prime contained in this sequence. Then $2n+2i > n+k$, for if not, then by Bertrand's postulate (for Tchebychef's proof see Landau, *Primzahlen*, v. I, pp. 89–92) there exists a prime q such that $n+i < q < 2n+2i \leqq n+k$, and this is contrary to the supposition. So $n+i$ is relatively prime to all others.

Solved also (partially) by W. E. Briggs, T. A. Brown, W. F. Cheney, Jr., D. C. B. Marsh, Leo Moser, C. R. Phelps, Azriel Rosenfeld and Seymour Haber.

Editorial Note. The problem as originally given [1953, 192] was restated because sequences lacking the property that at least one member is relatively prime to all others have already been given. S. S. Pillai (*Proc. of the Indian Acad. of Sci., Sec. A*, v. ii (1940) pp. 6–12) proved that every set with $k < 16$ possesses the property, but sequences with $k \geqq 16$ exist which lack it. A. Brauer (*Bull. Amer. Math. Soc.* (1941), pp. 328–331) proved that sets which do not have the property exist for all $k \geqq 16$. The simplest such set, noted by several solvers, is 2184, 2185, $\cdots$, 2200.

It was the Proposer's hope that an elementary proof of the second part would be submitted. His own treatment depends upon the lemma: *There are more than*

$$\frac{c_1 x}{\log x \log \log x}$$

primes $p \leqq x$ such that all prime factors of $p-1$ are greater than $(\log x)^2$. The proof of the lemma is not elementary and is complicated. It uses Bruns' method and the fact that for every $d<(\log x)^k$ and $(a, d)=1$, the number of primes $p \equiv a \pmod{d}$, $p<x$ equals

$$(1+o(1)) \frac{x}{\phi(d) \log x} \cdot$$

It can be shown that for an infinite subset q_i of the primes described in the lemma, the numbers q_i-1 satisfy the requirements of the problem.

A Convergent Series whose Partial Sums Have Distinct Real Roots

4529 [1953, 192]. *Proposed by C. D. Olds, San Jose State College, California*

Can one find a convergent series $a_0+a_1x+a_2x^2+\cdots$, where the a's are real and positive and such that all the roots of each of the equations

$$a_0 + a_1x + a_2x^2 = 0$$
$$a_0 + a_1x + a_2x^2 + a_3x^3 = 0$$
$$\cdots\cdots\cdots\cdots\cdots$$

are real?

I. *Solution by S. H. Gould, Purdue University.* Suppose that the equation $a_0+a_1x+\cdots+a_nx^n=0$, with positive coefficients, has the n distinct real roots $r_1, r_2, \cdots, r_n$. Then

$$a_nx + a_{n-1}x^2 + \cdots + a_0x^{n+1} = 0$$

has the $(n+1)$ distinct real roots

(1) $$\overset{-1}{r_1}, \overset{-1}{r_2}, \cdots, \overset{-1}{r_n}, 0.$$

Since the roots are continuous functions of the coefficients, the $(n+1)$ roots of

$$a_{n+1} + a_nx + \cdots + a_0x^{n+1} = 0$$

are, for sufficiently small $a_{n+1}>0$, arbitrarily close to (1), hence real and distinct. Their reciprocals are roots of

$$a_0 + a_1x + \cdots + a_{n+1}x^{n+1} = 0.$$

Thus the problem is solved by successively choosing a_{n+1} small enough to meet the above condition for $n=1, 2, \cdots$, and also small enough to ensure convergence of the series.

II. *Solution by I. N. Baker, Adelaide University, Australia.* Consider the particular example

$$f(x) = 1 + \frac{x}{4^1} + \frac{x^2}{(4^2)^2} + \cdots + \frac{x^n}{(4^n)^n} + \cdots$$

which converges for all x. For the nth degree polynomial

$$f_n(x) = 1 + \frac{x}{4} + \cdots + \frac{x^n}{4^{n^2}},$$

we can show that

(1) $$f_n(-4^{4r}) > 0 \qquad \text{when } n \geqq 2r,$$

(2) $$f_n(-4^{4r+2}) < 0 \qquad \text{when } n \geqq 2r + 1.$$

In fact, we have

$$f_n(-4^{2r}) = \sum_{k=0}^{\infty} (-1)^k T_k, \qquad T_k = 4^{k(2r-k)}.$$

The largest T_k is T_r and the terms decrease in magnitude as we leave T_r on either side, being alternately positive and negative. The terms on either side are all numerically $\leqq T_r/4$. We conclude that the sums of terms to the left and right respectively of T_r are each numerically $\leqq T_r/4$. Therefore $f_n(-4^{2r})$ has the sign of the rth term, *i.e.* $+$ for r even, $-$ for r odd. This argument applies only when the term T_r appears on the right, *i.e.* when $r \leqq n$, or when the conditions (1) and (2) are satisfied.

Now consider the equation $f_n(x) = 0$.

(a) If n is even, $n = 2r$, we have

$$f_n(0) > 0,\ f_n(-4^4) > 0, \cdots, f_n(-4^{4r}) > 0,$$
$$f_n(-4^2) < 0,\ f_n(-4^6) < 8, \cdots, f_n(-4^{4r-2}) < 0,$$

whence there are n real roots r_i such that

$$0 > r_1 > -4^2 > r_2 > -4^4 > \cdots > -4^{4r-2} > r_n > -4^{4r}.$$

(b) If n is odd, $n = 2r+1$, we have $2r$ roots exactly as above, all $> -4^{4r}$. But we have also $f_n(-4^{4r}) > 0$, $f_n(-\infty) < 0$. Therefore there is one additional real root which is less than -4^{4r}. This makes up the full number of $n = 2r+1$ real (negative) roots.

The above proof also shows that $|f_n(-4^{2r})| > 2/4 \cdot 4^{r^2}$ and hence as $n \to \infty$, $f_n(-4^{2r})$ approaches a limit L of sign $(-1)^r$ and different from zero. Thus $f(x)$ is an entire function which vanishes at least once in every interval $(-4^{4(r+1)}, -4^{4r})$.

No significance is to be attached to the use of 4 as base. Any real number greater than 2 will serve.

Also solved by R. H. Breusch, Leonard Carlitz, Myles McConnon and L. F. Boron, and O. E. Stanaitis.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to E. P. Vance, Oberlin College, Oberlin, Ohio.

Math Is Fun. By Joseph Degrazia. New York, Emerson Books, Inc., 1954. xvii+159 pages. $2.75.

This book consists of trick problems, puzzles and, as the author says, "brain teasers." Its main purpose is to furnish mathematical recreation and to stimulate interest in mathematics. The book is divided into two parts. The first part contains clearly stated problems with little or no hint as to the solution. The second part is devoted to carefully written solutions to most of the problems.

The problems are classified into chapters which serve to facilitate the reader in finding the particular type in which he may be interested. Such chapters as Faded Documents, Cryptograms, Clock Puzzles, Speed Puzzles, Shopping Puzzles, Railroad Shunting Problems, Agriculture Problems, are suggestive of this classification. The book contains a great variety of problems ranging from the very simple to many that require logical thinking and very careful analysis for their solution. The amateur as well as the experienced mathematician will find plenty of food for thought.

There are one hundred seventy-nine problems classified into seventeen chapters. Some are quite old and well known to mathematicians, but there is an abundance of new ones. There are many illustrations which are unique and interesting as well as helpful in clarifying the meaning of some of the problems. This is especially true of the chapter on railroad problems and the one related to squares.

Since no knowledge of mathematics beyond arithmetic and elementary algebra is needed for the solution to most of the problems, many of them could be used to add interest to high school algebra and arithmetic. College classes in elementary mathematics will find much material for thought in this volume. The mathematician or layman looking for recreation in mathematics will find an abundance of material suitable for such pastime.

The author is to be congratulated on producing such a well classified volume. It should be an interesting addition to any high school or college library and to many private libraries.

S. W. McINNIS
University of Florida

Stability Theory of Differential Equations. By Richard Bellman. New York, McGraw-Hill Book Company, Inc., 1953. xiii+166 pages, $5.50.

This book is concerned with real solutions of real differential equations and the behavior of these solutions as the independent variable increases without limit. The properties of the solutions of greatest interest are boundedness, asymptotic behavior, oscillation, and stability. Very little of the material is to be found in other textbooks since most of it is either original with the author or was taken by him from research papers in the literature.

There has long been a need for an up-to-date text in English to serve as a basis for a graduate course in the theory of ordinary differential equations. This book should help to meet this need as well as to stimulate others to carry on investigations of some of the unsolved problems in this very fascinating and increasingly important area of analysis. The book is not intended to be encyclopedic in scope but rather as introductory to the modern theory of stability and asymptotic behavior of linear and non-linear differential equations.

The book opens with a detailed treatment of the linear system using matrix methods. The results in matrix theory required are derived from the beginning assuming no previous acquaintance with matrix theory. After the linear system with constant and almost-constant coefficients has been treated in considerable detail, the results of the linear theory are used to derive the results of Poincaré and Liapounoff concerning the stability of nonlinear systems. A survey of the important results concerning the boundedness, stability, and asymptotic behavior of the second-order, linear differential equations is presented. The last two chapters are devoted to some important nonlinear differential equations whose solutions may be completely described as far as asymptotic behavior is concerned. Any discussion of periodic solutions of nonlinear equations is omitted since such a discussion would require the use of more advanced analytic and topological tools, and the author desires to keep the book at a somewhat more elementary level.

Considerable pain has been taken in the writing and organizing of this book. Each of the seven chapters is divided into sections with appropriate headings and is closed by a bibliography classified as to the section of the chapter concerned. The various definitions, theorems, lemmas and corollaries are carefully labeled and italicized. Proofs are presented in a pleasing and direct style, and a considerable number of well chosen exercises are distributed throughout the text. The book contains a minimum number of typographical errors. It should prove to be a most valuable text for the student of analysis as well as an excellent reference book for the student interested in applications.

J. J. L. Hinrichsen
Iowa State College

Introduction to the Theory of Statistics. By Victor Goedicke. New York, Harper and Brothers, 1953. 12+286 pages. $4.50.

The first 130 pages of this 286-page textbook cover quite satisfactorily the objectives of statistics, the preparation of a frequency tabulation, the use of logarithms and the slide rule, the basic laws of probability, the fundamentals of the normal curve, and methods of computing such basic sample statistics as are usually used to describe central tendency, dispersion, skewness, and kurtosis. An introduction to the use of the method of moments is also included.

Chapter 8 devotes 15 pages to the fitting of a straight line to data by methods of least squares and master points, and two pages to curvilinear curve fitting. Throughout the book problems are interspersed at strategic points and the answers are immediately available in the back of the book. This should be helpful to anyone wishing to study by himself. Chapter 9 quite thoroughly covers simple linear correlation in 43 pages and paves the way for a discourse on multiple and partial correlation in Chapter 12. Chapters 10 and 11 delve into sampling and reliability, and the testing of statistical hypotheses. A final short chapter called "statistics and common sense" contains some excellent advice to beginning students (and others). However, this reviewer is not as willing as the author seems to be to believe that Professor Kinsey established the validity of his original sampling technique in his first book on sexual behavior through the method cited on page 259.

With scores of modern and up-to-date examples and illustrations of the practical use of simple correlation readily available, it was a surprise to find Chapter 9 on simple correlation starting off with a diagram of data showing the association between cricket chirps and temperature compiled in 1927, and also one of the association of legislative ability and brain weight from a study made in 1932.

It seems to be the standard practice in writing elementary books on statistics to wave the magic wand called "normal distribution" over a situation and then quickly get on with an array of techniques and computations in which the early assumption is soon lost. How often one reads a statement similar to the one on page 166: "If we assume that the errors of prediction, $y-y_p$, will be distributed normally, we can solve all such problems by means of the normal curve tables." Quite true! But this assumption made in the use of the standard error of estimate may easily lead a beginning student into the idea that all that is required is to make such an assumption, and then proceed forthwith to make an estimate from the regression line, backing up his estimate with a probability of correctness obtained from the table of areas under the normal curve. The feeling may be established that somehow through the magic of mathematics all will be well once that assumption of normality is made. It is not made clear in Chapter 9 that the assumption of normality takes a lot for granted, that the sample regression line is not the true regression line of the universe, and that the sample

standard error of estimate is not the true standard error of estimate of the universe. The problem of estimating from a curve takes on a different aspect when these points are understood.

In Chapters 10 and 11 are further illustrations of the use of the broad assumption of normality in the population without sufficient emphasis and restatement of the danger in the indiscriminate use of this assumption. Answers are not found in Chapters 9, 10, and 11 to such questions as: 1—how can one tell whether or not use of the assumption of normality is justified; 2—what alterations need to be made in the formulae or tables used when the normal assumption is not applicable; 3—what can one do to improve the situation if the underlying assumption cannot be used. More stress should be put on the importance of randomization of samples, homogeneity of data, and basic knowledge of the population sampled. Principal emphasis should obviously be put on situations in which we do not know either the population mean or the population standard deviation, but the student should be made aware of other situations in which some knowledge of these parameters is available.

A few minor points noted were: the use of the word "proved" seems a bit strong when used on pages 216–218 in connection with the testing of statistical hypotheses; perhaps a better word could be found. In more than a score of places throughout the book the word "data" is used in the singular. On p. 111, 10th line, table 6–91 should be 6–9–1. On p. 219, in line beginning "If we replace . . . ," radical signs were omitted over the two denominators. On p. 271, the 4th formula is 11–5–1, not 10–5–1. Finally, out of 36 textbooks from the reviewer's shelves this was the only one using a bar above all expressions of averages, one of four using small x-bar for the arithmetic mean. Lack of standardization in symbolism is one of the curses of communication in the language of statistics.

G. I. BUTTERBAUGH
University of Washington

Mathematics and Statistics for Economists. By Gerhard Tintner. New York, Rinehart and Company, Inc. 1953. xiv+363 pp. $6.50.

This text, by an authority in the field of econometrics, is addressed to the student of economics who, with only high school training in algebra and trigonometry, wishes "to acquire some of the mathematical equipment necessary nowadays, for a serious study of economics."

The book is divided into three main sections: (1) Some Applications of Elementary Mathematics to Economics, (2) Calculus, (3) Probability and Statistics. Part 1 contains short chapters on functions and graphs, linear and quadratic equations, logarithms, progressions, determinants, and linear difference equations with constant coefficients. Part 2 contains the usual introductory material on differentiation (including such topics as functions of several variables, partial differentiation, homogeneous functions) and concludes with a short chapter on integration. Part 3 discusses the definition and laws of prob-

ability, random variables, probability distributions, sampling theory, tests of statistical hypotheses, regression, correlation, and index numbers.

In order to cover this variety of topics the author has had to write concisely in the text proper. The reader is helped by the compensatory influence of many carefully worked out illustrative examples which, in each section, follow the text material and precede the list of exercises. It is clear that the author has taken great care in collecting exercises which serve not only to give practice in the routine manipulations involved in getting answers, but which at the same time amply demonstrate the wide usefulness of mathematics in economics. This is a theme that runs throughout the book. Such economic ideas as demand and supply functions, market equilibrium, marginal cost and revenue, elasticity of demand, consumer's surplus, as well as many others, are treated in both the text and exercises. That these exercises will attract the economist and make his study of mathematics more interesting is undeniable.

For the teacher who plans to use the book, this poses a serious problem. There will be a temptation to hurry through the text material in order to consider the exercises. We know that all too many elementary mathematics courses are nothing but problem solving sessions. A few definitions, an intuitive argument which "convinces" the student of the validity of a theorem or method, an illustrative example or two, and then an assignment which requires only that the student mimic these examples so as to obtain answers which check with those at the back of the book. Such a course, whether it be for engineers or economists, does not succeed in training students to apply mathematics intelligently in their own field. The basic subject matter must be given time. The logic and development of the subject must be considered and given its due place in the classroom and in the homework assignment. For this reason, the reviewer regrets to note that the care taken by the author in selecting exercises seems not to have been lavished on the text proper.

In this connection, the following items seem noteworthy: (i) The arithmetic in the example on p. 87 needs to be reworked. (ii) The requirement that $M\neq 0$ should be added in the statement of the proposition on p. 103. (iii) The infinite series for e makes a sudden appearance (p. 110) without any previous, or later explanation of what such an infinite series means. (iv) On p. 123 the author asserts that if to a positive increment Δx there corresponds a positive (negative) increment Δy, then y' must be positive (negative). The example considered immediately following is $y=x^2$, but although the slope is calculated for four different points, the consideration of the slope at $x=0$, which shows the above statements to be false, is omitted. (v) The use of the word "means" in the sentence (p. 183), "The last expression in this formula is the definite integral, and it means $F(b)-F(a)$, $\cdots$" is especially misleading to the reader who has just seen, in the preceding line that this "expression" is *defined* as the limit of a certain sum. (vi) On p. 205 the author refers to the uniformly distributed random variable in the interval (2, 6) as the "random variable x which can assume all possible values between 2 and 6 with *equal* probability." The student's

difficulty in understanding what is involved here is compounded by the assertion which follows, *viz.*, "this circumstance yields the probability density, $p(x)=1/4$, $2 \leqq x \leqq 6$." The "proof" that is given merely shows that the definite integral of $p(x)$ from 2 to 6 is equal to one, *i.e.*, that $p(x)$ is *a* probability density, not that it is *the* probability density of x. One problem here is that the author has not defined or discussed probabilities of events for which the underlying sample space is not discrete. (vii) The independence of two events is mentioned, but the independence of random variables is not defined. Nevertheless, this concept is used (p. 213, p. 215). (viii) The statement concerning the meaning of the confidence limits obtained in example 1, p. 252, is incorrect.

An instructor experienced in both mathematics and economics can overcome such difficulties. This book should then prove to be a useful textbook, not only for the future econometrician but for all students of economics.

SAMUEL GOLDBERG
Oberlin College

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

THE INTERNATIONAL SCHOLARS FORUM

The International Scholars Forum has been organized to facilitate the publication of manuscripts likely to have a limited sale. The members of the Advisory Board of the Forum are as follows: Professor J. A. De Haas, Claremont Men's College; Director Philip Munz, Rancho Santa Ana Botanic Garden; Professor W. T. Jones, Pomona College; Professor Edward Weismiller, Pomona College; President Frederick Hard, Scripps College; Librarian David Davies, Honnold Library, Associated Colleges at Claremont.

The Board has entered into an agreement with Martinus Nijhoff of the Hague to receive manuscripts, appraise them, and make recommendations regarding publication. For further information write to Librarian of the Honnold Library, Claremont, California.

SUMMER SESSIONS

The following institutions announce advanced courses in mathematics for the summer of 1954:

Duke University. June 9 to July 17: Professor Gergen, algebra from an advanced standpoint; Professor Dressel, advanced calculus I. July 20 to August 27: Professor Carlitz, advanced calculus II. July 19 to 23: laboratory conference for teachers of science and mathematics.

Kent State University. June 21 to July 30: Professor Olson, differential equations I; Professor Bush, introduction to modern algebra; Assistant Professor Brumfield, advanced methods of teaching mathematics in high school. August 2 to September 3: Professor Jenkins, probability; Professor Brooks, history of mathematics.

Michigan State College. June 23 to July 30: Professor Grove, advanced calculus I; Mr. Kraft, analysis of variance; Professor Nordhaus, differential equations, solid and spherical geometry; Professor Olkin, elements of statistics, correlation analysis; Professor Robinson, theory of numbers, foundations of geometry. June 23 to August 20: Professor Hannan, statistical methods in engineering; Professor Hill, advanced calculus III, complex variable I; Professor Katz, time series and index numbers, design of experiments; Professor Parkus, mathematics of engineering, vector analysis, boundary value problems.

Northwestern University. June 18 to August 10: Professor Holyoke, theory of equations, engineering mathematics III; Professor Curtis, differential equations; Professor Dwass, probability; Professor Reid, functions of a complex variable; Professor Boothby, the continuum.

University of California at Los Angeles. June 21 to July 30: Professor Coddington, functions of a complex variable; Drs. Motzkin, Teichroew and Wasow, seminar in numerical analysis; Professor Straus, partial differential equations of mathematical physics.

PERSONAL ITEMS

Los Angeles City College announces that the Mathematics Department of the College with the cooperation of the mathematics teachers and high school students of the Los Angeles City Schools is publishing a Mathematics Newsletter. The objective of the Newsletter is to stimulate interest in mathematics among high school students. Further information about the Newsletter may be obtained from Mr. Samuel Skolnik, Los Angeles City College.

The University of Cincinnati announces that Dr. Alexander Peyerimhoff of the University of Giessen has joined the University research group which is investigating the theory of Nörlund means and its applications under the sponsorship of the United States Air Force through the Office of Scientific Research of the Air Research and Development Command. Dr. Peyerimhoff will work with the group during the current semester. Dr. Wolfgang Jurkat of the University of Tübingen and Professors H. D. Lipsich and C. N. Moore of the University are continuing to work with the group. Professor Konrad Knopp served with the group during the first semester of 1953–54.

The University of Connecticut reports that a new Department of Statistics

has been established in the College of Arts and Sciences of the University. Professor Geoffrey Beall is Chairman of the Department.

Mr. James Bates, formerly a graduate research mathematician at Visibility Laboratory, Scripps Institute of Oceanography, is now a staff mathematician with Military Operations Research Division, Lockheed Aircraft Corporation, Burbank, California.

Mr. L. F. Boron has been appointed to an instructorship at the University of Colorado.

Assistant Professor Truman Botts of the University of Virginia has received a Ford Foundation Grant and is on leave for the academic year 1953–54.

Mr. B. A. Chiappinelli, previously a mathematician for the Rand Corporation, Santa Monica, California, is now with Hughes Aircraft Company, Culver City, California, as a business systems analyst.

Dr. E. P. Coleman, formerly visiting professor, has been appointed Professor in the Department of Engineering, University of California at Los Angeles.

Dr. M. S. Demos has been appointed to a professorship at Drexel Institute of Technology.

Mr. P. C. Fife, previously a graduate student at the University of California, is now a physicist with Sandia Corporation, Albuquerque, New Mexico.

Mr. E. I. Gale has accepted a position at Brockville Bible College.

Associate Professor E. L. Godfrey of Defiance College has been promoted to a professorship.

Mr. F. D. Grogan has been promoted from the position of Army Chemical Corps Resident Inspector at Shwayder Brothers, Inc., Denver, Colorado, to the position of Chief, Arsenal Inspection Section, Rocky Mountain Arsenal, Denver, Colorado.

Mr. F. P. Harding, formerly a student at Marquette University, is teaching at Clarenceville High School, Livonia, Michigan.

Mr. J. E. Householder, previously a graduate student at the University of Arizona, has been appointed to a part-time instructorship at the University of Colorado.

Dr. M. A. Hyman, who returned recently from Holland where he was a Fulbright Scholar, has accepted a position in the Atomic Power Division of the Westinghouse Electric Corporation, Washington, D. C.

Dr. E. B. Leach has been appointed to an instructorship at Case Institute of Technology.

Associate Professor Mary A. Lee of Sweet Briar College is on leave of absence and has accepted a position as Assistant Mathematician at the Rand Corporation.

Mr. Stanislaw Leja, previously with the Ford Motor Company, Buffalo, New York, has been appointed to an instructorship at Cornell University.

Dr. John McCarthy of Princeton University has been appointed to an acting assistant professorship at Stanford University.

Mr. G. W. Medlin of Wake Forest College has been promoted to an assistant professorship.

Professor C. T. Molloy of Polytechnic Institute of Brooklyn has accepted a position as Head of Physics Research, Vitro Corporation of America.

Dr. E. S. Northam of Michigan State College has a position as a mathematician with the Bendix Aviation Corporation, Detroit, Michigan.

Mr. D. G. O'Connor has been promoted to the position of Technical Engineer in Planning Engineering, International Business Machines Corporation, Endicott, New York.

Dr. R. H. Owens of Brown University has accepted a position as Physical Sciences Coordinator with the Office of Naval Research, Pasadena, California.

Assistant Professor R. P. Peterson of the University of Washington has been appointed to an assistant professorship at the University of California, Riverside.

Professor Emeritus George Pólya of Stanford University has been appointed to a visiting professorship at the Swiss Federal Institute of Technology.

Miss Filomena R. Reyes has been appointed to an instructorship at the University of the East, Manila, Philippines.

Mrs. Joy Russek, previously at New York University, has been appointed to an instructorship at the University of Buffalo.

Dr. D. E. Sanderson has been appointed to an instructorship at Iowa State College of Agriculture and Mechanic Arts.

Associate Professor M. E. Shanks of Purdue University has been promoted to a professorship.

Mr. M. M. Slotnick, previously chief mathematician with Humble Oil and Refining Company, Houston, Texas, is now Consultant Geophysicist with Standard Vacuum Oil Company, New York City.

Associate Professor C. D. Smith of the University of Alabama has been promoted to a professorship.

Assistant Professor Frances M. Suter of Roanoke College is now Registrar at Peace College.

Professor A. W. Tucker of Princeton University gave a course in Combinatorial Topology in the autumn of 1953 at Haverford College under the Philips Bequest.

Mr. B. K. Youse, previously a graduate assistant at the University of Georgia, has been promoted to an instructorship.

Professor Emeritus J. L. Coolidge of Harvard University died on March 5, 1954. He was a charter member and a former president of the Association; he also served as vice-president, as editor of this MONTHLY and as a member of the Board of Governors.

Professor Emeritus L. E. Dickson of the University of Chicago died on January 17, 1954. He was a charter member of the Association and an honorary life member since 1941.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE FEBRUARY MEETING OF THE LOUISIANA-MISSISSIPPI SECTION

The thirty-first annual meeting of the Louisiana-Mississippi Section of the Mathematical Association of America was held at Tulane University, New Orleans, Louisiana, on February 19 and 20, 1954. Professor M. E. Gillis, Chairman of the Section, presided at the afternoon session. Professor B. O. Van Hook presided at the evening session and Professor M. M. Ohmer presided at the Saturday morning session.

There were one hundred eight persons registered including the following sixty-three members of the Association:

R. E. Allan, T. A. Bickerstaff, W. H. Brothers, Jr., Elsie T. Church, W. H. Cleveland, G. J. Corley, Myrtis Davis, Margaret R. Davis, M. P. Dossey, W. L. Duren, Jr., D. O. Etter, P. L. Ford, Elizabeth Freas, L. M. Garrison, M. E. Gillis, F. L. Griffin, J. S. Griffin, Jr., A. C. Grimes, J. P. Gwin, Jessie M. Hoag, S. T. Hu, John Jones, Jr., L. H. Kanter, H. T. Karnes, C. G. Killen, Margaret M. LaSalle, Z. L. Loflin, Saunders MacLane, A. C. Maddox, J. W. McClimans, Betty McKnight, R. A. Miller, B. Ernest Mitchell, B. Evans Mitchell, D. E. Morrill, D. R. Morrison, S. B. Murray, M. M. Ohmer, Arthur Ollivier, R. L. O'Quinn, B. J. Pettis, T. J. Pignani, W. G. Preble, P. K. Rees, T. L. Reynolds, F. A. Rickey, L. A. Rife, A. A. Ritchie, Wm. M. Sanders, D. R. Scholz, H. F. Schroeder, L. L. Scott, S. W. Shelton, Jr., A. L. Shields, W. H. Spragens, Jr., R. A. Stokes, V. B. Temple, W. B. Temple, W. E. Timon, Jr., B. B. Townsend, Eugenia Trapp, B. O. Van Hook, A. D. Wallace.

The following officers were elected for the coming year: Chairman, Professor J. W. McClimans, Southeastern Louisiana College; Louisiana Vice-Chairman, Professor P. K. Rees, Louisiana State University; Mississippi Vice-Chairman, Professor R. A. Miller, University of Mississippi; Secretary-Treasurer, Professor Z. L. Loflin, Southwestern Louisiana Institute.

The invited speaker for the meeting was Professor F. L. Griffin, Newcomb College. His address on Friday evening was entitled "Some Mathematical Sectors of Biology." At the Saturday morning session he spoke on the topic "Is Mathematics a Science?"

The following papers were presented:

1. *A Diophantine matrix equation*, by Professor John Jones, Jr., Mississippi Southern College.

M. H. Ingraham and H. C. Trimble considered the matrix equation $XA=BX+C$. W. V. Parker considered the matrix equation $AX=XB$. W. E. Roth found a solution to the matrix equation $AX^2+BX+XC=D$, and also found a necessary and sufficient condition for a solution to matrix equations of the types $AX-YB=C$, and $AX-XB=C$. The purpose of this paper is to find a necessary and sufficient condition for a Diophantine matrix equation of the type $A(\lambda)X(\lambda)-Y(\lambda)B(\lambda)=C(\lambda)$ to have a solution, $X(\lambda)$, $Y(\lambda)$, where the elements of the n by n matrices $A(\lambda)$, $B(\lambda)$, $C(\lambda)$, $X(\lambda)$, and $Y(\lambda)$ are analytic functions of a complex variable λ in a region R.

2. *Unitary multiples of a matrix*, by Professor B. Evans Mitchell, Louisiana State University.

A canonical form for one-sided unitary multiples of a matrix (the analogue of the Hermite

form) is developed. Two-sided unitary multiples of a commutative set $A_1, \cdots, A_m$ of matrices are considered briefly in connection with Frobenius' theorem on the characteristic roots of $f(A_1, \cdots, A_m)$.

3. *A note on the zeros of the ultraspherical polynomials*, by Professor L. H. Kanter, Mississippi State College.

Let $P_n(x, \lambda)$ denote the ultraspherical polynomial of degree n, and let $x_\nu(\lambda)$ denote the νth zero of $P_n(x, \lambda)$. By making use of a well-known result of Stieltjes on the sign of the derivative of $x_\nu(\lambda)$, certain separation theorems for the zeros of $P_n(x, \lambda)$ and its parametric derivative (with respect to λ) are obtained.

4. *Characterization of knots*, by Professor Margaret M. LaSalle, Southwestern Louisiana Institute.

If the equations of the crossings of a knot are given, the knot diagram can be constructed and if the knot diagram is given the equations of the crossings can be obtained. The ϵ-equivalence of matrices is defined and is used as a criterion for knot equivalence (J. W. Alexander, *Topological Invariants of Knots and Links*, Trans. A. M. S., v. 30, 1928, pp. 275–306). This work was done in connection with the Topology Seminar under Professor Young at the University of Michigan, Summer Session, 1953.

5. *A hybrid system of coordinates*, by Professor B. Ernest Mitchell, University of Mississippi.

A hybrid system of coordinates is obtained when we select the abscissa of the Cartesian system and the radius vector of the polar system. The frame of reference is also hybrid. In this system asymmetry lends itself to simplicity rather than symmetry in each of the original systems. In a certain position of asymmetry the conic has an equation of the first degree, while a straight line in general position is of the second degree. The system is tied in with a certain conservative field of force.

6. *Applications of the scalar product in geometry*, by Mr. D. E. Morrill, University of Mississippi.

1. The vector equation $\lambda \cdot \rho = c$ exhibits a pole-polar relationship relative to a circle, center, the point of reference, radius $\sqrt{c}$; λ is a line vector (not a line bound vector), ρ is a point vector.
2. $(\rho-\alpha)\cdot(\rho-\beta)=0$ is the equation of a circle constructed on the line adjoining the termini of α and β as a diameter.
3. $\rho \cdot \sigma = c$ is the equation of a congruence of circles, to wit, the circles orthogonal to the circle, center the point of reference, radius $\sqrt{c}$. The congruence is hyperbolic when $c>0$, parabolic when $c=0$, and elliptic when $c<0$.

7. *Skidding cycloids*, by Professor V. B. Temple, Louisiana College.

The general equations of the hypo- and epicycloids can be written respectively in parametric form as:

$$x = \frac{a}{n}\left[(n \mp 1)\cos\theta \pm \cos\left(\frac{n}{\lambda} \mp 1\right)\theta\right],$$

and

$$y = \frac{a}{n}\left[(n \mp 1)\sin\theta - \sin\left(\frac{n}{\lambda} \mp 1\right)\theta\right],$$

where a is the radius of the directrix circle, $n>1$, λ a constant $\neq 0$, and θ the parametric angle.

If: $\lambda=1$, we have the pure cycloids, positive and non-skidding;
$\lambda=-1$, the negative of the pure cycloids, and skidding;
$\lambda>1$ and $\lambda<-1$, + and − skidding respectively, and lagging;
$0<\lambda<1$ and $-1<\lambda<0$, + and − skidding, and advancing.

In particular if:

$2\lambda=\pm n$, we have lagging hypo- and epi-ellipses;
$\lambda=\pm n$, we have lagging hypo- and epi-circles.

8. *A non-clanoidal space*, by Professor A. D. Wallace, Tulane University.

Let X denote the "curve" $y=\sin x^{-1}$, $0<x\leqq(2\pi)^{-1}$, together with the segment from $(0, -1)$ to $(0, 1)$ on the axis of ordinates. It is shown that X does not admit a continuous associative multiplication with unit.

9. *Two analytic geometries for freshmen*, by Professor W. L. Duren, Jr., Tulane University.

The conventional analytic geometry is the Euclidean geometry of the plane, that is, the study of the invariants under Euclidean motion. There is another geometry of the plane, which is usually called "graphs," which may be described technically as the cartesian product of the affine lines and the affine line. This is the study of the invariants of the cartesian plane under the group of affine transformations:

$$x' = ax + b$$
$$y' = cy + d, \qquad a \neq 0, c \neq 0.$$

The second geometry is actually more prevalent in elementary mathematics than Euclidean geometry. In this paper a few results from these geometries are compared.

10. *Some mathematical sectors of biology*, by Professor F. L. Griffin, Tulane University.

While the life sciences are predominantly non-mathematical, numerous fragments of their literature embody uses of mathematics at the level of calculus or above. Roughly classified these fall into the following ten fields: metabolic or biochemical problems, growth and senescence, genetics, population problems, light biology, biophysical problems, anatomy and biometry, forest and agricultural experimentation, neurology, psychometrics. Professor Griffin listed many instances of these sorts with typical formulas employed in the literature, and commented on some of the more interesting and significant investigations.

11. *Is mathematics a science?*, by Professor F. L. Griffin, Tulane University.

A number of writers, notably the late Professor C. J. Keyser, have held that mathematics is not a science but is identical with formal logic, having no subject matter except propositional forms and their relations. In Professor Griffin's opinion the subject matter of mathematics includes also concrete systems like pure number and conceptual space, used as objectifications or models of abstract systems. In creating a new body of subject matter defined essentially by its postulates, a mathematician acts in the same manner as an artist; in exploring the further properties of an already created body of subject matter, a mathematician proceeds in the same manner as a natural scientist. Abstract deductions have great value for the study of concrete systems.

Z. L. LOFLIN, *Secretary*

EMPLOYMENT OPPORTUNITIES

University of Houston. Rank and salary depend on qualifications. Ph.D., Applied Math. or Math. Statistics preferred. Math. Dept.

University of New Hampshire, Durham, New Hampshire. Graduate Assistantships in Mathematics available to competent students; includes full remission of tuition.

The MONTHLY is devoting this space to paid announcements of employment opportunities for mathematicians. The text of such announcements should be in want-ad form and must be in the hands of the editor (C. B. Allendoerfer, Mathematics Department, University of Washington, Seattle 5, Wash.) before the first day of the month preceding the issue in which the notice is to appear. Announcements should indicate the academic rank or similar description of the opening, but should not mention a specific salary. Blind ads are permissible which direct replies to a specifix box number in care of the Mathematical Association of America, Buffalo 14, N. Y. In order to conserve space and achieve uniformity, the privilege is reserved to rearrange advertisements. Advertisers will be billed by the Association at the rate of $1.50 per line. Rates for display advertising may be obtained from the Advertising Manager.

CALENDAR OF FUTURE MEETINGS

Thirty-fifth Summer Meeting, University of Wyoming, Laramie, Wyoming, August 30–31, 1954.

Thirty-eighth Annual Meeting, University of Pittsburgh, Pittsburgh, Pennsylvania, December 30, 1954.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Marshall College, Huntington, West Virginia, May 1, 1954.
ILLINOIS, Knox College, Galesburg, May 14–15, 1954.
INDIANA, Rose Polytechnic Institute, Terre Haute, May 1, 1954.
IOWA, Iowa State College, Ames, April 30–May 1, 1954.
KANSAS
KENTUCKY, University of Kentucky, Lexington, May 8, 1954.
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, University of Maryland, College Park, May 1, 1954.
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA, Hamline University, St. Paul, May 8, 1954.
MISSOURI, University of Missouri, Columbia, May 7, 1954.
NEBRASKA
NORTHERN CALIFORNIA
OHIO
OKLAHOMA, Oklahoma City University, October 29, 1954.
PACIFIC NORTHWEST, Reed College, Portland, Oregon, June 18, 1954.
PHILADELPHIA, Princeton University, Princeton, New Jersey, November 27, 1954.
ROCKY MOUNTAIN, Colorado Agricultural and Mechanical College, Fort Collins, April 30–May 1, 1954.
SOUTHEASTERN
SOUTHERN CALIFORNIA
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE, College for Teachers at Albany, May 1, 1954.
WISCONSIN, State Teachers College, Eau Claire, May 8, 1954.

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

THE AMERICAN

MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 61 NUMBER 6

CONTENTS

JUNE-JULY 1954

The AMERICAN MATHEMATICAL MONTHLY

(Founded in 1894 by Benjamin F. Finkel)

EDITORIAL CORRESPONDENCE should be addressed to the Editor, C. B. Allendoerfer, Department of Mathematics, University of Washington, Seattle 5, Washington.

ADVERTISING CORRESPONDENCE should be addressed to J. F. Randolph, Morey Hall, University of Rochester, Rochester 3, N. Y.

NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the Monthly should be sent to the Secretary-Treasurer, H. M. Gehman, University of Buffalo, Buffalo 14, N. Y.

Entered as second class matter at the post office at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

Subscription Price: To Members, $4 a Year, To Others, $5 a Year.

Published by the Association at Menasha, Wisconsin, and Seattle, Wash. during the months of January, February, March, April, May, June–July, August–September, October, November, December.

PRINTED IN THE UNITED STATES OF AMERICA

WHAT IS AN ANGLE?*

HANS ZASSENHAUS, McGill University

1. Introduction. In this paper I plan to present to you my ideas on how the measurement of angles may be suitably introduced in our time. I am, however, fully aware that some of you rather desire an "angle" from which to look at what is taking place in modern algebra. I ask you, therefore, to accept with patience my diversions from the geometrical angle to the philosophical one and back to the subject matter of this paper.

As possible concepts on which the definition of an angle may be based there come to mind:

1) A pair of intersecting straight lines,
2) A pair of rays from the same origin,
3) An open convex part of the plane bounded by a pair of rays with the same origin, called *angular space*,
4) A rotation,
5) A circular arc,
6) An isosceles triangle.

Any one of these six concepts may serve to develop the notion of angles in the plane and, in fact, has played its role at one time or another. However, before accepting what other times have found suitable we want to understand the situation thoroughly. After consulting the books we come together and try to develop a picture which can be perceived and assimilated by *our* minds. This picture will, quite likely, bear features of every one of the six concepts mentioned above.

Due to the fact that so many concepts of what an angle might be are currently circulating, we feel an urge to abandon the visual concepts altogether and occupy ourselves merely with the essential properties of angles which actually do not depend on the special geometrical figures from which the angles originate. The response to this urge consists of the *axiomatic approach.*

We take it for granted that there is such a thing as an angle and we try to expose in simple terms its fundamental properties from which all other properties will follow as logical consequences.

What is simple? Is it a good quality or a bad one? The answer to the second question depends very much on the answer to the first. I will not pretend that there is either merit or reward in being a simpleton, although it seems to be a healthy state. However, a few ideas, well connected and forcefully expounded, do a lot more than a bagful of tricks. This, in my opinion, is a psychological law. We rejoice in repetitions carried on to higher levels, but we are unwilling to pursue new ways of thinking where there is no necessity. I think that simplicity in the approach to the solutions of our problems is a great human virtue and no sacrifice should be too big to attain it.

---

* This paper was originally an address delivered to the Students' Mathematical Society at McGill University.

2. Axioms. If we think of an angle as associated with a rotation we come to consider *oriented angles*. Also, the existence of an addition of angles satisfying all the laws of an additive group is suggested. On the other hand the concept of an angle as being associated with an angular space suggests the idea that angles can be ordered according to their size. The property of order and the group property seem to contradict each other since the repeated addition of positive angles would lead to negative ones if addition could be carried out unrestrictedly. Hence, we have to be careful in the handling of the addition. Mustering our experiences with angles as they present themselves in ordinary geometrical experience we arrive at the following axioms:

(A) There are certain geometrical things called *angles* among which there is given an equality relation satisfying the three rules of reflexivity, symmetry and transitivity. There are at least two different angles. We denote angles by small Greek letters.

(B) For certain ordered pairs of angles, α, β, a third angle called the sum of α and β is defined and may be denoted by $\alpha+\beta$ so that:

(1) $\alpha+\beta=\alpha'+\beta'$ whenever $\alpha=\alpha'$, $\beta=\beta'$ and $\alpha+\beta$ or $\alpha'+\beta'$ is defined.

(2) If $\alpha+\beta$ and $\beta+\gamma$ are defined and if at least one of $(\alpha+\beta)+\gamma$, $\alpha+(\beta+\gamma)$ is defined, then both are defined and they are equal, *i.e.*, $(\alpha+\beta)+\gamma=\alpha+(\beta+\gamma)$.

(3) There is a zero angle, 0, satisfying $\alpha+0=\alpha$ for all angles α.

(4) For every angle α there is an opposite angle $-\alpha$ satisfying

$$\alpha+(-\alpha)=0.$$

(C) Every angle, α, has a signature, sgn α, which is either 1 (positive angles), 0 (zero angles), or -1 (negative angles) so that:

(1) $\operatorname{sgn}(\alpha+\beta)=\operatorname{sgn}\alpha=\operatorname{sgn}\beta$ if $\operatorname{sgn}\alpha=\operatorname{sgn}\beta$ and $\alpha+\beta$ is defined.

(2) $\alpha+\beta$ is defined if $\operatorname{sgn}\alpha\neq\operatorname{sgn}\beta$.

(3) $\beta+\alpha$ is defined if $\alpha+\beta$ is defined and $\operatorname{sgn}(\beta+\alpha)=\operatorname{sgn}(\alpha+\beta)$.

(4) Every non-empty system of positive angles has a least upper bound.

Regarding (C4) we define: $\alpha\geqq\beta$ if either $\alpha=\beta$, or $1=\operatorname{sgn}\alpha\neq\operatorname{sgn}\beta$, or $\alpha+(-\beta)>0$.

For a given non-empty set, S, of angles, an upper bound is defined as an angle λ for which $\lambda\geqq\alpha$ for all $\alpha\in S$. An upper bound, λ, of S is called the least upper bound of S (l.u.b. S) if for any upper bound λ' of S, $\lambda'\geqq\lambda$.

3. Model. Before we study the implications of the axioms A, B and C let us outline the analytical model to which they are related.

The points, P, of the Cartesian plane are given as ordered pairs of real numbers x_1, x_2 called the coördinates of P: $P=(x_1, x_2)$.

The distance between two points $P=(x_1, x_2)$ and $Q=(y_1, y_2)$ is defined as the non-negative real number

$$\overline{PQ}=\left|\sqrt{(y_1-x_1)^2+(y_2-x_2)^2}\right|.$$

The rigid movements of the Cartesian plane are defined as distance-preserving maps: $P \to \sigma P$ of the set of all points into itself so that $\overline{\sigma P \sigma Q} = \overline{PQ}$.

It follows that the coördinates x_1', x_2' of the image σP are related to the coördinates x_1, x_2 of P by the formulae:

$$\begin{aligned} x_1' &= a_1(\sigma) + c(\sigma)x_1 - \epsilon s(\sigma)x_2, \\ x_2' &= a_2(\sigma) + s(\sigma)x_1 + \epsilon c(\sigma)x_2, \end{aligned}$$

where $a_1(\sigma)$, $a_2(\sigma)$, $c(\sigma)$, and $s(\sigma)$ are real numbers depending only on σ and subject to the conditions:

$$\begin{aligned} c(\sigma)^2 + s(\sigma)^2 &= 1, \\ \epsilon &= \pm 1. \end{aligned} \tag{1}$$

Among the rigid movements the most prominent are the rotations about the origin characterized by $\epsilon = +1$, $a_1(\sigma) = a_2(\sigma) = 0$, which form an abelian additive group if the addition of two rotations σ, τ is defined by applying one after the other:

$$(\sigma + \tau)P = \sigma(\tau P). \tag{2}$$

Equivalent to this fact are the addition theorems:

$$\begin{aligned} s(\sigma + \tau) &= s(\sigma)c(\tau) + c(\sigma)s(\tau), \\ c(\sigma + \tau) &= c(\sigma)c(\tau) - s(\sigma)s(\tau), \end{aligned} \tag{3}$$

which follow from our definitions.

By application of a rotation σ about the origin to the points of the upper half-plane H:

$$x_2 > 0$$

we obtain the *open half-planes* σH. The intersection of two open half-planes is called an *angular space*. The angular spaces may be characterized as those point sets of the Cartesian plane which are open, convex, and conic with respect to the origin. An angular space is either empty or it is bounded by two open rays from the origin and the origin itself.

We define an *oriented angle* as a system consisting of an ordered pair of rays r_1, r_2 from the origin and an angular space A bounded by r_1, r_2 and the origin; it may be denoted by (r_1, r_2, A). Also, the system of two equal open rays r,r and the empty angular space Λ is defined as an oriented angle and is called a zero angle denoted by 0 or by (r,r,Λ).

We define $(r_1, r_2, A) = (r_1', r_2', A')$ if and only if there is a rotation σ so that $\sigma r_1 = r_1'$, $\sigma r_2 = r_2'$, $\sigma A = A'$, which may be written $(r_1', r_2', A') = \sigma(r_1, r_2, A)$.

We define for any three rotations ρ, σ, τ about the origin

$$\rho(r_1, r_2, A) + \sigma(r_2, r_3, B) = \tau(r_1, r_3, C)$$

provided that either $A\cap B=\Lambda$, $A\cup r_2\cup B=C$ or $B\cap C=\Lambda$, $B\cup r_3\cup C=A$ or $A\cap C=\Lambda$, $A\cup r_1\cup C=B$. Furthermore: $-(r_1, r_2, A)=(r_2, r_1, A)$. Also $(r_1, r_2, A)>0$ if there is a rotation σ so that σr_1 is the positive x_1-axis: and σA is a non-empty subset of the upper half-plane. Finally: $(r_1, r_2, A)>(r_1', r_2', A')$ if $(r_1, r_2, A)>0$ and $(r_2', r_1', A')>0$, or $(r_1, r_2, A)+(r_2', r_1', A')>0$.

In other words an angle is meant to be an ordered pair of open rays from the origin together with the angular space in between. This space is uniquely determined by the two rays except for the extreme case of two opposite rays when two angles are determined: a positive angle μ and the angle $-\mu$. Due to the properties of rotations and our definitions all the axioms A, B and C can be verified.

It follows that μ is a maximum angle and $-\mu$ is a minimum angle. We remark that for two positive angles (r_1, r_2, A) and (r_1, r_3, B) the inequality $(r_1, r_2, A)>(r_1, r_3, B)$ is equivalent to $A\supset B$ (*i.e.*, B is a proper subset of A).

For a set S of positive angles $S=(r, r_S, A_S)$ the angular space belonging to the least upper bound of S is obtainable as the union of all the angular spaces A_S. The previously mentioned characterization of angular spaces may serve to prove that the union of the angular spaces A_S is an angular space used for establishing the existence of the least upper bound of S.

4. Measurement. After we have agreed on the essential properties of angles as laid down in the axioms A, B and C the problem arises as to whether angles can be measured. By this we mean the problem of setting up a 1-1 correspondence between angles and certain real numbers so that the addition and order relations are preserved:

$$\alpha \longrightarrow m(\alpha)$$

so that from $\alpha+\beta=\gamma$ it follows that $m(\alpha)+m(\beta)=m(\gamma)$ and for each angle it is true that:

$$\operatorname{sgn} m(\alpha) = \operatorname{sgn} \alpha.$$

This task will be made very much easier if we agree on a suitable set of axioms describing the ordered module formed by the real numbers. In other words, once we begin to axiomatize our experiences with numbers and figures we are quite naturally guided to unifying systems of axioms which help us in our task to make transitions from one branch of mathematics to another.

As a suitable system of axioms for the real axis we can take the following:

The ordered module formed by the real numbers is characterized as a set R of at least two elements in which the addition is uniquely defined subject to the associative law: $a+(b+c)=(a+b)+c$, the law of the existence of a zero element 0 satisfying $a+0=a$, the existence of a negative, $-a$, for each element a of R satisfying $a+(-a)=0$.

Furthermore for each element a of R one and only one of the three relations $a>0$, $a=0$, $-a>0$ holds so that

(1) from $a>0$, $b>0$ there follows $a+b>0$;

(2) if $a+b>0$ then $b+a>0$;

(3) each subset of R bounded from above has a least upper bound, if we define, as usual, $a>b$ if and only if $a+(-b)>0$;

(4) for each positive element a there is a positive element b satisfying $a>b>0$.

The condition (3) is equivalent to

(3a) there is a fixed positive element μ of R so that every positive element of R is equal to a finite sum over positive elements each not greater than μ. Every subset of positive elements not greater than μ has a least upper bound. In this form the condition (3) can be easily verified in the course of the ensuing construction.

We notice that the preceding set of axioms describing the real module is quite similar to the system of axioms describing our concept of angle. The main difference is in the fact that two real numbers can always be added whereas two angles cannot always be added. This suggests that our angles are in 1-1 correspondence to only a part of the real axis, say an interval between the upper bound μ and the lower bound $-\mu$ or a part of it. We may prove this directly by using the idea of the Dedekind section. However, there is another way available by first extending our concept of an angle before we try to measure angles. The need for this extension is made clear by considering the purely geometrical origin of our notion of angle which does not comply with the needs of kinematics. For example, consider the system of the fast rotating record disk and the slow-moving gramophone needle. One complete revolution of the disk produces a different situation since meanwhile the needle has slightly changed its position. It does not matter whether the rotation is performed straightforwardly or in leaps and bounds, back and forth. Hence we are led to the concept of kinematical angles ranging over a wider range than the geometrical angles, which may be produced by the unrestricted addition of geometrical angles.

We define:

A *kinematical angle* is a formal sum of geometrical angles $\alpha_1, \alpha_2, \cdots, \alpha^r$ written as $\alpha_1 \dot{+} \alpha_2 \dot{+} \cdots \dot{+} \alpha_r$ or $\dot{\sum}_{i=1}^r \alpha_i$ in order to distinguish the formal nature of the $\dot{+}$ addition from the algebraic nature of the $+$ addition. The number r of summands ranges over the natural numbers. We define

$$\dot{\sum}_{i=1}^r \alpha_i = \dot{\sum}_{j=1}^s \beta_j$$

if the formal sum on the right-hand side can be obtained from the formal sum on the left by a finite number of *reductions* and anti-reductions. Here, a reduction means the replacement of two consecutive formal summands α_i, α_{i+1} by one formal summand $\alpha_i + \alpha_{i+1}$ provided this sum is defined. An anti-reduction is defined to be the converse of a reduction. We define the addition of kinematical angles by the rule

$$(\alpha_1 \dot{+} \alpha_2 \dot{+} \cdots \dot{+} \alpha_r) \dot{+} (\beta_1 \dot{+} \beta_2 \dot{+} \cdots \dot{+} \beta_s)$$
$$= \alpha_1 \dot{+} \alpha_2 \dot{+} \cdots \dot{+} \alpha_r \dot{+} \beta_1 \dot{+} \beta_2 \dot{+} \cdots \dot{+} \beta_s.$$

We convince ourselves that the kinematical angles according to our definition form an additive group. As zero element we have 0 from before. The negative of the kinematical angle $\alpha_1 \dot{+} \alpha_2 \dot{+} \cdots \dot{+} \alpha_r$ is given by $-\alpha_r \dot{+} -\alpha_{r-1} \dot{+} \cdots \dot{+} -\alpha_1$. We see at once that every formal sum by mere reduction can be reduced either to 0 or to a formal sum over positive terms only or to a formal sum over negative terms only. We will use this observation in order to define an ordering of the kinematical angles by calling them 0, positive or negative corresponding to the three possible results of mere reduction just enumerated. This, however, presupposes that the result of the reduction of any formal sum equal to a given one will always lead eventually to the same type of reduced sum. We prove by induction over r:

LEMMA 1. *If a formal sum* $S = \dot{\sum}_{i=1}^{r} \alpha_i$ *can be reduced to* 0 *then every reduction eventually leads to* 0.

Let us notice that every reduction diminishes the number of formal summands by 1 so that after at most $r-1$ reductions an irreducible sum will be obtained. If $r=1$ then $\alpha_1=0$. Let $r>1$ and assume that the lemma is true for all formal sums of less than r terms which can be reduced to 0. We distinguish several cases according to the way in which the first step of a reduction, I, leading to 0 interferes with the first step of another reduction, II. We indicate by dots those terms which are not concerned by the reduction under consideration. An arrow points from a formal sum to another which is obtained from it by a reduction. We distinguish the following cases:

(a)

$$\begin{tikzcd}
 & S = \cdots \alpha \dot{+} \beta \cdots \gamma \dot{+} \delta \cdots \arrow[dl, "\mathrm{I}"'] \arrow[dr, "\mathrm{II}"] & \\
S_{\mathrm{I}} = \cdots (\alpha + \beta) \cdots \gamma \dot{+} \delta \cdots \arrow[dr] & & S_{\mathrm{II}} = \cdots \alpha \dot{+} \beta \cdots (\gamma + \delta) \cdots \arrow[dl] \\
 & S' = \cdots (\alpha + \beta) \cdots (\gamma + \delta) \cdots . &
\end{tikzcd}$$

Since S_{I} has only $r-1$ terms it follows from the inductional assumption that every reduction procedure applied to S_{I} winds up with 0. Since S' can be obtained by reduction from both S_{I} and S_{II} it follows that S_{II} can be reduced to 0. Hence, by inductional assumption, every reduction procedure applied to S_{II} winds up with 0. Hence the reduction procedure II applied to S winds up with 0.

We argue similarly in the other cases which are merely outlined by inserting the links connecting S_{I} and S_{II}. In enumerating these cases we observe that the given formal sum of r terms cannot consist of merely negative terms since reduction would always lead again to formal sums of the same type so that 0 would never be obtained. Furthermore if α, β and γ are three consecutive terms

of S so that $\alpha+\beta, \beta+\gamma$ exist but neither $(\alpha+\beta)+\gamma$ nor $\alpha+(\beta+\gamma)$ exists, then either α, β and γ are all positive or all negative. This is because none of the three angles α, β, γ vanishes. Also if β and γ are of the same signature, α of opposite signature, then $\beta+\gamma$ will be of the same signature as β and γ and hence $\alpha+(\beta+\gamma)$ will exist contrary to our assumption. Similarly it follows that it is impossible that α and β have the same and γ the opposite signature.

(b)

$$\begin{tikzcd}
& S=\cdots\alpha\dot{+}\beta\dot{+}\gamma\cdots \arrow[dl, "\mathrm{I}"'] \arrow[dr, "\mathrm{II}"] & \\
S_{\mathrm{I}}=\cdots(\alpha+\beta)\dot{+}\gamma\cdots \arrow[dr] & & S_{\mathrm{II}}=\cdots\alpha\dot{+}(\beta+\gamma)\cdots \arrow[dl] \\
& \cdots(\alpha+\beta+\gamma)\cdots. &
\end{tikzcd}$$

(c)

$$\begin{tikzcd}
& S=\cdots\xi\dot{+}\eta\cdots\alpha\dot{+}\beta\dot{+}\gamma\cdots \arrow[dl, "\mathrm{I}"'] \arrow[d] \arrow[dr, "\mathrm{II}"] & \\
S_{\mathrm{I}}=\cdots\xi\dot{+}\eta\cdots(\alpha+\beta)\dot{+}\gamma\cdots & \cdots(\xi+\eta)\cdots\alpha\dot{+}\beta\dot{+}\gamma\cdots \arrow[dl] \arrow[dr] & S_{\mathrm{II}}=\cdots\xi\dot{+}\eta\cdots\alpha\dot{+}(\beta+\gamma)\cdots \\
\cdots(\xi+\eta)\cdots(\alpha+\beta)\dot{+}\gamma\cdots & & \cdots(\xi+\eta)\cdots\alpha\dot{+}(\beta+\gamma)\cdots.
\end{tikzcd}$$

In cases (d) and (e) the ϵ's represent the signatures of the corresponding angles.

(d)

$$\begin{tikzcd}
& S=\cdots\overset{-\epsilon}{\eta}\dot{+}\overset{\epsilon}{\alpha}\dot{+}\overset{\epsilon}{\beta}\dot{+}\overset{\epsilon}{\gamma}\cdots \arrow[dl, "\mathrm{I}"'] \arrow[dr, "\mathrm{II}"] & \\
S_{\mathrm{I}}=\cdots\eta\dot{+}(\alpha+\beta)\dot{+}\gamma\cdots & \cdots(\eta+\alpha)\dot{+}\beta\dot{+}\gamma\cdots \arrow[dl] \arrow[dr] & S_{\mathrm{II}}=\cdots\eta\dot{+}\alpha\dot{+}(\beta+\gamma)\cdots \\
\cdots(\eta+\alpha+\beta)\dot{+}\gamma\cdots & & \cdots(\eta+\alpha)\dot{+}(\beta+\gamma)\cdots.
\end{tikzcd}$$

(e)

$$\begin{tikzcd}
& S=\cdots\overset{\epsilon}{\alpha}\dot{+}\overset{\epsilon}{\beta}\dot{+}\overset{\epsilon}{\gamma}\dot{+}\overset{-\epsilon}{\zeta}\cdots \arrow[dl, "\mathrm{I}"'] \arrow[dr, "\mathrm{II}"] & \\
S_{\mathrm{I}}=\cdots(\alpha+\beta)\dot{+}\gamma\dot{+}\zeta\cdots & \cdots\alpha\dot{+}\beta\dot{+}(\gamma+\zeta)\cdots \arrow[dl] \arrow[dr] & S_{\mathrm{II}}=\cdots\alpha\dot{+}(\beta+\gamma)\dot{+}\zeta\cdots \\
\cdots(\alpha+\beta)\dot{+}(\gamma+\zeta)\cdots & & \cdots\alpha\dot{+}(\beta+\gamma+\zeta)\cdots.
\end{tikzcd}$$

Other cases, apart from the trivial case $S_{\mathrm{I}}=S_{\mathrm{II}}$ and complete reversal of the order of addition do not occur.

LEMMA 2: *If a formal sum $S=\dot{\sum}_{i=1}^{r}\alpha_i$ is equal to* 0 *then it can be reduced to* 0.

We apply induction over the number, n, of reductions and anti-reductions which are necessary to carry S over into 0.

For $n=1$ the lemma is clear.

The first step leads to a formal sum S' which according to the inductional assumption can be reduced to 0. We have either $S\to S'$ in which case S can be reduced to 0 obviously or $S'\to S$ in which case S can be reduced to 0 by lemma 1.

Let sgn $\beta=1$. If sgn $\alpha=1$ then $\beta+-\alpha$ exists as well as $-\alpha+\alpha$. Then, $\beta+(-\alpha+\alpha)=\beta+0=\beta$, hence also $(\beta+-\alpha)+\alpha=\beta$ and $\gamma=\alpha+(\beta+-\alpha)$ is defined and has the same signature as β. If sgn $\alpha=-1$ then $\alpha+\beta$, $-\alpha+\alpha$ exist,

$(-\alpha+\alpha)+\beta=0+\beta=\beta$ so that also $-\alpha+(\alpha+\beta)=\beta$ and $\gamma=(\alpha+\beta)+-\alpha$ exists and has the same signature as β. It follows that $\alpha\dot{+}\beta\dot{+}-\alpha$ can be reduced to γ and hence is positive. Repeated application of this argument yields that

$$\dot{\sum}_{i=1}^{r}\alpha_i \dot{+} \beta \dot{+} \dot{\sum_{i=r}^{1}} - \alpha_i$$
$$= \alpha_1 \dot{+} \alpha_2 \dot{+} \cdots \dot{+} \alpha_r + \beta + -\alpha_r + -\alpha_{r-1} + \cdots + -\alpha_1$$

can be reduced to one positive term.

Since $S\dot{+}\beta\dot{+}-S=-(S\dot{+}-\beta\dot{+}-S)$, it follows that $\operatorname{sgn}(S\dot{+}\xi\dot{+}-S)=\operatorname{sgn}\xi$ for every geometrical angle ξ.

If

$$\operatorname{sgn}\beta_1 = \operatorname{sgn}\beta_2 = \cdots = \operatorname{sgn}\beta_q =$$

then

$$\operatorname{sgn}(S \dot{+} \beta_j + - S) = \operatorname{sgn}\beta_j = \epsilon$$
$$\epsilon = \operatorname{sgn}(\dot{\sum}(S \dot{+} \beta_j \dot{+} - S)) = \operatorname{sgn}(S + \dot{\sum}\beta_j + - S)$$
$$\operatorname{sgn}(S \dot{+} T + - S) = \operatorname{sgn} T$$
$$\operatorname{sgn}(S \dot{+} T) = \operatorname{sgn}(-S + (S + T) + - (-S)) = \operatorname{sgn}(T + S).$$

It follows that the additive group A of kinematical angles can be ordered by the rule: $T>S$ if and only if $T\dot{+}-S$ is positive.

The kinematical angle corresponding to a given geometrical angle α has the same signature as α. Furthermore it follows that the correspondence between the geometrical angle α and the kinematical angle α is 1-1 and preserves addition as well as signature. In other words we have succeeded in extending the set of geometrical angles to the ordered additive group of the kinematical angles.

If $0<s<\alpha$ then $s=\dot{\sum}_{i=1}^{r}\alpha_i$, $\alpha\dot{+}-S=\dot{\sum}_{j=1}^{q}\beta_j$ with $\operatorname{sgn}\alpha_i=\operatorname{sgn}\beta_j=1$,

$$-S = -\alpha \dot{+} \dot{\sum}_{j=1}^{q}\beta_j = (-\alpha + \beta_1) \dot{+} \beta_2 + \cdots + \beta_q$$
$$= ((-\alpha + \beta_1) + \beta_2) \dot{+} \beta_3 \dot{+} \cdots \dot{+} \beta_q = \cdots .$$

Hence every positive kinematical angle which is not greater than a geometrical angle is itself geometrical. Denoting by μ the least upper bound of all geometrical angles we verify the condition 3a for the ordering of real numbers.

If there is no smallest positive geometrical angle, then the ordered additive group A of the kinematical angles satisfies all the axioms of the real module R. Since these axioms characterize R up to an order preserving isomorphism it follows that there is an order preserving isomorphism m of A onto R. In other words there is a measurement of the kinematical angles. This leads also to a measurement of the geometrical angles.

The order preserving isomorphism m is uniquely determined up to an order

preserving automorphism of R which is obtained by the multiplication of each real number by a fixed positive number. In other words, the measurement of angles is unique up to a positive factor of proportionality. A normalization is obtained by prescribing the measure $m(\mu)$. This must be a positive real number.

The Babylonians introduced $m(\mu)=180$ because of the relatively many divisors of this number. The artillery uses $m(\mu)=3200$ because of the high power of 2 which divides it as well as a suitable power of 10. It has been suggested to set $m(\mu)=\frac{1}{2}$ corresponding to $m(2\mu)=1$ for the full circuit but this convention is nowhere in use. Instead of taking 180 degrees many instruments are graduated into 200 degrees corresponding to $m(\mu)=200$. This convention assigns 100 degrees to the quadrant which is in accordance with the general tendency to introduce the decimal system everywhere. Finally, the transcendental number $\pi=3.14159\cdots$ has been widely used for the normalization of $m(\mu)$. This can be derived from the local property:

$$\underset{\alpha>0}{\text{l.u.b.}} \frac{s(R_\alpha)}{m(\alpha)} = 1,$$

where R_α denotes the rotation about the origin through the angle α.

For this measure we define

$$\begin{aligned} \sin m(\alpha) &= s(R_\alpha) \\ \cos m(\alpha) &= c(R_\alpha). \end{aligned} \tag{3}$$

We convince ourselves that the map

$$S = \sum_{i=1}^{r} \alpha_i \to R_{\alpha_1} \dotplus R_{\alpha_2} \dotplus \cdots \dotplus R_{\alpha_r} = R_s$$

is a homomorphic map of the group $\mathfrak{A}$ of the kinematical angles onto the group $\mathfrak{R}$ formed by the rotations about the origin. This must be so because the reductions and anti-reductions in $\mathfrak{A}$ correspond to identical substitutions in $\mathfrak{R}$.

Hence we may extend (3) by

$$\begin{aligned} \sin m(s) &= s(R_s) \\ \cos m(s) &= c(R_s). \end{aligned} \tag{4}$$

We then obtain the addition theorems and, in fact, all essential properties of the trigonometric functions in terms of a suitable angular measure.

If there is a smallest positive angle, say μ_0, then it follows by a well known argument that every angle is a multiple of μ_0. In this case all angles are commensurable, which is of course the only case realizable in practice. Again we find that, up to a constant factor of proportionality λ, there is only one angular measurement.

It is realized as $m(g\mu_0)=\lambda g (g=0, \pm 1, \pm 2, \cdots, \lambda>0)$. The corresponding plane trigonometry would be related to the regular polygon of $2n$ sides and vertices where $\mu=n\mu(0)$.

I would just point out that our construction of the module formed by the kinematical angles which has them generated by the geometrical angles is closely related to the construction of a covering group by Otto Schreier. In fact we can use our construction for a proof of the well known theorem that every 1-parameter group germ generates the additive group of the real numbers after suitable choice of the parameter. This proof avoids entirely the extraction of square roots; apart from our construction only some simple theorems on continuous functions need be used.

5. Conclusion. In closing this paper I can say this much in favor of the axiomatic method: it forces us to concentrate on the first principles and gives us back that humble attitude towards the unknown which is in constant danger of being lost under the weight of too many detailed results which we are often compelled to swallow whole. However, the axiomatic approach formally appears merely as a nice way of organizing things unknown and things known. The upshot of these two statements is that the axiomatic treatment is, as is everything else in the development of our science, the fruit of a particular historical situation, and that there is no fear that yet another historical situation may provide us with new points of view amplifying and revitalizing the axiomatic approach. Nor need you fear that the axiomatic treatment endangers your originality. It is entirely up to you to suit yourself with such foundations in a particular branch of mathematics as are most readily adaptable to your own problems. For example, I am quite sure that at some future time another mathematician will turn up and give his ideas on what an angle ought to be; ideas which will be quite different from those given today. His only obligation will be that his conception comprehends those concepts of angle and angular measurement which have been previously adopted and I might say it would be nice of him to extend our knowledge in the process. So why then are we doing this type of mathematics so far exceeding the necessity of making computations for daily use? Is it because we learn the principles governing effective computations and thereby do them better? Or because of the beauty of the logical pattern which evolves? Or because we desire to get a more powerful grip of the strings by which the isolated facts of mathematical experience are related? Or, simply because we must?

COMMENTS ON MATHEMATICAL EDUCATION*

J. W. LASLEY, JR., University of North Carolina

The need for closer cooperation between high school, college, and university teachers is receiving more attention at the present time than perhaps at any time in the past. Recognition of this need is not confined to the teaching profes-

* Written in December, 1953.

sion but is definitely high in the thinking of our government and industry. Essentially, the basic objective here is to find students with capacity for mathematical training while they are in the secondary schools, give them proper training and guidance throughout a program of study leading through the university, and thereby supply the nation with an adequate number of capable and well-trained mathematicians. Efforts to meet the need just cited are being made, in varying degrees of adequacy, by many educational institutions in the country. It is thought that reports on several of these might serve a number of good purposes, among which are: provide basis for over-all appraisal of the extent to which the need is being met, suggest variations in approaches being used in a given locality, lead to a closer cooperation between institutions that are seriously concerned about the matter. The present note reports some of the recent approaches used at the University of North Carolina.

1. High school mathematics contest. The University holds each year a high school mathematics contest with awards for the winner and recognition by honorable mention for the runners-up. The local Pi Mu Epsilon chapter makes an award to the winner. It is contemplated that scholarships be provided for this meritorious achievement. A thousand high school seniors participate in this contest. On High School Day when many hundreds of high school students come to the University, this contact with the schools of the State is greatly broadened. Opportunity is provided for meeting the mathematics staff, seeing the places where future university students will study, looking at models, exhibits, *etc.*, tending to quicken their interest in mathematical studies.

2. Contacts between high school and university teachers. The North Carolina Education Association has a mathematics section. Its meeting occurs concurrently with that of the larger body. It is felt here that a more specific get-together of this group with the University would be mutually beneficial. Consequently, this year such a meeting is projected for the first week in March.* It is to be a weekend session with both formal talks and small discussion groups. Three of these sub-groups will meet for afternoon, night, and morning-after sessions. Each will have one particular phase of high school-college relationship subjected to common scrutiny. It is hoped that this program, sparked by two or three formal addresses, will attract a hundred or more high school teachers.

The University is seeking a closer cooperation with the State Department of Public Instruction in the area of mathematics. In the early spring of this year an interested group from our mathematics staff met with a representative of that Department with a view to better understanding of our mutual problems at the high school teaching level. This representative proved to be eager, sensitive, and aware of many of these common problems. We sat down at table and talked these matters out at length. We decided to send a selected few of our staff to some of the meetings which this representative holds periodically with the county groups of high school mathematics teachers. In a few weeks a start

was made on this project, and it is our intent to follow up with similar meetings in many sections of the State. We found the teachers in these county centers entirely cooperative and interested. There was no spirit of recrimination evident. We met on the basis of a common job to do and a mutual sharing of the difficulties which it presented. We, at any rate, found these associations worthwhile.

3. Distribution of mathematical information. The General Electric Company of Schenectady, New York, has issued recently, in addition to its bimonthly News Digest, two pamphlets: one entitled "Why Study Math?," the other "Math at General Electric." These brochures are in the modern tempo, with direct appeal, jazzed up with cartoons. Few high school students could read them unmoved. These pamphlets were featured in the regular department bulletin board exhibits of interesting current mathematical materials. There was sent out in September of this past year the first of these to teachers of mathematics in all the colleges and high schools of the State. Copies of the second of these pamphlets are now in process of being similarly mailed. The potential for interest in this contact is believed to be considerable.

4. Special courses and fellowships. For some time now the mathematics courses offered in the summer session have been directed in a substantial part to high school teachers. Last summer the offerings in algebra designed for teachers were revamped into a course in special topics in modern algebra. This was part of a concerted move on the part of mathematics together with the several sciences to give an interlocking program to insure a full program for the teacher-student. During the ensuing summer this course in algebra is to be re-offered with alterations. A somewhat similar course in geometry will be added. This coming summer all courses are to be set up on an 80-minute 5-day schedule, with two such courses constituting a full-time load.

The Du Pont Company of Wilmington, Delaware, upon learning of the efforts being made at this university for better high school-college relationships, set up through the School of Education twelve Du Pont Fellowships for the summer session of 1954. A committee of representatives of the mathematics department and the several science departments together with the Dean of the School of Education are administering this fellowship program. It is heartening to observe that the cooperation between the Dean and the department representatives is all one could desire. Each of these fellowships carries a compensation of $225 in addition to coverage of tuition and fees. The tenure is for twelve weeks. The work is to be centered within subject matter areas, at least one of which must be mathematics or a science. The awards are not restricted to this state.

hence

$$\triangle BPE \sim \triangle APC,$$

and

(B) $$\measuredangle PAC = \measuredangle PBE.$$

(A) and (B) are the desired result. Similarly we can show that

$$\measuredangle QBC = \measuredangle QCA = \measuredangle QAB.$$

THE ELEMENTARY TRANSCENDENTAL FUNCTIONS

W. F. EBERLEIN, University of Wisconsin

It is difficult to differentiate the undefined.

1. Introduction. We outline a unified approach to the circular and hyperbolic functions motivated by the traditional geometric considerations (Figs. 1a and 1b), yet leading immediately to a sound analytic definition. The treatment in whole or part seems adaptable to modern calculus courses in which integration appears early and some knowledge of trigonometry is presupposed.

2. Geometric approach to the circular and hyperbolic functions. Consider the unit circle (right branch of the equilateral hyperbola)

(1a) $$x^2 + y^2 = 1$$

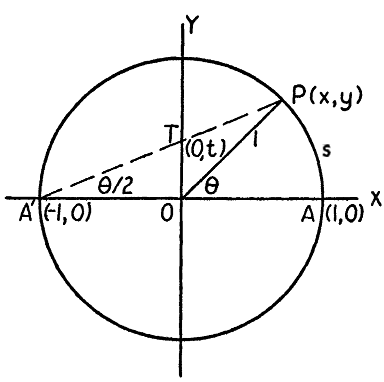


FIG. 1a

(1b) $$x^2 - y^2 = 1$$

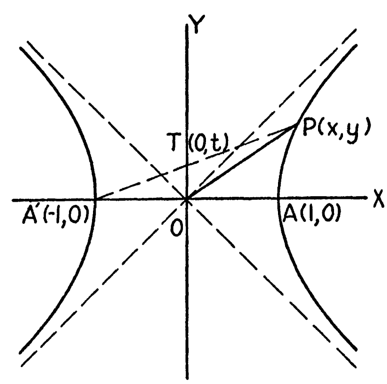


FIG. 1b

Letting θ denote twice the signed area of the sector OAP, regard the coordinates (x, y) of the variable point P as functions of θ:

(2a) $$x = \cos\theta,\quad y = \sin\theta.$$

[$\theta = s = \widehat{AP}$ is then the radian measure of $\angle AOP$.]

(2b) $$x = \cosh\theta,\quad y = \sinh\theta.$$

In particular

(3a) $$\cos 0 = 1,\quad \sin 0 = 0.$$

(3b) $$\cosh 0 = 1,\quad \sinh 0 = 0.$$

It follows from Figure 1 that in both cases

(4) $$\theta = xy - 2\int_1^x ydx.$$

Taking differentials in equations (1) and (4) yields

(5a) $$0 = xdx + ydy,$$

(5b) $$0 = xdx - ydy.$$

(6) $$d\theta = xdy + ydx - 2ydx = xdy - ydx.$$

Now solve equations (5) and (6) for the unknowns dx and dy:

(7a) $$dx = -\,yd\theta,\quad dy = xd\theta.$$

(7b) $$dx = yd\theta,\quad dy = xd\theta.$$

Hence

(8a) $$D_\theta \cos\theta = -\sin\theta,\quad D_\theta \sin\theta = \cos\theta.$$

(8b) $$D_\theta \cosh\theta = \sinh\theta,\quad D_\theta \sinh\theta = \cosh\theta.$$

To justify the formal differentiation consider the line $A'P$ joining A': $(-1, 0)$ and the point P: (x, y), where $P \neq A'$. Let $A'P$ intersect the y axis in the point T: $(0, t)$. It follows from the figure that

(9a) $$t = \tan\frac{\theta}{2},$$

?

(10) $$t = y/(1 + x).$$

Solving for the intersections of the line $A'T$ (10) and the curves (1) yields for the coordinates of P:

(11a) $$x = \frac{1 - t^2}{1 + t^2},\quad y = \frac{2t}{1 + t^2}\qquad (-\infty < t < \infty)$$

(11b) $$x = \frac{1 + t^2}{1 - t^2},\quad y = \frac{2t}{1 - t^2},\qquad -1 < t < 1).$$

Equations (11) define a one-to-one differentiable parametric mapping of the open interval $-\infty < t < \infty$ $(-1 < t < 1)$ onto the unit circle minus the point A' (the right branch of the rectangular hyperbola):

$$\text{(12a)} \quad dx = -\frac{4t}{(1+t^2)^2}\,dt, \qquad dy = \frac{2(1-t^2)}{(1+t^2)^2}\,dt.$$

$$\text{(12b)} \quad dx = \frac{4t}{(1-t^2)^2}\,dt, \qquad dy = \frac{2(1+t^2)}{(1-t^2)^2}\,dt.$$

Since (4) now defines θ as a differentiable function of t, equations (5) and (6) are justified. Substitute the values (11) and (12) into (6) to obtain*

$$\text{(13a)} \quad d\theta = \frac{2dt}{1+t^2}, \qquad \text{(13b)} \quad d\theta = \frac{2dt}{1-t^2}.$$

Note that equations (7) and (8) follow directly from (11), (12), and (13).

Since $\theta = 0$ when $t = 0$,

$$\text{(14a)} \quad \theta = 2\int_0^t \frac{du}{1+u^2} \quad (-\infty < t < \infty), \qquad \text{(14b)} \quad \theta = 2\int_0^t \frac{du}{1-u^2} \quad (-1 < t < 1).$$

The equations (9a), (11a), and (13a) are the familiar substitutions used to integrate rational functions of $\sin x$, $\cos x$. Moreover, since it follows from Figure 1a that $\theta = \frac{1}{2}\pi$ when $t = 1$ and $\theta \to \pi$ when $t \to \infty$,

$$\text{(15)} \quad \frac{\pi}{4} = \int_0^1 \frac{du}{1+u^2}, \qquad \frac{\pi}{2} = \int_0^\infty \frac{du}{1+u^2}.$$

To obtain an analytic theory of the circular and hyperbolic functions we need only retrace our steps carefully.

3. Analytic theory of the circular functions. Define π by the first of equations (15). The second equation then follows on making the change of variable $u = v^{-1}$:

$$\int_1^\infty \frac{du}{1+u^2} = \lim_{t\to\infty} \int_1^t \frac{du}{1+u^2} = \lim_{t\to\infty} \int_{t^{-1}}^1 \frac{dv}{1+v^2} = \int_0^1 \frac{dv}{1+v^2}.$$

Equation (14a) defines θ as an odd increasing function of $t(-\infty < t < \infty)$ with positive derivative (13a). Invert (14a) to obtain t as an odd increasing differentiable function of $\theta(-\pi < \theta < \pi)$. Equations (11a), (2a) now define $\cos\theta$, $\sin\theta$ as

* If θ is defined by the arc length s, (13a) appears as a consequence of the relation $ds^2 = dx^2 + dy^2$.

functions of θ over the open interval $-\pi<\theta<\pi$.† Cos θ, sin θ extend to continuous functions over the closed interval $-\pi\leqq\theta\leqq\pi$ on setting

$$\cos(\pm\pi) = \lim_{\theta\to\pm\pi} \cos\theta = \lim_{t\to\pm\infty} \frac{1-t^2}{1+t^2} = -1,$$

$$\sin(\pm\pi) = \lim_{\theta\to\pm\pi} \sin\theta = \lim_{t\to\pm\infty} \frac{2t}{1+t^2} = 0.$$

It is then legitimate to extend $\cos\theta$, $\sin\theta$ to arbitrary real values of θ by the requirement

$$\begin{aligned} \sin(\theta+2\pi) &= \sin\theta, \\ \cos(\theta+2\pi) &= \cos\theta. \end{aligned} \tag{16a}$$

The relations $\cos(-\theta)=\cos\theta$, $\sin(-\theta)=-\sin\theta(-\pi<\theta<\pi)$ resulting from (11a) are clearly preserved under extension.

Consider now the differentiation formulae (8a). For the interval $-\pi<\theta<\pi$ they follow again from (11a), (12a), and (13a). Their validity for $\theta=\pm\pi$ results from a familiar application of the law of the mean: For example, $\cos\pi-\cos\theta = -(\pi-\theta)\sin\xi\ (\theta<\xi<\pi)$; whence the left derivative

$$\cos'(\pi)=\lim_{\theta\to\pi}(\cos\pi-\cos\theta)/(\pi-\theta) = -\lim_{\xi\to\pi}\sin\xi=0$$

exists and has the correct value. The validity of (8a) for unrestricted values of θ then follows from the periodicity condition (16a).

Consider the remaining equations. (3a) and (10) follow directly from (11a); (1a) requires an additional continuity and periodicity argument. The equations (5a), (6a), (7a), (12a), (13a) leading from (4) to (14a) were previously justified, whence (4) and (14a) define the same quantity θ.* Note finally that the relation (9a) follows from (10) via the double angle corollaries of the addition formulae:

LEMMA:

$$\begin{aligned} u(\theta) &\equiv \sin(\theta+\theta_0) - \sin\theta\cos\theta_0 - \cos\theta\sin\theta_0 = 0; \\ v(\theta) &\equiv \cos(\theta+\theta_0) - \cos\theta\cos\theta_0 + \sin\theta\sin\theta_0 = 0. \end{aligned} \tag{19a}$$

Proof: (8a) implies that $u'(\theta)=v(\theta)$, $v'(\theta)=-u(\theta)$, whence $D_\theta(u^2+v^2)=2uv-2uv=0$. Hence $u^2+v^2\equiv[u(0)^2+v(0)^2]=0$ by (3).

4. Analytic theory of the hyperbolic and exponential functions. The development of the hyperbolic functions is even simpler in that no continuity or

---

† $t=1$ or $\theta=\pi/2$ is then the smallest positive zero of $x(t)=\cos\theta$. This result is usually taken as the definition of π in the power series approach to the circular functions.

* Changing to the variable t in (4) leads to an integral which is difficult to transform directly into (14).

periodicity arguments are required. Equation (14b) defines θ as an odd increasing function of $t(-1<t<1)$ with positive derivative (13b). That $\theta\uparrow\infty$ when $t\uparrow 1$ follows readily from inequalities of the type

$$\int_{1-1/n}^{1-1/(2n)} \frac{du}{(1+u)(1-u)} > \frac{1}{2}\int_{1-1/n}^{1-1/(2n)} \frac{du}{1-u} \geqq \frac{1}{2}\int_{1-1/n}^{1-1/(2n)} \frac{du}{n^{-1}} = \frac{1}{4}\,.$$

Invert (14b) to obtain t as an odd increasing differentiable function of θ $(-\infty<\theta<\infty)$. Equations (11b), (2b) then define $\cosh\theta$, $\sinh\theta$ for *all* real values of θ, and $\cosh(-\theta)=\cosh\theta$, $\sinh(-\theta)=-\sinh\theta$. Equations (8b) and the remaining equations now follow even more readily than before.

To obtain the missing equation (9b) introduce the function

$$E(\theta) = \cosh\theta + \sinh\theta = x + y = \frac{1+t}{1-t} > 0 \qquad (-1<t<1), \tag{17b}$$

and observe that

$$\begin{aligned} E'(\theta) &= E(\theta),\\ E(0) &= 1. \end{aligned} \tag{18b}$$

Moreover, $E(\theta)\uparrow\infty$ when $\theta\uparrow\infty\,(t\uparrow 1)$, while $E(\theta)\downarrow 0$ when $\theta\downarrow-\infty\,(t\downarrow -1)$.

Now set $e=E(1)$ and identify $E(\theta)$ with e^θ in standard fashion via the addition formula:

$$E(\theta_1+\theta_2) = E(\theta_1)E(\theta_2). \tag{19b}$$

LEMMA: *Let $E(\theta)$ be any differentiable function of θ satisfying the equations $E'(\theta)=aE(\theta)(-\infty<\theta<\infty)$ for some constant a, and $E(0)=1$. Then $E(\theta)$ satisfies* (19b).

Proof: Set $u(\theta)=E(\theta+\theta_0)E(-\theta)$. Then $u'(\theta)=aE(\theta+\theta_0)E(-\theta)-aE(\theta+\theta_0)\cdot E(-\theta)=0$. Hence $u(\theta)\equiv u(0)$, or $E(\theta+\theta_0)E(-\theta)=E(\theta_0)$. The symmetric form (19b) results on writing $\theta_0=\theta_1+\theta_2$, $\theta=-\theta_2$.

Noting finally that (17b) implies the familiar formulae

$$\begin{aligned} \cosh\theta &= \tfrac{1}{2}(e^\theta + e^{-\theta}),\\ \sinh\theta &= \tfrac{1}{2}(e^\theta - e^{-\theta}), \end{aligned} \tag{20b}$$

solve(17b) for t to obtain

$$t = \frac{e^\theta - 1}{e^\theta + 1} = \frac{\frac{1}{2}(e^{\theta/2} - e^{-\theta/2})}{\frac{1}{2}(e^{\theta/2} + e^{-\theta/2})} \equiv \tanh\frac{\theta}{2}\,. \tag{9b}$$

The function $w=\ln v(v>0)$ may be defined in elementary fashion by the equation $v=e^w\cdots$.

5. Euler's formula and the Fourier constant. The fundamental relations between the circular and hyperbolic functions are just the missing analogues

(16b), (17a), (18a), (20a). Under the formal changes of variable $t=-is$, $u=-iv$, where $i=\sqrt{-1}$, (14a) and (11a) become:

$$i\theta = 2\int_0^s \frac{dv}{1-v^2},$$

$$\cos\theta = \frac{1+s^2}{1-s^2} = \cosh i\theta,$$

$$\sin\theta = -i\frac{2s}{1-s^2} = -i\sinh i\theta.$$

The relation $e^{i\theta}=\cosh i\theta+\sinh i\theta$ then yields Euler's formula:

(17a) $$e^{i\theta} = \cos\theta + i\sin\theta = \frac{1+it}{1-it}\cdot$$

Now take (17a) as a definition and observe that

(16b) $$e^{i(\theta+2\pi)} = e^{i\theta};$$

(18a) $$D_\theta e^{i\theta} = ie^{i\theta}, \quad e^{i0} = 1;$$

(20a) $$\cos\theta = \frac{1}{2}(e^{i\theta}+e^{-i\theta}). \quad \sin\theta = \frac{1}{2i}(e^{i\theta}-e^{-i\theta}).$$

Since the lemma of section 4 remains valid for complex-valued functions of a real variable, (18a) implies

(19a′) $$e^{i(\theta_1+\theta_2)} = e^{i\theta_1}\cdot e^{i\theta_2},$$

which reduces to the addition formula (19a) on separating the real and imaginary parts. Moreover, (9a) can now be derived by solving (17a) for t.

Finally, the significance of the isolated definition (15) of π lies in its connection with the Fourier inversion formula:

$$F(u) = \int_{-\infty}^{\infty} e^{iux}f(x)dx,$$

$$f(x) = \frac{1}{C}\int_{-\infty}^{\infty} e^{-iux}F(u)du.$$

[The first equation defines $F(u)$ as a continuous function of u if $f\in L'(-\infty,\infty)$, in particular. The additional assumption that $F\in L'(-\infty,\infty)$ implies that the second equation holds at all points of continuity of $f(x)$.] In the classical theory the constant C is defined by the integral

$$C = 2\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx.$$

In the modern theory C appears as a normalizing factor for the dual group measure on the (self-dual) additive group of the reals.* In either case the problem is to prove that $C=2\pi$ by real variable arguments.

Set $f(x)=e^{-|x|}$. Equation (17a) or elementary integrations by parts yield

$$F(u) = 2\int_0^{\infty} e^{-x} \cos uxdx = \frac{2}{1+u^2}\cdot$$

Then

$$1 = f(0) = \frac{1}{C}\int_{-\infty}^{\infty} F(u)du = \frac{2}{C}\int_{-\infty}^{\infty} \frac{du}{1+u^2}\cdot$$

Hence

$$C = 2\int_{-\infty}^{\infty} \frac{du}{1+u^2} = 2\pi$$

by (15).

* For an introduction to the theory of harmonic analysis on commutative groups *cf.* the author's article Spectral theory and harmonic analysis, Proceedings of the Symposium on Spectral Theory and Differential Problems, Stillwater, Oklahoma, 1951, pp. 209–219.

THE EXPECTED NUMBER OF COMPONENTS UNDER A RANDOM MAPPING FUNCTION

MARTIN D. KRUSKAL, Princeton University

Let S be a finite set of N elements and $f(x)$ a function on S into S. A subset T of S is called *invariant* if $f^{-1}(T)=T$, or equivalently if $f(T)\subset T$ and $f^{-1}(T)\subset T$. It is straightforward to prove that if T and U are invariant, then so are the union, the intersection, and the complements of T and U. It follows immediately that the minimal non-null invariant subsets of S, called the *components of S under $f(x)$*, form a decomposition of S, *i. e.*, that every element of S belongs to exactly one component.

If $f(x)$ is one-to-one and thus a permutation of S, then the components are just the cycles of the permutation. In the general case, a component consists of a cycle together with a number of trees attached to the elements of the cycle.

Now let $f(x)$ be a random function whose values (for different arguments) are independently distributed with uniform probability $1/N$ over S. (In other

words, $f(x)$ has probability N^{-N} of being any specific mapping function.) N. Metropolis and S. Ulam* raise the problem of determining the expected number of components of S. This problem is solved here.

For any function $f(x)$ we may determine the components of S systematically in the following way. Starting with an arbitrary element x_1, if $f(x_1) \neq x_1$ we choose $x_2 = f(x_1)$, if $f(x_2) \neq x_1$ and $f(x_2) \neq x_2$ we choose $x_3 = f(x_2)$, and so on until we come to the first element x_j whose image $f(x_j)$ is one of the j elements already defined. We then call these elements the *first partial component*, and start over, beginning with an arbitrary element x_{j+1} different from the j distinct elements chosen so far. The sequence starting with x_{j+1} is continued until we reach the first element whose image is already either in the sequence or in the first partial component. If this image is in the sequence then we form the *second partial component* out of the sequence elements and start over. If it is in the first partial component, however, then we enlarge the first partial component by the addition of the sequence elements, and again start over. In either case, another arbitrary starting element generates a sequence which grows either until it runs into itself, thereby forming a new *partial component*, or else until it runs into an already existent partial component, in which case it is added on to that partial component. This process is continued until the set S is exhausted. The components of S are then just the partial components at the end.

We now apply this process to the random function $f(x)$ described above. Let $p_n(s)$ denote the probability that the growing sequence has exactly s elements just after the nth element x_n has been chosen from S. Now x_1 necessarily constitutes a sequence of exactly one element, so we have the initial condition

$$p_1(1) = 1, \qquad p_1(s) = 0 \qquad \text{for } 2 \leqq s \leqq N. \tag{1}$$

For $n > 1$, x_n can constitute a sequence of one element only if the sequence at the previous step went into a partial component (whether new or old), *i. e.*, only if $f(x_{n-1})$ was one of the $n-1$ elements $x_1, x_2, \cdots, x_{n-1}$. We have therefore the boundary condition

$$p_n(1) = \frac{n-1}{N} \qquad \text{for } 2 \leqq n \leqq N. \tag{2}$$

Moreover, x_n can be the sth element $(s > 1)$ of a growing sequence only if at the previous step the sequence had $s-1$ elements and $f(x_{n-1})$ was one of the $N-n+1$ elements not chosen by then, so we have the recursion relation

$$p_n(s) = \frac{N-n+1}{N} p_{n-1}(s-1) \qquad \text{for } 2 \leqq n \leqq N,\ 2 \leqq s \leqq N. \tag{3}$$

---

* A property of randomness of an arithmetical function, this MONTHLY, vol. 60, p. 252. They call trees what are here called components. Our usage of the words "tree" and "component" is consistent with that of graph theory, if a graph is formed by considering the elements of S as points and drawing an edge from every element x to its image $f(x)$.

Let P_n be the probability of forming a new partial component during the step of going to the image of x_n. If there are s elements in the sequence at that point in the process, the probability of forming a new partial component is the probability s/N that the image will be in the sequence, so that

$$P_n = \sum_{s=1}^{N} \frac{s}{N} p_n(s). \tag{4}$$

Using formulas (2), (3), and (4) and the obvious fact that

$$\sum_{s=1}^{N} p_n(s) = 1 \qquad \text{for } 1 \leqq n \leqq N, \tag{5}$$

we proceed to derive a recursion relation for P_n. For $2 \leqq n \leqq N$ we have $p_{n-1}(N) = 0$ and hence

$$\begin{aligned} P_n &= \frac{1}{N}\cdot\frac{n-1}{N} + \sum_{s=2}^{N} \frac{s}{N}\cdot\frac{N-n+1}{N} p_{n-1}(s-1) \\ &= \frac{n-1}{N^2} + \frac{N-n+1}{N} \sum_{s=1}^{N} \frac{s+1}{N} p_{n-1}(s) \\ &= \frac{n-1}{N^2} + \frac{N-n+1}{N}\left(P_{n-1} + \frac{1}{N}\right) \\ &= \frac{N-n+1}{N} P_{n-1} + \frac{1}{N}\cdot \end{aligned} \tag{6}$$

We wish to find E, the expected number of components. Since partial components once formed remain distinct, clearly

$$E = \sum_{n=1}^{N} P_n. \tag{7}$$

From recursion relation (6) we could already write E as an explicit sum. (Or just from formula (4) we could have written E as a double sum, since $p_n(s)$ can easily be found explicitly from formulas (1), (2), and (3).) We proceed differently, however, and find a simpler summation formula and on the way an integral representation for E which is useful in determining its asymptotic behavior.

We introduce the generating function

$$g(y) = \sum_{n=1}^{N} P_n y^n \tag{8}$$

and seek to convert the recursion relation (6) into a condition on $g(y)$. From formulas (1) and (4) we have $P_1 = 1/N$, and for convenience we define

$$P_{N+1} = \frac{1}{N}, \tag{9}$$

consistently with formula (6). Multiplying formula (6) through by y^n and summing over n from 2 to $N+1$ gives

$$
\begin{aligned}
g(y) - \frac{y}{N} + \frac{y^{N+1}}{N} &= \sum_{n=1}^{N} \frac{N-n}{N} P_n y^{n+1} + \frac{y^2 - y^{N+2}}{N(1-y)} \\
&= yg(y) - \frac{y^2}{N} g'(y) + \frac{y^2 - y^{N+2}}{N(1-y)},
\end{aligned} \tag{10}
$$

or

$$
g'(y) + N\frac{1-y}{y^2} g(y) = \frac{1-y^N}{y(1-y)} \cdot \tag{11}
$$

The solution of differential equation (11) is

$$
g(y) = y^N e^{N/y} \int_0^y u^{-N} e^{-N/u} \frac{1-u^N}{u(1-u)} du, \tag{12}
$$

the constant of integration being determined by the condition that $g(y)$ be regular at $y=0$. Hence we have for the expected number of components

$$
E = g(1) = \int_0^1 e^{N-N/u} \frac{u^{-N}-1}{u(1-u)} du, \tag{13}
$$

which, making the transformation $u=N/(N+z)$, may be more simply written

$$
E = \int_0^\infty \left[\left(1+\frac{z}{N}\right)^N - 1\right] e^{-z} \frac{dz}{z} \cdot \tag{14}
$$

Expanding $(1+z/N)^N$ by the binomial theorem and noting that

$$
\int_0^\infty z^m e^{-z} dz = m!, \tag{15}
$$

we immediately obtain

$$
E = \sum_{m=1}^{N} \frac{N!}{(N-m)!mN^m} \cdot \tag{16}
$$

Returning to formula (14), we observe that for each value of z, as N approaches infinity $(1+z/N)^N$ approaches e^z monotonically from below. Since the integral of the limiting integrand $(1-e^{-z})/z$ diverges at infinity, it is clear that E approaches infinity monotonically with N, as was to be expected. To determine the asymptotic behavior of E for large N, we define A by the condition

$$
E = \int_0^A (1-e^{-z}) \frac{dz}{z} \cdot \tag{17}
$$

Since the asymptotic behavior of E for large A is easy to determine, we seek the asymptotic behavior of A for large N. From formulas (14) and (17) we have

$$\int_0^A \left[1 - \left(1 + \frac{z}{N}\right)^N e^{-z}\right]\frac{dz}{z} = \int_A^\infty \left[\left(1 + \frac{z}{N}\right)^N - 1\right] e^{-z}\frac{dz}{z}, \tag{18}$$

or, setting $z = Aw$,

$$\begin{aligned}\int_0^1 [1 - e^{-N\{Aw/N - \log(1+Aw/N)\}}]\frac{dw}{w}\\ = \int_1^\infty [e^{-N\{Aw/N - \log(1+Aw/N)\}} - e^{-Aw}]\frac{dw}{w}.\end{aligned} \tag{19}$$

Now, A/N approaches zero as N approaches infinity, since otherwise the exponential in the left-hand integral of equation (19) would approach zero uniformly over every interval $0 < a \leqq w \leqq 1$ and therefore the left-hand integral itself would become infinite, whereas the right-hand integral would manifestly approach zero. Hence for any value of w

$$N\left\{\frac{Aw}{N} - \log\left(1 + \frac{Aw}{N}\right)\right\} = \frac{A^2w^2}{2N} - \frac{A^3w^3}{3N^2} + \cdots. \tag{20}$$

Accordingly, let*

$$L = \lim_{N\to\infty} \frac{A^2}{2N}. \tag{21}$$

Going to the limit under the integral signs in equation (19), which is easily justified, we obtain

$$\int_0^1 [1 - e^{-Lw^2}]\frac{dw}{w} = \int_1^\infty e^{-Lw^2}\frac{dw}{w}. \tag{22}$$

Transforming to the new variable of integration $v = Lw^2$ gives

$$\int_0^L [1 - e^{-v}]\frac{dv}{v} = \int_L^\infty e^{-v}\frac{dv}{v}, \tag{23}$$

whence

---

* Note that we are not really presupposing the convergence of $A^2/2N$, since the following argument applies to the limits superior and inferior of $A^2/2N$ (proving them equal) and, properly interpreted, to a supposed infinite value of L.

$$\begin{aligned}\log L &= \int_1^L \frac{dv}{v} = \int_1^\infty e^{-v}\frac{dv}{v} - \int_0^1 [1 - e^{-v}]\frac{dv}{v}\\ &= \int_0^\infty e^{-v}\log v\, dv,\end{aligned} \tag{24}$$

using integration by parts. But this last integral is known* to equal $-C$, where $C=0.5772\cdots$ is Euler's constant. For large N, therefore,

$$\log A = \tfrac{1}{2}(\log 2N - C) + o(1), \tag{25}$$

where $o(1)$ denotes a term vanishing in the limit. It would not be difficult to obtain further terms of the asymptotic series for $\log A$.

Returning to formula (17), it is clear that E behaves like $\log A$ for large A; indeed,

$$\begin{aligned}E - \log A &= \int_0^1 (1 - e^{-z})\frac{dz}{z} - \int_1^A e^{-z}\frac{dz}{z}\\ &= C + \int_A^\infty e^{-z}\frac{dz}{z} = C + o(1).\end{aligned} \tag{26}$$

The desired asymptotic behavior of the expected number E of components for large N is therefore

$$E = \tfrac{1}{2}(\log 2N + C) + o(1). \tag{27}$$

Formulas (16) and (27) may be compared to the corresponding results when $f(x)$ is a random permutation. In that case the expected number E of components (cycles) is given by†

$$E = \sum_{m=1}^{N} \frac{1}{m} = \log N + C + o(1). \tag{28}$$

* See for instance D. Bierens de Haan, Nouvelles tables d'intégrales définies, Leyden, 1867, table 256, formula (1).

† See for instance R. E. Greenwood, The number of cycles associated with the elements of a permutation group, this MONTHLY, vol. 60, p. 407.

THE RELATIONSHIP BETWEEN HARMONIC AND ANHARMONIC COLLINEATIONS IN A PLANE

HARI DAS BAGCHI and SHIB SANKAR SARKAR, Calcutta University

Introduction. A plane collineation (Ω) is said to be "regular" when there exist at least three *non-collinear* invariant points A, B, C, forming a triangle (Δ), such that Ω can be exhibited in the analytic form:

$$\text{(I)} \qquad \rho x' = \lambda x, \qquad \rho y' = \mu y, \qquad \rho z' = \nu z,$$

(ρ being a factor of proportionality), where (x, y, z) and (x', y', z') are respectively the triads of homogeneous or projective coordinates of a pair of corresponding points P, Q (referred to Δ as the *fundamental* triangle), and λ, μ, ν are certain constants, called the "roots" of Ω. In particular, Ω is called "anharmonic,"* when one of the three parameters λ, μ, ν is the geometric mean between the other two. More particularly, the "anharmonic" collineation is called "harmonic,"† when the "roots" are given by:

$$\lambda:\mu:\nu = -1:1:1 \quad \text{or} \quad 1:-1:1 \quad \text{or} \quad 1:1:-1.$$

The main purpose of the present paper is to reckon with the relationship between an "A.C." and an "H.C." The precise result to be proved is that the product of *two* H.C.'s is an A.C. and conversely an A.C. is always expressible—and that in an infinity of ways—as the product of two H.C.'s.

1. Starting with *two* given H.C.'s S_1, S_2, *viz.*,

$$S_1 \equiv \Pi_{O_1,L_1} \quad \text{and} \quad S_2 \equiv \Pi_{O_2,L_2},$$

let us now consider the two correlated product-collineations Π, Π', where

$$\Pi \equiv S_1 S_2 \quad \text{and} \quad \Pi' \equiv S_2 S_1.$$

Palpably the point of intersection (A) of the two base-lines L_1, L_2 (as shown in the adjoining figure) is an *invariant* point for both the collineations Π and Π'. To find the other two *invariant* points for either of them, we firstly note the two points K_1, K_2, where the lines L_1, L_2 cut the line $O_1 O_2$, and secondly mark the two *foci* (or *focal points*) B, C of the involution, defined uniquely by the point-

---

* If ξ, η, ζ denote respectively the cross-ratios of the *three* pencils of lines, *viz.*, $(b, c; AQ, AP)$, $(c, a; BQ, BP)$ and $(a, b; CQ, CP)$, where P, Q are as before a pair of corresponding points and a, b, c are the *three* sides of Δ, an "anharmonic" collineation,—often contracted as an A.C.—is definable alternatively as one for which two out of the three associated entities (ξ, η, ζ) are *equal*. The trivial case $(\xi=\eta=\zeta)$, answering to an "identity," may be left out of consideration.

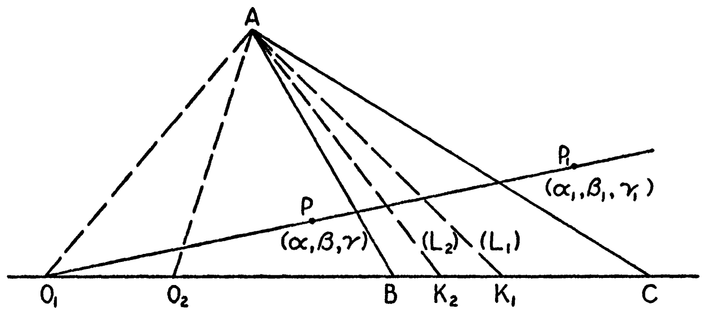
† A "harmonic" collineation—often contracted as an H.C.—is best defined as a particular variety of (1, 1)-correspondence between pairs of points P, Q, which are so related that when any one of these points (say, P) is joined to a certain *fixed* point (O) and the line OP cuts a certain *fixed* line (L) (not passing through O) at a point R, then Q must lie on the line OPR and the pairs O, R and P, Q must be harmonic conjugates. The H.C. is said to have O for its "centre" and L for its "axis" and is symbolised as $\Pi_{O,L}$, and can be put in the analytic form:

$$(\Pi_{O,L}): \quad \rho x' = -x, \quad \rho y' = y, \quad \rho z' = z,$$

provided that O is chosen as the *first* angular point (A) and any two distinct points (B, C) on L are chosen as the *second* and *third* angular points of the "triangle of reference" (Δ). For a *different ordering* of the three angular points of Δ, the H.C. can be made to assume either of the two forms:

$$(\rho x' = x, \ \rho y' = -y, \ \rho z' = z),$$
$$(\rho x' = x, \ \rho y' = y, \ \rho z' = -z).$$

pairs (O_1, K_1) and (O_2, K_2). Then the points B, C, being harmonically conjugate with respect to both the point-pairs (O_1, K_1) and (O_2, K_2), must simply interchange their positions under the operation of either S_1 or S_2. So no matter whether the operation of S_1 is preceded by or followed by that of S_2, each of the two points B, C will, after suffering a double interchange, remain absolutely *stationary*. Thus A, B, C are the three common invariant points of both the product collineations Π, Π'.



Now let us take the resulting invariant triangle ABC for the "triangle of reference." Then since the individual equations of AB, AC are $z=0$ and $y=0$, and the line-pair (AB, AC) is harmonic with each of the line-pairs (AO_1, AK_1) and (AO_2, AK_2), the equations of these four lines AO_1, AK_1, AO_2, AK_2 may respectively be taken as:

$$y = mz, \qquad y = -mz, \qquad y = nz, \qquad y = -nz,$$

where m and n are certain constants. Evidently the two points O_1, O_2 are $(0, m, 1)$ and $(0, n, 1)$.

If, then, an arbitrary point $P(\alpha, \beta, \gamma)$ be changed into $P_1(\alpha_1, \beta_1, \gamma_1)$ by the operation of S_1, the combined equation to the line-pair (AP, AP_1) is:

$$y^2 - \left(\frac{\beta}{\gamma} + \frac{\beta_1}{\gamma_1}\right) yz + \frac{\beta\beta_1}{\gamma\gamma_1} z^2 = 0. \tag{1}$$

Since the combined equation to the line-pair (AO_1, AK_1) is

$$y^2 - m^2z^2 = 0, \tag{2}$$

and (1) and (2) are harmonically conjugate, we must have

$$\frac{\beta_1}{\gamma_1} = m^2 \frac{\gamma}{\beta} \cdot \tag{3}$$

Further the collinearity of the three points O_1, P and P_1 imposes the condition:

$$
\begin{vmatrix} \dfrac{\alpha_1}{\gamma_1}, & \dfrac{\beta_1}{\gamma_1}, & 1 \\ \dfrac{\alpha}{\gamma} & \dfrac{\beta}{\gamma}, & 1 \\ 0, & m, & 1 \end{vmatrix} = 0. \tag{4}
$$

When (3) is substituted in (4), the latter equation takes the form:

$$
\left(\frac{\beta}{\gamma} - m\right)\left(\frac{\alpha_1}{\gamma_1} + m\frac{\alpha}{\beta}\right) = 0. \tag{5}
$$

Since $(\beta/\gamma) - m \neq 0$ for an arbitrary position of $P(\alpha, \beta, \gamma)$, (5) simplifies to:

$$
\frac{\alpha_1}{\gamma_1} = -m\frac{\alpha}{\beta}\cdot \tag{6}
$$

So combining (3) and (6), we get

$$
\alpha_1:\beta_1:\gamma_1 = -m\alpha : m^2\gamma : \beta.
$$

If we now represent the coordinates of the pair of corresponding points P, P_1 by the more familiar notations (x, y, z) and (x_1, y_1, z_1), then the H.C. (S_1) can be put in the analytical form:

$$
S_1: \quad \rho_1 x_1 = -mx, \quad \rho_1 y_1 = m^2 z, \quad \rho_1 z_1 = y, \tag{7}
$$

where ρ_1 is a factor of proportionality.

From symmetry S_2 can be written as:

$$
S_2: \quad \rho_2 x_2 = -nx, \quad \rho_2 y_2 = n^2 z, \quad \rho_2 z_2 = y, \tag{8}
$$

where ρ_2 has a similar meaning, and (x, y, z), and (x_2, y_2, z_2) are the initial and final positions of an *arbitrary* point.

By a simple algebraic superposition of the two transformations, we can easily obtain the product Π in the form:

$$
\Pi = S_1 S_2: \quad \rho x' = mnx, \quad \rho y' = n^2 y, \quad \rho z' = m^2 z. \tag{9}
$$

By a simple interchange of the two parameters m, n, we find Π' in the form:

$$
\Pi' \equiv S_2 S_1: \quad \rho x' = nmx, \quad \rho y' = m^2 y, \quad \rho z' = n^2 z. \tag{10}
$$

Now each of the two sets of equations (9) and (10) being of the *symbolic* form:

$$
\rho x' = \lambda x, \qquad \rho y' = \mu y, \qquad \rho z' = \nu z, \tag{11}
$$

where

$$
\lambda^2 = (mn)^2 = m^2 n^2 = \mu\nu,
$$

it follows that each of the two products Π, Π' is an A.C. This disposes of the first part of the premised proposition *viz.*, that *the product of two H.C.'s is an A.C.*

In order to test whether the commutative law holds or not we have simply to ascertain the condition under which the two sets of equations (9) and (10), defining (Π) and (Π'), may be *identical*. The requisite conditions are easily seen to be:

$$\frac{mn}{nm} = \frac{n^2}{m^2} = \frac{m^2}{n^2},$$

which imply that

$$m = +n \quad \text{or} \quad m = -n.$$

The first contingency is untenable, for the equality $m = +n$ implies that the points O_1, K_1 (Fig.) coincide *respectively* with O_2, K_2, leading ultimately to the coincidence of S_1 and S_2. Next the other contingency (*viz.*: $m = -n$) is possible only when the points O_1, K_1 coincide *respectively* with K_2, O_2.

Summing up the results, we can now assert that *the product of two H.C.'s conforms to the commutative law, when and only when the "centre" of either of them lies on the "axis" of the other.*

2. To examine the converse proposition, we observe that an arbitrary A.C. (T), given initially in the canonical form (11)—with *known* constants λ, μ, ν compatible with the relation $\lambda^2 = \mu\nu$—can be alternatively exhibited in the form (10), provided that the constants m, n are properly chosen. That is to say, *any* A.C. (T) can be expressed in the product form S_2S_1, where S_1, S_2 are the two H.C.'s given by (7) and (8).

Next to show that there are *infinitely many* solutions, we may modify S_1, S_2 by introducing an *arbitrary* parameter k as under:

$$S_1: \qquad \rho x' = -mx, \quad \rho y' = km^2z, \quad \rho z' = \frac{y}{k},$$

$$S_2: \qquad \rho x' = -nx, \quad \rho y' = kn^2z, \quad \rho z' = \frac{y}{k}.$$

These are all reflexions (H.C.'s), which lead to the *same* product, whatever be the value of the parameter k.

Recapitulating the results, we finally affirm that the product of two H.C.'s is an A.C. and that conversely every A.C. can be factorised into two H.C.'s, and that in an infinity of ways.

In conclusion we beg to express our indebtedness to the referee for his suggestions and helpful criticisms.

MATHEMATICAL NOTES

EDITED BY F. A. FICKEN, University of Tennessee

Material for this department should be sent to F. A. Ficken, University of Tennessee, Knoxville 16, Tenn.

ON THE POLYNOMIAL SOLUTIONS OF A RICCATI EQUATION

J. G. CAMPBELL, Albany, Kentucky and MICHAEL GOLOMB, Purdue University

1. The problem. In this note we propose to find the polynomial solutions of

$$Ay' = B_0 + B_1 y + B_2 y^2 \tag{1}$$

where A, B_0, B_1, B_2 are polynomials in x of degrees a, b_0, b_1, b_2, respectively. A general method is developed for finding all polynomial solutions, and some modifications suitable for special cases are given.

2. The degrees of polynomial solutions. Let m be the degree of a polynomial solution of (1). In a previous note [1] it was shown that if $a \neq 1+b_1$ then m is one of the five numbers $1+b_0-a$, $a-b_2-1$, b_0-b_1, $(b_0-b_2)/2$, b_1-b_2; if $a=1+b_1$ then $b_0-b_1 \leqq m \leqq b_1-b_2$. This result can be sharpened. Let

$$y = cx^m + c_1 x^{m-1} + \cdots + c_m, \qquad c \neq 0,$$

and let α, β_0, β_1, β_2 be the leading coefficients of A, B_0, B_1, B_2 respectively. For obvious reasons we assume $\alpha \neq 0$, $\beta_2 \neq 0$. Writing only the leading powers of each term in (1) we have

$$m\alpha c x^{a+m-1} + \cdots = \beta_0 x^{b_0} + \beta_1 c x^{b_1+m} + \beta_2 c^2 x^{b_2+2m} + \cdots. \tag{2}$$

Assume $a-1 \neq b_1$. If $m=1+b_0-a$ then b_0 is the largest exponent in (2). A value of m smaller than $1+b_0-a$ is not possible in this case since (2) would contain no term to balance $\beta_0 x^{b_0}$; the only possible m larger than $1+b_0-a$ makes $a+m-1=b_2+2m$, *i.e.*, $m=a-b_2-1$. In the same way it is seen that if $m=b_0-b_1$ then the only other possible value of m is b_1-b_2. If $a-1=b_1$ then the two cases considered coincide and nothing is changed unless, at the same time,

$$a - 1 = b_1,\ \frac{\beta_1}{\alpha} \text{ is a positive integer, } b_0 < b_1 + \frac{\beta_1}{\alpha},\ b_2 < b_1 - \frac{\beta_1}{\alpha}. \tag{3}$$

In this case, as is seen from (2), $m=\beta_1/\alpha$ is another possible degree for a polynomial solution. Finally, if none of the mentioned cases holds, but $m=(b_0-b_2)/2$, then no other degree is possible. Thus, we have

THEOREM 1. *The polynomial solutions of* (1) *have degrees from exactly one of the following classes:*

(i) $(b_0-b_2)/2$,
(ii) $1+b_0-a$, $a-b_2-1$,
(iii) b_0-b_1, b_1-b_2,

(iv) $1+b_0-a=b_0-b_1$, $a-b_2-1=b_1-b_2$, β_1/α.
Class (iv) *can occur only if conditions* (3) *are satisfied.*

If conditions (3) are satisfied then $m=\beta_1/\alpha$ is referred to as the "singular exponent."

It is seen that (1) can never have polynomial solutions of more than three different degrees. That three different degrees can occur is shown by the example given in [1].

3. The algorithm for the polynomials. The coefficients $c, c_1, \cdots, c_m$ of the polynomial solution y can now be determined by the following method. If m is not the singular exponent, then equating the coefficient of the highest power of x in (2) to zero gives a linear or quadratic equation for c, hence at most two possible values for c. To determine the other coefficients of y put $y=cx^m+y_1$ in (1) and find the following equation for y_1:

$$Ay_1' = (B_0 + B_1cx^m - Acmx^{m-1} + B_2c^2x^{2m}) + (B_1 + 2B_2cx^m)y_1 + B_2y_1^2.$$

This is another Riccati equation for y_1 with known polynomial coefficients. By proceeding as before we determine the leading coefficient c_1 of y_1, and $c_2, \cdots, c_m$ follow in similar fashion.

It remains to consider the case in which m is the singular exponent β_1/α. Equating the coefficient of the highest power of x in (2) to zero then gives $m\alpha c=\beta_1 c$; hence c remains undetermined in this first step. Comparing the coefficients of the powers x^{a+m-2} gives an equation

$$(m-1)\alpha c_1 + \cdots = \beta_1 c_1 + \cdots,$$

where the terms not written out contain c and possibly c^2, but not c_1. Since $(m-1)\alpha-\beta_1=-\alpha\neq 0$, c_1 is uniquely determined as a function of c, c^2. By proceeding in analogous fashion, we determine c_2 uniquely as a function of c, c^2 c^3, $c^4, \cdots$, and c_m uniquely as a function of $c, c^2, \cdots, c^{2m}$ by comparing the coefficients of the powers $x^{a-1}=x^{b_1}$ in (2). Finally, comparing the coefficients of $x^{b_1-1}, \cdots, x^0$ gives one or more equations for c alone, from which the possible values of c are determined. These, in turn, give the possible values of $c_1, \cdots, c_m$.

If it is known that (1) has more than one polynomial solution, the following algorithm gives all solutions. The difference $u=y-y_0$ between two polynomial solutions of (1) is a polynomial solution of the simplified Riccati equation

$$(1') \qquad Au' = (B_1 + 2y_0B_2)u + B_2u^2.$$

Suppose r (real or complex) is a zero of $u(x)$ of multiplicity p. Then r is a zero of the right-hand term of (1′) of multiplicity $\geqq p$, and since r is a zero of $u'(x)$ of multiplicity $p-1$, r must be a zero of $A(x)$. It follows that the polynomial solutions of (1′) are of the form

$$(4) \qquad u = k(x-r_1)^{p_1}(x-r_2)^{p_2}\cdots(x-r_k)^{p_k},$$

where the r_i are the zeros of $A(x)$, the p_i are non-negative integers, and k is a constant. Since for a given u, y_0 is immediately determined from (1′), we have

Theorem 2. *If* (1) *has more than one polynomial solution then the class of functions*

$$y = (Au' - B_1u - B_2u^2)/2B_2u, \tag{5}$$

where u is given by (4), *includes all polynomial solutions of* (1).

This algorithm is not applicable if (1) has only one polynomial solution. In this case (1′) has only the trivial solution $u=0$ and y_0 cannot be determined from (1′).

Finally we remark that if k distinct zeros $r_1, \cdots, r_k$ of $A(x)$ are known, then any polynomial solution of (1) of degree $\leqq k-1$ is readily determined. If $A(r_i)=B_1(r_i)=B_2(r_i)=0$ then it follows from (1) that no polynomial solution can exist unless also $B_0(r_i)=0$. In the latter case $x-r_i$ is a common factor of A, B_0, B_1, B_2, and may be eliminated from (1). Hence, we may assume that not both $B_1(r_i), B_2(r_i)$ vanish. Substitution of $x=r_i$ in (1) then gives a linear or quadratic equation for $y(r_i)$, hence at most two values for $y(r_i)$. When $y(r_1), \cdots, y(r_k)$ are known, the polynomial $y(x)$ of degree $\leqq k-1$ is found by interpolation.

4. An example. The Riccati equation

$$(x - x^3)y' = 4x - 4x^2 - (x^2 - 4x + 1)y - y^2 \tag{6}$$

has the four polynomial solutions: $2x$, $x-1$, $3x-1$, x^2+2x-1. By Theorem 1, the possible degrees are 0, 1, 2. The possible polynomials (4) are:

$$k_0, \quad k_1x, \quad k_2(1+x), \quad k_3(1-x), \quad k_4x(1+x), \quad k_5x(1-x),$$
$$k_6(1-x^2), \quad k_7x^2, \quad k_8(1+x)^2, \quad k_9(1-x)^2.$$

Putting $u=k_0$ in (5) we find the polynomial $-(x^2-4x+1+k_0)/2$ and direct substitution in (6) shows that this is not a solution for any k_0. Next put $u=k_1x$ in (5) and obtain the polynomial $\frac{1}{2}(4-k_1)x-1$, which on substitution in (6) gives $k_1=\pm 2$. Both $x-1$ and $3x-1$ are polynomial solutions. These solutions are obtained again if $k_2(1+x)$ and $k_3(1-x)$ are used for u in (5). Putting $u=k_4x(1+x)$ we find $k_4=-1$ and the solution x^2+2x-1. Putting $u=k_6(1-x^2)$ we find $k_6=-1$ and the solution $2x$. The remaining possible u's yield no further polynomial solutions of (6).

These solutions could also have been found from the values $y(0)=0$ or -1, $y(1)=1$ or 2, $y(-1)=-2$ or -4, which result from the substitution of $x=0, 1, -1$ in (6).

Reference

1. J. G. Campbell, A criterion for the polynomial solutions of a certain Riccati equation, this Monthly, vol. 59, 1952, pp. 388–389.

A SET-THEORETIC DESCRIPTION OF NORMAL TOPOLOGIES*

DAVID ELLIS, The University of Florida

1. Introduction. A matter of considerable interest in topology is the application of the topology of a space to obtain auxiliary structures by means of which the space may be studied algebraically. This general approach includes such diverse matters as homology and cohomology theories [**2**], homotopy theory [**3**], lattice-theoretic methods such as those of Stone [**7**] and Wallman [**10**], and algebraic characterizations of spaces by functions on them. Perhaps the most elegant example of this last type is the theory of Kaplansky [**4**] which characterizes compact Hausdorff spaces in terms of lattices of continuous functions. It is the object of the present note to characterize convergence in normal spaces algebraically (set-theoretically).

By a space we shall mean a set with certain distinguished subsets called open so that:

(i) The null set is open.

(ii) Any set-theoretic sum of open sets is open.

(iii) Any finite set-theoretic product of open sets is open.

(iv) The complement of any finite set is open. In the presence of (iv), (ii) implies

(i′) The entire space is open.

A set is called closed if and only if its complement is open. The complement of a set F is written $C(F)$. The smallest closed set which contains a set F is called the closure of F and written $\overline{F}$.

By a net [**5, 6**], we mean any mapping of a directed set [**1, 5, 6, 8**] into a set. Let $x(n):D\rightarrow S$ be a net mapping a directed set D into a set S. It is conventional to write x_n for $x(n)$. If S is a space, one says that x_n converges to $x\in S$, provided for each open set G in S with $x\in G$ there is $a\in D$ so that $b\geqq a$ implies $x_b\in G$. Kelley [**5**; page 9, no. 7] has shown that x is an accumulation point of a set $E\subset S$ if and only if there is a net taking values in $E-(x)$ and converging to x. Thus we have

LEMMA 1. *The topology of a space is completely determined by the convergence of nets in the space.*

By Lemma 1, then, if we find a structure which specifies the convergence of nets in a space this structure will be an adequate description of the topology of the space.

If A_n is a net on D whose values are subsets of a set S, we define

$$\liminf_n A_n = \bigcup_{a\in D} \bigcap_{b\geqq a} A_b \quad [\mathbf{1}; \text{page } 40, \text{no. } 2; \text{page } 60, \text{no. } 7].$$

This, we note, is a purely algebraic (or, if one prefers, set-theoretic) construction without reference to any topology on S.

* Presented to the Southeastern Section of the Mathematical Association of America, March 13, 1953.

2. The mapping $\mathfrak{k}:S\to\mathfrak{K}$. Let S be a space. If A and B are non-null subsets of S, a characteristic function for A and B is a continuous mapping $f:S\to[0, 1]$ of S into the closed unit interval so that $f(x)=0$ for $x\in A$ and $f(x)=1$ for $x\in B$. Clearly, if $f:S\to[0, 1]$ is a characteristic function for A and B, then $g(x)=1-f(x)$ is a characteristic function for B and A. We denote by K the class of all functions $f:S\to[0, 1]$ with the property: There are two non-null closed subsets A and B of S so that at least one of A and B contains a non-null open set and $f:S\to[0, 1]$ is a characteristic function for A and B. We denote by $\mathfrak{K}$ the Boolean algebra of subsets of K [1; page 153, paragraph 2] and define a mapping $\mathfrak{k}:S\to\mathfrak{K}$ by taking as $\mathfrak{k}(x)$ all those members of K which vanish on some open set containing x. We note that if $x_n:D\to S$ is a net in S, then $\mathfrak{k}(x_n):D\to\mathfrak{K}$ is a net in $\mathfrak{K}$.

LEMMA 2. *If x_n is a net in S which converges to $x\in S$, then* $\liminf_n \mathfrak{k}(x_n)\supset\mathfrak{k}(x)$.

Proof. Suppose $f\in\mathfrak{k}(x)$. Then f vanishes on some open set U with $x\in U$. But since x_n converges to x, there is $a\in D$ so that $b\geq a$ implies $x_b\in U$. Hence,

$$f \in \bigcap_{b\geq a} \mathfrak{k}(x_b) \subset \liminf_{n} \mathfrak{k}(x_n).$$

3. Additional terminology. Urysohn's Lemma. A space S is called completely regular [9; page 149, no. 42.1] provided there is a characteristic function for each pair A and B where A consists of a single point and B is a non-null closed set which does not contain the point of A. We shall write merely x for the set having only x as a member. S is called normal [9; page 79, no. 22.5; page 94, no. 28] provided when E and F are disjoint closed sets there are disjoint open sets G and H with $E\subset G$ and $F\subset H$. Any normal space is completely regular and we shall employ the stronger result [9; page 150, no. 43.3; page 149; no. 42.3; page 94, no. 28; page 79, no. 22.9]:

LEMMA 3 (*Urysohn's Lemma*). *If S is normal and A and B are disjoint, non-null, closed subsets of S, there exists a characteristic function for A and B.*

We note also that complete regularity (and, hence, normality) of S implies that S has the property of regularity [9; page 92, no. 29; page 81, no. 22.9; page 149, no. 43.2]: Any open set containing a closed set E contains the closure of an open set which contains E.

We say that two sets A and B are incomparable if neither contains the other. A class of sets which are pairwise incomparable is called totally unordered.

4. Main theorem.

LEMMA 4. *If S is completely regular and if x and y are distinct points of S, then $\mathfrak{k}(x)$ and $\mathfrak{k}(y)$ are incomparable.*

Proof. Suppose the hypotheses. Since S is completely regular, it is regular and there is an open set G with $y\in G\subset\overline{G}\subset C(x)$. Also, since S is completely regular, there is a characteristic function f for $\overline{G}$ and x. Thus, $f\in\mathfrak{k}(y)$ but $f\notin\mathfrak{k}(x)$.

Similarly, $\mathfrak{k}(x) \not\subset \mathfrak{k}(y)$.

LEMMA 5. *If S is normal and x_n is a net in S, and if $x \in S$ with* $\liminf_n \mathfrak{k}(x_n) \supset \mathfrak{k}(x)$, *then x_n converges to x.*

Proof. Suppose the hypotheses and the contradiction of the conclusion. Then there is an open set U with $x \in U$, and $a \in D$ implies there is $b \in D$ with $b > a$ and $x_b \in C(U)$. Since S is normal, it is regular, and there is an open set G with $x \in G \subset \overline{G} \subset U$. Also, since S is normal, Lemma 3 applies and there is a characteristic function f for the pair $\overline{G}$ and $C(U)$. But $f \in \mathfrak{k}(x)$ and $f \notin \liminf_n \mathfrak{k}(x_n)$ since $f(x_b) = 1$ for some x_b with b exceeding any preassigned $a \in D$. Thus, Lemma 5 is proved by contradiction.

Collecting the results of Lemmas 2, 4, and 5 we have

THEOREM. *If S is a normal space, there is a biuniform mapping $\mathfrak{k}: S \to \mathfrak{K}$ into a totally unordered subset of a set algebra $\mathfrak{K}$ so that if x_n is a net in S then x_n converges to $x \in S$ if and only if* $\liminf_n \mathfrak{k}(x_n) \supset \mathfrak{k}(x)$.

In view of Lemma 1, this theorem accomplishes the objective of algebracizing the topology of normal spaces.

References

1. Garrett Birkhoff, Lattice Theory (Rev. Ed.), Am. Math. Soc., New York, 1948.
2. S. Eilenberg and N. Steenrod, Foundations of Algebraic Topology, Princeton University Press, Princeton, New Jersey, 1952.
3. S. T. Hu, Homotopy Theory (Mimeographed), Tulane University, New Orleans, 1949.
4. I. Kaplansky, Lattice of continuous functions, Bull. Amer. Math., Soc., vol. 53, 1947, pp. 617–622.
5. J. L. Kelley, Convergence in topology, Duke Math. Jour., vol. 17, 1950, pp. 277–283.
6. E. J. McShane, Partial orderings and Moore-Smith limits, This MONTHLY, vol. 59, 1952, pp. 1–11.
7. M. H. Stone, Applications of the theory of Boolean rings to general topology, Trans. Am. Math. Soc., vol. 41, 1937, pp. 375–481.
8. J. W. Tukey, Convergence and Uniformity in Topology, Princeton University Press, Princeton, New Jersey, 1940.
9. R. Vaidyanathaswami, Treatise on Set Topology, Indian Math. Soc., Madras, 1947.
10. H. Wallman, Lattices and topological spaces, Annals of Math., vol. 39, 1938, pp. 112–126.

CONGRUENCES FOR THE NUMBER OF n-GONS FORMED BY n LINES

L. CARLITZ, Duke University

1. Introduction. Let g_n denote the number of polygons of n sides (including degenerate cases) formed by a network of n lines. Robinson [3] proved that g_n satisfies the recurrence

$$g_{n+1} = ng_n + \tfrac{1}{2}n(n-1)g_{n-2}, \tag{1.1}$$

when $g_1 = g_2 = 0$, $g_3 = 1$. For proofs of the relation

$$\lim_{n=\infty} \frac{u_n^2}{n} = \frac{4}{e^{3/2}\pi} \qquad (g_n = \tfrac{1}{2}(n-1)!u_n),$$

see [4].

In the present note we establish certain congruences satisfied by g_n. If m is a fixed integer greater than or equal to 1 we show first that

$$(1.2) \qquad g_{n+m} \equiv g_m g_n \pmod{m_0} \qquad (n \geqq 1),$$

where $m_0 = m$ for m odd, $m_0 = \frac{1}{2}m$ for m even. Formula (1.2) implies $g_{n+mk} \equiv g_m^k g_n$ so that the period (mod m_0) is mk when $g_n^k \equiv 1$. More generally if we define

$$(1.3) \qquad \Delta^r g_n = \sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} g_{n+sm} g_{(r-s)m},$$

then we show that

$$(1.4) \qquad \Delta^r g_n \equiv 0 \pmod{m_0^{[(r+1)/2]}} \qquad (n \geqq 1),$$

where $[x]$ denotes the greatest integer less than or equal to x. To put (1.2) in more explicit form it is necessary to know the residue of g_m (mod m). For $m = p$, a prime greater than 2, we show that

$$(1.5) \qquad g_p \equiv \tfrac{1}{2}(p-1) \pmod{p}.$$

The above results are suggested by similar results satisfied by the number of three-line latin rectangles [1], [2].

2. Proof of (1.2). Replacing n by $n+m$ in (1.1) we get

$$g_{n+m+1} = (n+m)g_{n+m} + \tfrac{1}{2}(n+m)(n+m-1)g_{n+m-2},$$

so that

$$(2.1) \quad \Delta g_{n+1} = n\Delta g_n + m g_{n+m} + \tfrac{1}{2}n(n-1)\Delta g_{n-2} + \tfrac{1}{2}m(m+2n-1)g_{n-m+2},$$

where $\Delta g_n = g_{n+m} - g_n g_m$. Clearly (2.1) implies

$$(2.2) \qquad \Delta g_{n+1} \equiv n\Delta g_n + \tfrac{1}{2}n(n-1)\Delta g_{n-2} \pmod{m_0},$$

where m_0 has the same meaning as in (1.2). Now for $n = m$, (1.1) yields

$$(2.3) \qquad \Delta g_1 = g_{m+1} = m g_m + \tfrac{1}{2}m(m-1)g_{m-2} \equiv 0.$$

Similarly we have

$$(2.4) \quad \Delta g_2 = g_{m+2} = (m+1)g_{m+1} + \tfrac{1}{2}(m+1)m g_{m-1} \equiv 0,$$

$$(2.5) \quad \Delta g_3 = g_{m+1} - g_m = (m+2)g_{m+2} + \tfrac{1}{2}(m+2)(m+1)g_m - g_m \equiv 0.$$

But (2.2) together with (2.3), (2.4), (2.5) imply

$$\Delta g_n \equiv 0 \pmod{m_0} \qquad (n \geqq 1), \tag{2.6}$$

which is equivalent to (1.2).

3. Proof of (1.4). The proof of (1.4) is somewhat more elaborate. In the first place we require an extension of (2.1). Using the notation (1.3), we have for $n \geqq 3$

$$\begin{aligned}\Delta^r g_{n+1} &= n\Delta^r g_n + \tfrac{1}{2}n(n-1)\Delta^r g_{n-2} + rm\Delta^{r-1}g_{n+m} \\ &\quad + \tfrac{1}{2}rm(m+2n-1)\Delta^{r-1}g_{n+m-2} + \tfrac{1}{2}r(r-1)m^2\Delta^{r-2}g_{n+2m-2},\end{aligned} \tag{3.1}$$

as is easily verified. We shall prove that

$$\Delta^{2r-1} g_n \equiv 0 \equiv \Delta^{2r} g_n \pmod{m_0^r} \qquad (n \geqq 1). \tag{3.2}$$

For $r=1$, (3.2) is a consequence of (2.6). We therefore assume that (3.2) holds up to and including the value $r-1$. Then (3.1) implies

$$\Delta^r g_{n+1} \equiv n\Delta^r g_n + \tfrac{1}{2}n(n-1)\Delta^r g_{n-2} \pmod{m_0^{[(r+1)/2]}},$$

so that it suffices to prove (3.2) for $n=1, 2, 3$.

Now for $n=1$, we have, using (1.1) and (3.1),

$$\begin{aligned}\Delta^r g_1 &= \sum_{s=1}^{r} (-1)^{r-s}\binom{r}{s}(smg_{sm} + \tfrac{1}{2}sm(sm-1)g_{sm-2})g_{(r-s)m} \\ &= mr\sum_{1}^{r}(-1)^{r-s}\binom{r-1}{s-1}g_{sm}g_{(r-s)m} \\ &\quad + \tfrac{1}{2}m^2r(r-1)\sum_{2}^{r}(-1)^{r-s}\binom{r-2}{s-2}g_{sm-2}g_{(r-s)m} \\ &\quad + \tfrac{1}{2}m(m-1)r\sum_{1}^{r}(-1)^{r-s}\binom{r-1}{s-1}g_{sm-2}g_{(r-s)m} \\ &= mr\Delta^{r-1}g_m + \tfrac{1}{2}m^2r(r-1)\Delta^{r-2}g_{2m-2} + \tfrac{1}{2}m(m-1)r\Delta^{r-1}g_{m-2}.\end{aligned}$$

Hence by the inductive hypothesis we infer that (3.2) holds for $n=1$. In exactly the same way

$$\Delta^r g_2 = \Delta^r g_1 + mr\Delta^{r-1}g_{m+1} + \tfrac{1}{2}m^2r(r-1)\Delta^{r-2}g_{2m-1} + \tfrac{1}{2}m(m-1)r\Delta^{r-1}g_{m-1},$$

and therefore (3.2) holds for $n=2$.

As for $n=3$, we have

$$\begin{aligned}\Delta^r g_3 &= 2\Delta^r g_2 + mr\Delta^{r-1}g_{m+2} + \tfrac{1}{2}m^2r(r-1)\Delta^{r-2}g_{2m} \\ &\quad + \tfrac{1}{2}m(m+3)r\Delta^{r-1}g_m + \Delta^r g_0,\end{aligned} \tag{3.3}$$

where we have put

$$(3.4) \qquad \Delta^r g_0 = \sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} g_{sm} g_{(r-s)m}, \qquad (g_0 = 1).$$

Now it is clear from (3.4) that

$$(3.5) \qquad \Delta^{2r-1} g_0 = 0$$

and

$$(3.6) \qquad \Delta^{2r} g_0 = 2\Delta^{2r-1} g_m.$$

Hence for r odd it follows from (3.3) and (3.5) that $\Delta^r g_3 = 0$, while for r even the same result is a consequence of (3.3), (3.6) and the inductive hypothesis. This completes the proof of (3.2).

4. Proof of (1.5). By formula (2) of [4] we have

$$(4.1) \qquad 1 + \sum_{1}^{\infty} \frac{g_n x^n}{n!} = (1 - x)^{-1/2} \exp\left(-\tfrac{1}{4}(x^2 + 2x)\right),$$

which is indeed an easy consequence of (1.1). Now (4.1) implies the following explicit expression for g_m:

$$(4.2) \qquad g_m = \sum_{r+s+2t=m} (-1)^{s+t} \frac{m!}{r!s!(2t)!} C_r C_t 2^{-s-t},$$

where $C_r = 1\cdot 3\cdot 5 \cdots (2r-1)$. For $m = p$, an odd prime, it is evident that all terms in the right member of (4.2) are divisible by p except the term corresponding to $r=0$, $s=p$, $t=0$. We accordingly get

$$g_p \equiv -2^{-p} \equiv \frac{p-1}{2} \pmod{p},$$

which proves (1.5).

Combining (1.5) and (1.2) we see that

$$(4.3) \qquad g_{n+p} \equiv -\tfrac{1}{2} g_n \pmod{p}, \qquad (n \geqq 1).$$

5. The following small table of the g_n and their residues was kindly supplied by John Riordan.

n	0	1	2	3	4	5	6	7	8	9	10
g_n	1	0	0	1	3	12	70	465	3507	30016	286884
mod 3	1	0	0	1	0	0	1	0	0	1	0
4	1	0	0	1	3	0	2	1	3	0	0
5	1	0	0	1	3	2	0	0	2	1	4

References

1. L. Carlitz, Congruences connected with three-line latin rectangles, Proc. Amer. Math. Soc., vol. 4, 1953, pp. 9–11.
2. John Riordan, A recurrence relation for three-line latin rectangles, this MONTHLY, vol. 59, 1952, pp. 159–162.
3. Robin Robinson, A new absolute geometric constant? this MONTHLY, vol. 58, 1951, pp. 462–469.
4. Robinson's constant (editorial note), this MONTHLY, vol. 59, 1952, pp. 296–297.

ON THE EVALUATION OF DIRICHLET'S INTEGRAL

S. K. LAKSHMANA RAO, Indian Institute of Science

The well-known multiple integral

$$\int\int_{(R_n)}\cdots\int x_1^{\alpha_1-1}x_2^{\alpha_2-1}\cdots x_n^{\alpha_n-1}(1-x_1-\cdots-x_n)^{\alpha_0-1}dx_1dx_2\cdots dx_n,$$

where R_n is the region defined by $x_1\geqq 0$, $x_2\geqq 0$, $\cdots$, $x_n\geqq 0$, $x_1+x_2+\cdots+x_n\leqq 1$, and α_0, α_1, $\cdots$, α_n are real positive constants, is usually evaluated by the use of Dirichlet's transformation. The integral is expressed as a product of Beta functions and the procedure does not at any stage suggest the relation between the Beta and Gamma functions.

The following is an alternative procedure for evaluating the above integral by the use of Laplace Transforms. It has the advantage that when $n=1$ the procedure provides a proof of the relation between the Beta and Gamma functions.

Let $f_0(x), f_1(x), \cdots, f_n(x)$, be $n+1$ functions whose Laplace Transforms *viz.*, $\int_0^\infty e^{-px}f_r(x)dx$ $(r=0, 1, \cdots, n)$ exist. We have by the convolution theorem:

$$L(f_0(x)*f_1(x)*\cdots*f_n(x)) = Lf_0(x)\cdot Lf_1(x)\cdots Lf_n(x). \tag{1}$$

Now $f_0(x)*f_1(x)*\cdots*f_n(x)$ can be written as

$$\int_0^x f_n(x_n)dx_n[f_0(x)*f_1(x)*\cdots*f_{n-1}(x)]_{(x-x_n)} = \cdots = \cdots$$

$$= \int_0^x f_n(x_n)dx_n\int_0^{x-x_n} f_{n-1}(x_{n-1})dx_{n-1}\cdots$$

$$\int_0^{x-x_n-\cdots-x_2} f_1(x_1)f_0(x-x_n-\cdots-x_2-x_1)dx_1.$$

The inverse of (1) is

$$f_0(x)*f_1(x)*\cdots*f_n(x) = L^{-1}\{Lf_0(x)\cdot Lf_1(x)\cdots Lf_n(x)\}.$$

Therefore

$$\int_0^x f_n(x_n)dx_n \int_0^{x-x_n} f_{n-1}(x_{n-1})dx_{n-1} \cdots$$

$$\int_0^{x-x_n-\cdots-x_2} f_1(x_1)f_0(x - x_n - \cdots - x_2 - x_1)dx_1$$

$$= L^{-1}\{Lf_0(x)\cdot Lf_1(x) \cdots Lf_n(x)\}.$$

Now take

$$f_r(x) = x^{\alpha_r - 1} \qquad (r = 0, 1, \cdots, n),\ \alpha_r > 0. \tag{2}$$

Then

$$\int_0^x x_n^{\alpha_n - 1} dx_n \int_0^{x-x_n} x_{n-1}^{\alpha_{n-1}-1} dx_{n-1} \cdots$$

$$\int_0^{x-x_n-\cdots-x_2} x_1^{\alpha_1-1}(x - x_1 - \cdots - x_n)^{\alpha_0-1}dx_1 = L^{-1}\left\{\frac{\Gamma(\alpha_0)}{p^{\alpha_0}}\cdot\frac{\Gamma(\alpha_1)}{p^{\alpha_1}} \cdots \frac{\Gamma(\alpha_n)}{p^{\alpha_n}}\right\}$$

$$= \frac{\Gamma(\alpha_0)\Gamma(\alpha_1)\cdots\Gamma(\alpha_n)}{\Gamma(\alpha_0 + \alpha_1 + \cdots + \alpha_n)} x^{\alpha_0+\alpha_1+\cdots+\alpha_n-1}.$$

When $x=1$ the above relation reduces to

$$\int_0^1 x_n^{\alpha_n-1}dx_n \int_0^{1-x_n} x_{n-1}^{\alpha_{n-1}-1} dx_{n-1} \cdots$$

$$\cdot \int_0^{1-x_n-\cdots-x_2} x_1^{\alpha_1-1}(1 - x_1 - x_2 - \cdots - x_n)^{\alpha_0-1}dx_1 = \frac{\Gamma(\alpha_0)\Gamma(\alpha_1)\cdots\Gamma(\alpha_n)}{\Gamma(\alpha_0 + \alpha_1 + \cdots + \alpha_n)}$$

i.e.,

$$\int\int_{R_n} \cdots \int x_1^{\alpha_1-1} x_2^{\alpha_2-1} \cdots x_n^{\alpha_n-1}(1 - x_1 - \cdots - x_n)^{\alpha_0-1} dx_1 dx_2 \cdots dx_n$$

$$= \frac{\Gamma(\alpha_0)\Gamma(\alpha_1)\cdots\Gamma(\alpha_n)}{\Gamma(\alpha_0 + \alpha_1 + \cdots + \alpha_n)}.$$

When $n=1$ this can be written

$$\int_0^1 x_1^{\alpha_1-1}(1 - x_1)^{\alpha_0-1}dx_1 = \frac{\Gamma(\alpha_0)\Gamma(\alpha_1)}{\Gamma(\alpha_0 + \alpha_1)}; \quad i.e., \quad B(\alpha_1, \alpha_0) = \frac{\Gamma(\alpha_0)\Gamma(\alpha_1)}{\Gamma(\alpha_0 + \alpha_1)}.$$

If instead of (2) we choose

$$f_r(x) = x^{\alpha_r-1} \quad (r = 1, \cdots, n), \qquad f_0(x) = f(x) = g(1 - x)$$

then we obtain in the same way the more general relation

$$\int\int_{R_n} \cdots \int x_1^{\alpha_1-1} x_2^{\alpha_2-1} \cdots x_n^{\alpha_n-1} g(x_1 + x_2 + \cdots + x_n) dx_1 \cdots dx_n$$
$$= \frac{\Gamma(\alpha_1)\Gamma(\alpha_2) \cdots \Gamma(\alpha_n)}{\Gamma(\alpha_1 + \alpha_2 + \cdots + \alpha_n)} \int_0^1 t^{\alpha_1+\alpha_2+\cdots+\alpha_n-1} g(t) dt.$$

CLASSROOM NOTES

EDITED BY G. B. THOMAS, Massachusetts Institute of Technology

All materials for this department should be sent to G. B. Thomas, Department of Mathematics, Massachusetts Institute of Technology, Cambridge 39, Mass.

A NATURAL APPROACH TO THE FUNDAMENTAL THEOREM OF THE INTEGRAL CALCULUS

J. P. HOYT, U. S. Naval Academy

A great deal of the work of elementary mathematics is concerned with operations and functions which are inverses of each other. One of the things to emphasize in teaching mathematics is how the knowledge of one of two inverse operations or inverse functions facilitates the use of the other. With this emphasis in mind, it is natural to see if summations can be effected by means of differences. That such is the case is well known to students of the calculus of finite differences but I have not seen this technique carried over to the calculus (of infinitesimal differences). Without discussing whys and wherefores, suffice it to say that I have used the following procedure to help first year students appreciate the fundamental theorem of the integral calculus.

We will assume that the student is familiar with the usual concepts and formulae that precede the fundamental theorem except that he need not to have been exposed to differentiation of the trigonometric, logarithmic, or exponential functions. In particular we will assume that he knows

$$(1)\quad \text{If } f(x) = x^n,\ \Delta f(x) = f'(x)\overline{\Delta x} + f''(x)\frac{\overline{\Delta x}^2}{2!} + \cdots + f^{(n)}(x)\frac{\overline{\Delta x}^n}{n!},$$

n a positive integer.

$$(2)\quad \text{If } f(x) = x^n,\ df(x) = f'(x)\Delta x = \Delta f(x) - o(\Delta x), \qquad n \text{ a positive integer.}$$

Statement (1) is useful as giving a preview of Maclaurin's theorem and also as a practical way of finding Δx^n easily.

We then show that if the closed interval (a, b) is divided into n sub-intervals by the points $a=x_0<x_1<x_2< \cdots <x_n=b$ that

$$\begin{aligned}\sum \Delta x_i = \sum (x_{i+1} - x_i) = (x_1 - x_0) + (x_2 - x_1) + \cdots \\ + (x_n - x_{n-1}) = x_n - x_0 = b - a.\end{aligned} \tag{3}$$

(Here, as in all uses of $\sum$ that follow, we assume the summation is from $i=0$ to $i=n-1$ unless otherwise noted.) Statement (3) is true regardless of the manner of division, but from this point on we will assume the sub-intervals are of equal length so that $\Delta x_{i+1}=\Delta x_i$.

More generally, if $f(x)$ is a function continuous throughout the closed interval (a, b) and if the interval is divided as above,

$$\sum \Delta f(x_i) = f(x_n) - f(x_0) = f(b) - f(a), \qquad \text{for any } n. \tag{4}$$

We illustrate this last statement to the class by taking a function such as x^3 over a small interval such as $(2, 6)$.

We then point out that the simple type of summation given in (4) is rarely encountered in mathematical applications but that the two types which are most frequently encountered, namely $\sum f(x_i)$ and the limit of $\sum f(x_i)\Delta x_i$ as n increases without limit can often be evaluated by (4). Evaluation of the latter type is the main concern of the integral calculus and we now show by considering a special case how this can be effected.

Suppose we wish to find (this can be motivated in the usual manner by "area under a curve")

$$\lim_{n\to\infty} \sum 3x_i^2\Delta x_i, \qquad x_0 = 2, \qquad x_n = 6, \qquad \Delta x_i = x_{i+1} - x_i = \frac{6-2}{n}\cdot \tag{5}$$

(In all summations which follow, unless otherwise indicated, $x_0=2$, $x_n=6$, $\Delta x_i=x_{i+1}-x_i=(6-2)/n$.)

Now we know that $dx^3=3x^2\Delta x$ while $\Delta x^3=3x^2\overline{\Delta x}+3x\overline{\Delta x}^2+\overline{\Delta x}^3$ (this latter is not quickly recognized by the average student but is easily found by (1)).

From (4) we know that

$$\sum \Delta x_i^3 = x_n^3 - x_0^3 = 6^3 - 2^3 \tag{6}$$

for any integral n. Hence

$$\sum 3x_i^2\overline{\Delta x_i} + \sum 3x_i\overline{\Delta x_i}^2 + \sum \overline{\Delta x_i}^3 = 6^3 - 2^3 \tag{7}$$

for any integral n.

Hence

$$\lim_{n\to\infty} \left[\sum 3x_i^2\overline{\Delta x_i} + \sum 3x_i\overline{\Delta x_i}^2 + \sum \overline{\Delta x_i}^3\right] = 6^3 - 2^3. \tag{8}$$

It is easily shown, without using any summation formulae, that

$$\lim_{n\to\infty} \left[\sum 3x_i\overline{\Delta x_i}^2 + \sum \overline{\Delta x_i}^3\right] = 0 \tag{9}$$

so that

$$\lim_{n\to\infty} \sum 3x_i^2\overline{\Delta x_i} = \sum \Delta x_i^3 = 6^3 - 2^3. \tag{10}$$

In (10) we have a special case of the fundamental theorem which we may express symbolically as

$$\lim_{n\to\infty} \sum df(x_i) = \sum \Delta f(x_i) = f(x_n) - f(x_0) \tag{11}$$

or, introducing the integral notation, as

$$\lim_{n\to\infty} \sum df(x_i) = \sum \Delta f(x_i) = \int_{x_0}^{x_n} df(x) = f(x_n) - f(x_0). \tag{12}$$

As an illustration of the finite type of summation we might use the following:

$$\text{Find } 2^2 + 2.5^2 + 3^2 + 3.5^2 + \cdots + 51.5^2. \tag{13}$$

In our summation notation, this is written

$$\sum_{i=0}^{99} x_i^2, \qquad x_0 = 2, \qquad x_n = 52, \qquad x_{i+1} - x_i = \tfrac{1}{2}, \qquad n = 100. \tag{14}$$

To express this in a form for which (4) and (12) are applicable, we divide and multiply by $3\Delta x_i$ where $\Delta x_i = \frac{1}{2}$ so that

$$\sum x_i^2 = \frac{2}{3}\sum 3x_i^2\Delta x_i = \frac{2}{3}\left[\sum \Delta x_i^3 - \sum 3x_i\overline{\Delta x_i}^2 - \sum \overline{\Delta x_i}^3\right] \tag{15}$$

$$= \frac{2}{3}\left[\int_2^{52} 3x^2dx - \frac{3}{2}\overline{\Delta x}\int_2^{52} 2x\,dx + \frac{1}{2}\sum \overline{\Delta x_i}^3\right] \tag{16}$$

$$= 92,387.5.$$

In like manner we can derive the general formula for the sum of n squares

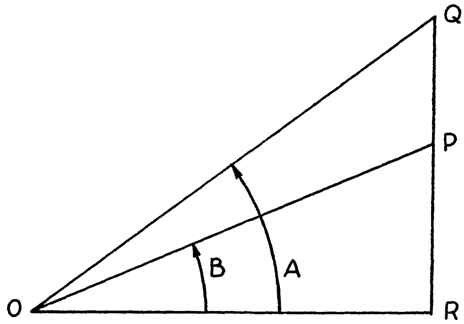
$$a^2 + (a+d)^2 + (a+2d)^2 + \cdots + b^2 = abn + \frac{nd^2}{6}(n-1)(2n-1), \tag{17}$$

or the formula for the sum of n mth powers, and even the Euler-Maclaurin summation formula.

A DERIVATION OF sin $(A-B)$ AND cos $(A-B)$

M. J. PASCUAL, Siena College

Since most texts use a rather tricky diagram to derive the formulas for the sin $(A+B)$ and cos $(A+B)$, it might be worth while to first derive the formulas for sin $(A-B)$ and cos $(A-B)$, and from these obtain the others. The derivation which follows assumes that the sine and cosine laws have been previously covered, and such is usually the case. A distinct advantage in the following method, other than the absence of any auxiliary lines, is the fact that it is natural to derive the sin $(A-B)$ by means of the sine law, and the cos $(A-B)$ by means of the cosine law. I'm sure it would be much more easily remembered by students.



By sine law in triangle OPQ

$$\frac{\sin(A-B)}{QP}=\frac{\sin\left(\frac{\pi}{2}-A\right)}{OP}=\frac{\cos A}{OP}$$

$$\sin(A-B)=\frac{QP\cos A}{OP}$$

$$=\frac{(QR-PR)\cos A}{OP}$$

$$=\frac{QR\cos A}{OP}-\frac{PR}{OP}\cos A$$

$$=\frac{QR}{OP}\cdot\frac{OR}{OQ}-\sin B\cos A$$

$$=\frac{QR}{OQ}\cdot\frac{OR}{OP}-\sin B\cos A$$

$$=\sin A\cos B-\sin B\cos A$$

By cosine law in triangle OPQ

$$\cos(A-B)$$

$$=\frac{\overline{OP}^2+\overline{OQ}^2-\overline{QP}^2}{2\overline{OP}\,\overline{OQ}}$$

$$=\frac{\overline{OP}^2+\overline{OQ}^2-(\overline{QR}-\overline{PR})^2}{2\overline{OP}\,\overline{OQ}}$$

$$=\frac{\overline{OP}^2-\overline{PR}^2+\overline{OQ}^2-\overline{QR}^2+2\overline{QR}\,\overline{PR}}{2\overline{OP}\,\overline{OQ}}$$

$$=\frac{\overline{OR}^2+\overline{OR}^2+2\overline{QR}\,\overline{PR}}{2\overline{OP}\,\overline{OQ}}$$

$$=\frac{\overline{OR}}{\overline{OQ}}\cdot\frac{\overline{OR}}{\overline{OP}}+\frac{\overline{QR}}{\overline{OQ}}\cdot\frac{\overline{PR}}{\overline{OP}}$$

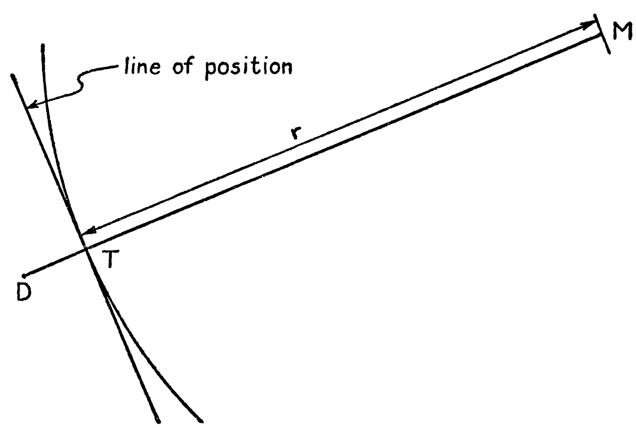
$$=\cos A\cos B+\sin A\sin B.$$

EXPLAINING THE LINE OF POSITION

R. P. BOAS, JR., Northwestern University

In principle, a circle of position can be obtained by drawing on a globe a circle with center at the geographical position of a star and radius equal to the star's zenith distance. This fact is easily grasped by a student at the beginning of his study of celestial navigation; but it is often not easy for him to see how it is related to the methods used in practice for drawing a line of position.

If we had a large globe and suitable drawing instruments, we could carry out the construction directly. (This has been suggested as a practical method [*Fortune*, January 1943].) If we tried to do the same thing on a large flat chart, the circle of position would usually not appear as a circle on the chart. This, however, is a minor difficulty, because we need only the part of the circle of position which passes near our approximate position. This part will, except for negligible errors, be represented by part of a circle of the same center and radius. A more serious difficulty is that to get both the center of the circle and our position on the same chart, we need either an inconveniently large chart, or one on too small a scale for accurate work. If we use, as we do in practice, a small chart on a large scale, we have to draw on the chart an arc of a circle whose radius is known and whose center is a known but inaccessible point.



To draw a circle of known radius r and known but inaccessible center M, we can take any convenient auxiliary point D on our chart, and determine the direction of M from D and the distance MD (these are the azimuth and zenith distance of the star whose geographical position is M, obtained by solving the astronomical triangle). Suppose that $MD > r$. Then a line drawn from D in the direction of M will be a radius (produced) of the circle of position. The circle will meet MD at a point T such that $DT = MD - r$. (DT is the altitude intercept.) Since in most cases the part of the circle of position which is on our chart will practically coincide with the tangent to the circle at T, we obtain the line

of position by drawing a line through T perpendicular to DM. If $MD<r$, $DT=r-MD$.

In this presentation, the "assumed position" (point D) takes its proper place as merely an auxiliary and arbitrary point. The discussion of "what we should see if we were at the assumed position," appearing in some textbooks, is an unnecessary complication.

The method just explained, that of St. Hilaire, depends on the fact that a point and a direction determine a straight line. A straight line is also determined by two points; it is this determination which is used in Sumner's original method for drawing a line of position.

RELATIVE MAXIMA AND MINIMA OF FUNCTIONS OF TWO OR MORE VARIABLES

A. S. HENDLER, Rensselaer Polytechnic Institute

Of our current calculus texts that include a treatment of partial derivatives, a number fail to include any discussion of relative extremes of functions of two variables. Some discuss only necessary conditions. Others add to necessary conditions a statement without proof of sufficient conditions. And a few to be sure do present sufficient conditions with a proof usually based on Taylor's theorem. Every text and every first course, however, certainly includes both necessary and sufficient conditions for relative extremes of functions of one variable. The purpose of this paper is to present an elementary derivation, that should have wide appeal among beginning students, of the usual sufficient conditions for relative extremes of functions of two variables, and to generalize this derivation to functions of more than two variables.

We take as our basis

THEOREM 1. *If:*

1. *$f(x)$ belongs to C^2,* $a\leqq x\leqq b$,
2. $f'(c) = 0$, $a<c<b$,
3. $f''(c)>0$,

then $f(c)$ is a relative minimum value of $f(x)$.

DEFINITION 1. *$f(a, b)$ is a relative minimum value of $f(x, y)$ in a region R if and only if $f(a, b)<f(x, y)$ for all (x, y) in R satisfying the inequality $0<(x-a)^2+(y-b)^2<\delta$ for some positive δ.*

If $f(x, y)$ belongs to C^2 in some region R and (a, b) is an interior point of R then

$$F(s) = f(a + s \cos \alpha, b + s \sin \alpha)$$

belongs to C^2, $-d\leqq s\leqq d$.

SOME STANDARD PROBLEMS IN INTEGRATION SIMPLIFIED

D. G. DUNCAN, University of Arizona

Students are often confused by the first applications of the fundamental theorem of the integral calculus, simply because the first problems they encounter involve either integrals with which they are not sufficiently familiar, or situations in which the basic idea of integration is lost in the details of analytical geometry required to set up the problems.

In the following examples, the intrinsic geometry of the figure involved, rather than an imposed coordinate system, is used. This leads to simpler and more intuitive solutions than those usually found in the texts.

(a) The standard derivation of the formula for the area of a circle involves the integral: $\int_0^a y\,dx = \int_0^a \sqrt{a^2-x^2}\,dx$.
However, by an obvious application of the fundamental theorem to a different element of integration we have: $A = \int_0^R 2\pi r\,dr = \pi R^2$.

(b) An integral for determining the area of a sphere is easily set up with the aid of Fig. 1; it is:

$$A = 2\int_0^{\pi/2} 2\pi R \sin\theta(Rd\theta) = 4\pi R^2.$$

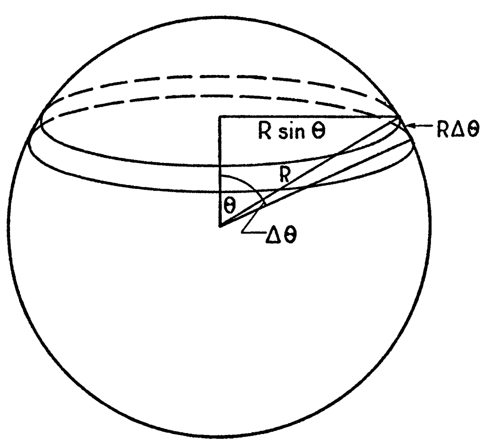


FIG. 1

(c) If we think of a sphere as composed of a number of concentric spherical shells of thickness Δr, its volume is determined immediately by the integral: $V = \int_0^R 4\pi r^2\,dr = \frac{4}{3}\pi R^3$.

(d) From the formula for the area of the lateral surface of a right circular cone, $A = \pi rs$ (s = slant height), the volume of a cone is easily computed

by taking concentric conical shells of thickness Δr (cos α) as the elements of integration (Fig. 2).

$$\begin{aligned} V &= \int_0^R (\pi r s)(\cos \alpha dr) \\ &= \int_0^R \pi r^2 \cot \alpha dr \\ &= \tfrac{1}{3}\pi R^3 \cot \alpha = \tfrac{1}{3}\pi R^2 H. \end{aligned}$$

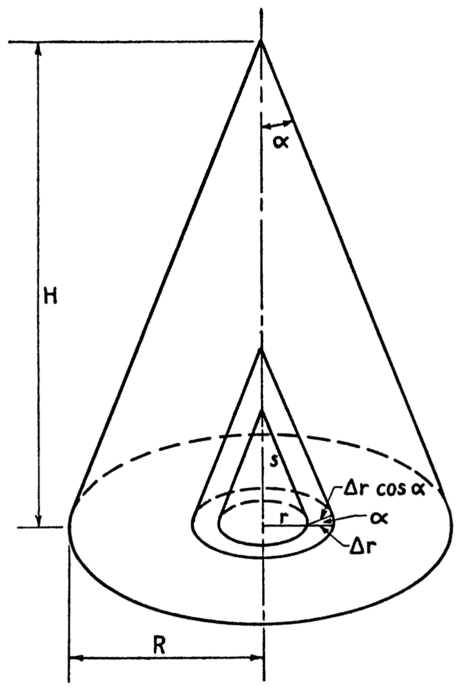


FIG. 2

(e) By the method of (b) above, we find the curved area of a spherical segment of one base to be $A = 2\pi R^2(1-\cos\theta)$. As in (c) above, this area may be used to calculate the volume of a spherical cone:

$$V = \int_0^R 2\pi r^2(1-\cos\theta)\, dr = \tfrac{2}{3}\pi R^3\,(1-\cos\theta).$$

Examples of the above type invariably help the student to visualize the process of integration and to cope with problems he may meet in the physical sciences which involve integration but do not correspond to a specific text-book situation.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, State University of New York

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Harpur College, Endicott, New York. This department welcomes problems believed to be new and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1121. *Proposed by W. D. Serbyn, Carnegie Institute of Technology*

Let E be any collection of n integers, not necessarily distinct. Show that there exists a non-empty subcollection $F\subseteq E$ such that the sum of the integers contained in F is divisible by n.

E 1122. *Proposed by P. B. Johnson, Occidental College and Haverford College*

Perfectly rigid playing cards are piled on the edge of a table with the pile slanting up away from the table. How far from the edge of the table can the pile be made to extend without falling to the floor?

E 1123. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn*

Given a square $N\times N$ point lattice, show that it is possible to draw a polygonal path passing through all the N^2 lattice points and consisting of $2N-2$ segments. Can it be done with less than $2N-2$ segments?

E 1124. *Proposed by L. J. Lander and J. L. Selfridge, U. C. L. A.*

Let $(a; 0)=1$ and $(a; n+1)=a^{(a;n)}$. Find all solutions of $(a; m)=(b; n)$ in integers a, b, m, n all greater than 1.

E 1125. *Proposed by Walter James, University of Minnesota*

Determine the coefficients B_j^p for the following sum:

$$\sum_{s=1}^{n} s^p = \frac{1}{(n-1)!}\sum_{j=1}^{p}\frac{B_j^p(n+j)!}{j+1}.$$

SOLUTIONS

Series of Reciprocals

E 1091 [1953, 711]. *Proposed by J. Lambek, McGill University, and Leo Moser, University of Alberta*

Given a sequence of integers $n_1<n_2<\cdots<n_i<\cdots$ such that for $j>i$ the decimal representation of n_j does not begin (on the left) with the decimal representation of n_i. Prove that

$$\sum_i 1/n_i \leqq 1+1/2+1/3+\cdots+1/9.$$

Solution by P. B. Johnson, Occidental College and Haverford College. Obviously one can assume that the n_i are positive. If the integers $1, 2, 3, \cdots, 9$ are all in the set, there can be no others, and equality follows. To fix our ideas, suppose 3 is not in the set. Successive inequalities of the type $1/30+1/31+1/32+\cdots+1/39<10/30=1/3$ shows that the sum of the reciprocals of all allowable numbers beginning with 3 is less than 1/3. Similar argument shows that the sum of the reciprocals of all permissible numbers beginning with $k<10$ is less than $1/k$. Hence the desired result.

Also solved by J. W. Baldwin, J. L. Botsford, W. E. Briggs, A. R. Hyde, H. N. Laden, D. C. B. Marsh, Joseph Muskat, L. A. Ringenberg, and H. F. Trotter.

The solution and result are easily generalized for any base of representation. For base 2 we find that the sum of reciprocals of a satisfactory sequence is at most 1.

Orthic and Tangential Triangles

E 1092 [1953, 711]. *Proposed by N. A. Court, University of Oklahoma*

The homothetic center of the orthic and tangential triangles of a given triangle (T) (see the proposer's *College Geometry*, 2nd ed., p. 98, art. 191) is the pole of the orthic axis of (T) with respect to the circumcircle of (T).

Solution by the Proposer. Let the sides BC, CA, AB of $(T)=ABC$ meet the orthic axis $D'E'F'$ of (T) in the points D', E', F', respectively. The polar of the point D' with respect to the circumcircle (O) of (T) passes through the harmonic conjugate of D' with respect to the points B, C of (O), which point is the foot D of the altitude AD of (T) (*ibid*, p. 241, ex. 1). The polar of D' for (O) also passes through the pole A'' of the side BC with respect to (O). Thus the polar of D' is the line $A''D$ which joins the vertex A'' of the tangential triangle $A''B''C''$ of (T) to the vertex D of the orthic triangle DEF of (T).

Now the point D' lies on the orthic axis $D'E'F'$, hence $A''D$ passes through the pole L of $D'E'F'$ for the circle (O). Similarly the polars $B''E$, $C''F$ of the points E', F' for (O) pass through the point L. Hence the proposition.

Also solved by G. B. Charlesworth, Joseph Langr, and Victor Thébault.

Thébault pointed out that the ratio of homothety is $2R \cos A \cos B \cos C$.

Limit of a Sequence

E 1093 [1953, 711]. *Proposed by H. S. Wilf, Nuclear Development Associates, White Plains, N. Y.*

Define $S_0=1$, $S_1=3$, $S_{n+1}=2S_n^2-1$ for $n\geqq 1$. Find

$$\lim_{n\to\infty} \frac{S_n}{2^n S_0 S_1 \cdots S_{n-1}}\cdot$$

I. *Solution by D. C. B. Marsh, University of Colorado.* From

$$S_n^2 - 1 = (S_n + 1)(S_n - 1) = 2S_{n-1}^2(2S_{n-1}^2 - 2) = 2^2S_{n-1}^2(S_{n-1}^2 - 1),$$

it follows that

$$S_n^2 - 1 = 2^{2(n-1)}S_{n-1}^2S_{n-2}^2 \cdots S_1^2(S_1^2 - 1),$$

whence

$$\frac{S_n}{2^nS_0S_1 \cdots S_{n-1}} = \frac{S_n}{2S_0\sqrt{\dfrac{S_n^2 - 1}{S_1^2 - 1}}} = \frac{\sqrt{S_1^2 - 1}}{2S_0}\sqrt{\frac{S_n^2}{S_n^2 - 1}},$$

which has as limit

$$\sqrt{(S_1^2 - 1)/2S_0} = \sqrt{2}.$$

II. *Solution by J. V. Whittaker, University of California, Los Angeles.* Let r be the positive root of the equation $\cosh x = 3$. Since $\cosh 2x = 2\cosh^2 x - 1$, we have $S_n = \cosh 2^{n-1}r$. Then

$$\begin{aligned}\frac{S_n}{2^nS_1 \cdots S_{n-1}} &= \frac{\cosh 2^{n-1}r}{2^n \cosh r \cdots \cosh 2^{n-2}r}\\ &= (1/2)\sinh r \coth 2^{n-1}r\\ &= \sqrt{2}\coth 2^{n-1}r.\end{aligned}$$

As $n \to \infty$, this quantity approaches $\sqrt{2}$, since $\coth x \to 1$ as $x \to \infty$.

Also solved by R. P. Bailey, J. W. Baldwin, L. F. Boron, J. L. Botsford, Bernice Brown, G. B. Charlesworth, F. J. Duarte, M. P. Epstein, H. M. Feldman, Harry Goheen, A. S. G. Grant, A. R. Hyde, M. S. Klamkin, Viktors Linis, C. F. Pinzka, D. C. Russell, Paul Schillo, M. R. Spiegel, O. E. Stanaitis, Chih-yi Wang, L. E. Ward, Jr. and L. E. Ward, Sr. (jointly), and the proposer.

Construction of a Function

E 1094 [1953, 712]. *Proposed by Azriel Rosenfeld, Columbia University*

(1) Construct a function defined everywhere on a closed (or open) interval which takes on each of its values exactly twice on this interval.

(2) Prove that no such function can be continuous.

Solution by the Proposer. (1) It is trivially easy to construct such a function on a *half*-open interval. A one-one mapping from this to the given closed (open) interval (along the lines, for example, of the example given in Kamke, *Theory of Sets*, pp. 14–15) will give the desired function.

(2) This is a special case of the following: A continuous function which takes on no value more than twice must take on some value exactly once.

Proof: Obviously the function cannot be even locally constant. Let I be some closed interval in which it is defined, and M the maximum value which the function f takes on I. If $f=M$ nowhere else, we are done; if not, let m be the minimum which f takes on between the two points where $f=M$. Then evidently between these two points f takes on all the values $m<a\leqq M$ at least twice, and thus, by hypothesis, exactly twice. But the value m is taken on only once, since for it to be taken on twice, either within or without the interval between the two maxima, f would have to take on some of the values $m<a<M$ still more times.

Also solved by Trevor Barker, J. L. Botsford, Vern Hoggatt, H. D. Lipsich and G. M. Merriman (jointly), J. D. Miller, Norman Miller, L. L. Pennisi, L. A. Ringenberg, W. L. Shepherd, O. E. Stanaitis, G. H. M. Thomas, and L. E. Ward, Jr.

Many interesting examples of functions satisfying (1) were offered. Lipsich and Merriman supplied examples showing that for such a function monotonicity on a range of continuity is not necessary, and also that boundedness, bounded variation, and integrability are not necessary either. They also produced an example of a function defined everywhere on a closed interval which takes on each of its values exactly k times on this interval, where k is any positive integer.

Rebuses

E 1095 [1953, 712]. *Proposed by Leon Bankoff and C. W. Trigg, Los Angeles, Calif.*

Translate each of the following sketches into a mathematical term. (For the sketches, see the proposal, this MONTHLY, vol. 60, p. 712.)

Contributions by Julian Braun, Bernice Brown, P. L. Chessin, A. R. Hyde, M. S. Klamkin, D. C. B. Marsh, Norman Miller, Walter Penny, and the proposers.

1. osculating circles, osculation, triple contact, infinity, degenerate curves, elliptical (lip-tickle)

2. reflection, analysis situs, concentration, positive definite form, image, rational function, homogeneous (homo genius), everywhere dense

3. inverse sine (sign in verse), negative sign, sine law, versine, ordered field, natural barrier, closed region

4. primitive triangle, primitive solution, triad, opposition, 2:1 ratio, elementary operations, addition and subtraction, substitution, law of the mean, intercept, transformation

5. pentagon (tag on pen), boundary values, null class, pencil (pen+sell), reduction to lower terms, reduced cubic

6. moment of inertia, imbedded, curl, nappe (nap), horizontal plane, partial covering, singular

7. point set, high point contact, tangent (tan gent), points in a plane, needle problem, fixed points, multiply connected surfaces, singular points

8. asymptotes (ass totes imp), amount, develop (devil up), first law of the mean

9. matrix (May+tricks), magic square, matrix inversion (if left to right is normal order), proof by mathematical induction (like pulling rabbits out of a hat)

10. absolutely convergent, positively decreasing, convergent, bounded and convergent

11. evolution (ape+missing link+man), linkage, open interval, discontinuity, infinite descent, discontinuous chain, evolute, tractrix (evolute of a catenary), law of excluded middle

12. combination (comb+eye+nation), ordered set (I use a [U.S.A.] comb).

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known text books or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4593. *Proposed by S. W. Golomb, Harvard University*

Let $0, a_1, a_2, \cdots, a_{p-1}$ be any complete residue system modulo the odd prime p. Show that

$$0, a_1, 2a_2, \cdots, (p-1)a_{p-1} \tag{1}$$

is never a complete residue system modulo p.

Show further that if any non-zero residue r is specified, the a_i's can be so chosen that every residue except r occurs in (1). What relation will the residue s occurring twice bear to r?

4594. *Proposed by Edgar Reich, Rand Corporation, Santa Monica, California*

If $f(z)$ is a complex-valued continuous function of the complex variable z, such that $f(z)=z$ whenever $|z|=1$, show that $f(z)$ has at least one zero in $|z|<1$.

4595. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn, New York*

If

$$[xD]^n = \sum_{r=1}^{n} A_{rn} x^{n-r+1} D^{n-r+1},$$

where D is the differential operator, determine A_{rn}.

4596. *Proposed by G. G. Lorentz, Wayne University, and M. S. MacPhail, Carleton College, Ottawa*

Given a sequence s_n, let

$$\sigma_{n,p} = \frac{s_n + s_{n+1} + \cdots + s_{n+p-1}}{p}. \tag{1}$$

We call s_n almost absolutely convergent (a.a.c.) if the series

$$\sum_{p=1}^{\infty} \left| \sigma_{n,p} - \sigma_{n,p+1} \right|$$

converge uniformly for $n=1, 2, \cdots$. Prove that:

1. A sequence s_n is a.a.c. if s_n is absolutely convergent, that is if $\sum |s_n - s_{n+1}| < \infty$.
2. If each a.a.c. sequence is absolutely summable by a method of summation $A=(a_{mn})$, then there is a bounded sequence which is absolutely A summable but is not a.a.c.

4597. *Proposed by Paul Erdös, University of Notre Dame*

Let $1=a_1<a_2<\cdots$ be an infinite sequence of integers. Let n be any integer and write $n=a_{i_1}+a_{i_2}+\cdots+a_{i_k}$ where a_{i_1} is the greatest $a \leqq n$, a_{i_2} the greatest $a \leqq n-a_{i_1}$, *etc.* (Since $a_1=1$ this representation is always possible.) Put $f(n)=k$ (*i.e.*, the number of summands representing n). Prove that if the upper density of the a's is 0, then

$$\lim_{x\to\infty} \frac{1}{x} \sum_{k=1}^{x} f(k) = \infty,$$

and if the lower density of the a's is >0, then

$$\overline{\lim_{x\to\infty}} \frac{1}{x} \sum_{k=1}^{x} f(k) < \infty.$$

(If $N(y)$ denotes the number of a's $\leqq y$, the upper density of the a's is $\overline{\lim}\, N(y)/y$ and the lower density is $\underline{\lim}\, N(y)/y$.)

SOLUTIONS

Optimum Strategy for an Addition Game

3651 [1933, 610]. *Proposed by Orrin Frink, Pennsylvania State University*

Two persons play a game using 24 cards numbered 1 to 6, placed face up (there being four cards of each number), by alternately turning over any card

not previously turned. The first player to make the sum of the numbers on the cards turned over exceed 30, loses. With best play, should the player who plays first always win, or always lose?

Solution by D. C. B. Marsh, Colorado University. The first player always wins if he leads with a 2 and uses the following strategy.

Initially, every card has a complement with respect to 7. A player certainly wins if his play brings the total to 30 (or to 29 if there are no 1's left). A primary strategy, then, is for the first player, A, to lead with 2 and follow each x played by B with $7-x$. It is still possible for B so to play that all 2's are exhausted before A has achieved a total of 30. Hence if B responds to the original lead of 2 with either 5 or 2, the strategy must be modified.

(1) Let the first three plays be 2-5-1. Then A should play $8-x$ to B's x at the first opportunity and $7-x$ thereafter. If B always plays 1, A should respond with 6 each time, reaching a total of 29, which wins in this case since all 1's are exhausted.

(2) If the first three plays are 2-2-2, A's responses to various possible further plays by B are best shown in a table, although the strategy in each case is quite transparent.

<table>
<tr><td>B</td><td colspan="5">6 2</td><td colspan="6">5 4 3</td></tr>
<tr><td>A</td><td colspan="5">2 6</td><td colspan="6">3 4 5</td></tr>
<tr><td>B</td><td colspan="2">6 4 3 1</td><td colspan="3">5</td><td>6 5 4 3</td><td colspan="3">1</td><td colspan="2">2</td></tr>
<tr><td>A</td><td colspan="2">1 3 4 6</td><td colspan="3">1</td><td>3 4 5 6</td><td colspan="3">6</td><td colspan="2">5</td></tr>
<tr><td>B</td><td>6 5 4 3</td><td>1</td><td>6 5 4</td><td>3</td><td>1</td><td>y</td><td>6 5 4 3</td><td>2</td><td>1</td><td>6 5 4 3</td><td>1</td></tr>
<tr><td>A</td><td>3 4 5 6</td><td>6</td><td>4 5 6</td><td>5</td><td>1</td><td>7−y</td><td>3 4 5 6</td><td>5</td><td>1</td><td>3 4 5 6</td><td>6</td></tr>
<tr><td>B</td><td></td><td>1</td><td></td><td>1</td><td>6 5 4 3 1</td><td></td><td></td><td>1</td><td>x</td><td></td><td>1</td></tr>
<tr><td>A</td><td></td><td>1</td><td></td><td>1</td><td>1 3 4 5 6</td><td></td><td></td><td>1</td><td>7−x</td><td></td><td>1</td></tr>
</table>

<table>
<tr><td>B</td><td colspan="9">1</td></tr>
<tr><td>A</td><td colspan="9">6</td></tr>
<tr><td>B</td><td>6 5 4</td><td colspan="3">3</td><td colspan="2">2</td><td colspan="3">1</td></tr>
<tr><td>A</td><td>4 5 6</td><td colspan="3">5</td><td colspan="2">6</td><td colspan="3">1</td></tr>
<tr><td>B</td><td>y</td><td>6 5 4 3</td><td>2</td><td>1</td><td>6 5 4 3</td><td>1</td><td>5 4 3</td><td>2 6</td><td>1</td></tr>
<tr><td>A</td><td>7−y</td><td>3 4 5 6</td><td>5</td><td>1</td><td>3 4 5 6</td><td>6</td><td>3 4 5</td><td>6 2</td><td>6</td></tr>
<tr><td>B</td><td></td><td></td><td>1</td><td>x</td><td></td><td>1</td><td>x</td><td>6 5 4 3 1</td><td>6 5 4 3 2</td></tr>
<tr><td>A</td><td></td><td></td><td>1</td><td>7−x</td><td></td><td>1</td><td>7−x</td><td>1 1 3 4 6</td><td>2 3 4 5 6</td></tr>
</table>

A Multiplication Theorem for Meromorphic Functions

4531 [1953, 193]. *Proposed by R. M. Redheffer, University of California, Los Angeles*

Let $f(z)=\sum a_n z^n$ have simple poles with positive residues at the points $d_m \neq 0$, and similarly for $g(z)=\sum b_n z^n$ at the points $e_m \neq 0$. Both functions are supposed regular elsewhere. Then the function $\sum a_n b_n z^n$ has simple poles with negative residues at the points $d_m e_n$ and no other finite singularities.

Solution by the Proposer. Define the operation $\cdot$ by $f\cdot g=\sum a_n b_n z^n$, f and g being as in the problem. Then one shows trivially

$$f\cdot g=g\cdot f, \qquad f\cdot(g+h)=f\cdot g+f\cdot h, \tag{1}$$

$$\frac{a}{z-b}\cdot f(z) = -\frac{a}{b}\, f\left(\frac{z}{b}\right) \qquad (a,\ b \text{ constant}) \tag{2}$$

$$\frac{p}{z-d}\cdot\frac{q}{z-e} = -\frac{pq}{z-de} \qquad (d,\ e,\ p,\ q \text{ constant}). \tag{3}$$

It suffices to get the result for an arbitrary (large) circle $|z|<R$. Let the residues of the poles for f and g be respectively f_i and g_i. We may write (making use of the hypothesis)

$$f(z)=\sum_{i=1}^{p}\frac{f_i}{z-d_i}+F(z) \tag{4}$$

where $d_1, d_2, \cdots, d_p$ are the (finite number of) poles of $f(z)$ inside the circle $|z|=R/d$ and $F(z)$ is regular for the same disk. Here d stands for the smallest of the numbers $(1, |d_i|, |e_i|)$. Writing down a relation like (4) for g and computing $f\cdot g$ we get, by (1),

$$f\cdot g=\sum\sum\frac{f_i}{z-d_i}\cdot\frac{g_i}{z-e_i}+\sum\frac{f_i}{z-d_i}\cdot G+\sum\frac{g_i}{z-e_i}\cdot F+F\cdot G.$$

By (3) the first sum has simple poles with negative residues at all points $d_m e_n$ which lie in $|z|<R$, and by (2) the next two sums are regular in this circle. Regularity of the last follows by computing the radius of convergence, which is $\geqq R/d \geqq R$. (It has been brought to the author's attention that this method is well known. See Borel, Sur les singularities des series de Taylor, *Bull. de la Soc. Mathem. de France*, t. XXVI, and Pincherle, A proposito di un recente teorema del sig. Hadamard, *Rendiconto de l'Ac. des Sc. de Bologne.* See also Hadamard's multiplication theorem, Titchmarsh, *Theory of Functions*, pp. 157–159. The sole novelty in the present treatment lies in the specialization to simple poles.)

Also solved by R. M. Breusch and O. E. Stanaitis.

Primitive Root (mod p)

4532 [1953, 193]. *Proposed by Leonard Carlitz, Duke University*

Let p be a prime >3 and let g denote a primitive root (mod p). Prove

1. $$1 + \sum_{r=1}^{(p-3)/2} \frac{(1+g)\cdots(1+g^r)}{(1-g)\cdots(1-g^r)} = \prod_{r=1}^{(p-3)/2} (1+g^r) \pmod{p}.$$

2. If $p \equiv 3 \pmod 4$, then

$$1 + \sum_{r=1}^{(p-3)/2} g^{r(r+1)/2} \frac{(1+g)\cdots(1+g^r)}{(1-g)\cdots(1-g^r)} = \prod_{r=1}^{(p-3)/4} (1+g^{2r}) \pmod{p};$$

if $p \equiv 1 \pmod 4$, the sum vanishes.

Solution by the Proposer. 1. We shall prove the following slightly more general result. Let $GF(p^n)$ denote a finite field of order p^n, $p>2$, and let $2m$ divide (p^n-1). Let α be a number of $GF(p^n)$ that belongs to the exponent $2m$, and let x be an indeterminate. Then

$$\prod_{r=1}^{m-1} (1+\alpha^r x) = 1 + \sum_{r=1}^{m-1} \frac{(1+\alpha)\cdots(1+\alpha^r)}{(1-\alpha)\cdots(1-\alpha^r)} x^r. \tag{1}$$

Proof. We recall the identity

$$\prod_{r=1}^{m-1} (1+q^r x) = 1 + \sum_{r=1}^{m-1} \frac{(1-q^{m-1})\cdots(1-q^{m-r})}{(1-q)\cdots(1-q^r)} q^{r(r+1)/2} x^r. \tag{2}$$

In (2) place $q=\alpha$; then, since $\alpha^m \equiv -1$, we have

$$(1-\alpha^{m-1})\cdots(1-\alpha^{m-r}) = (1+\alpha)\cdots(1+\alpha^r)\alpha^{-r(r+1)/2},$$

and (1) follows.

2. Let $2m$ divide (p^n-1) and let α belong to the exponent $2m$. We shall prove

$$1 + \sum_{r=1}^{m-1} \alpha^{r(r+1)/2} \frac{(1+\alpha)\cdots(1+\alpha^r)}{(1-\alpha)\cdots(1-\alpha^r)} = \begin{cases} \displaystyle\prod_{r=1}^{(m-1)/2} (1+\alpha^{2r}) & (m \text{ odd}) \\ 0 & (m \text{ even}). \end{cases} \tag{3}$$

Proof. We shall use the formula of Gauss (see, for example, G. B. Mathews, *Theory of Numbers*, p. 210.)

$$1 + \sum_{r=1}^{m-1} (-1)^r \frac{(1-x^{m-1})\cdots(1-x^{m-r})}{(1-x)\cdots(1-x^r)} = \begin{cases} \displaystyle\prod_{r=1}^{(m-1)/2} (1-x^{2r-1}) & (m \text{ odd}) \\ 0 & (m \text{ even}). \end{cases} \tag{4}$$

In (4) place $x=\alpha^{-1}$; then, since $\alpha^m\equiv -1$, we have

$$\frac{(1-\alpha^{-m+1})\cdots(1-\alpha^{-m+r})}{(1-\alpha^{-1})\cdots(1-\alpha^{-r})} = (-1)^r\alpha^{r(r+1)/2}\frac{(1+\alpha)\cdots(1+\alpha^r)}{(1-\alpha)\cdots(1-\alpha^r)},$$

so that the left member of (4) reduces to the left member of (3). Similarly the right member of (4) yields for m odd

$$\prod_{r=1}^{(m-1)/2}(1-\alpha^{-2r+1}) = \prod_{r=1}^{(m-1)/2}(1+\alpha^{m-2r+1}) = \prod_{r=1}^{(m-1)/2}(1+\alpha^{2r}).$$

This completes the proof of (3).

The Fermat Equation

4533 [1953, 267]. *Proposed by R. Kissling, Student, University of California, Berkeley*

Given $a^n+b^n=c^n$ with a, b, c, n integers and $a>b>1$, $n\geqq 2$; $\sigma_n(k)$ being the sum of the nth powers of all divisors of k, prove

$$(1)\qquad \left|1-\frac{\sigma_n(c)}{\sigma_n(a)+\sigma_n(b)}\right| < \frac{2n-1}{n(n-1)} < \frac{2}{n-1},$$

$$(2)\qquad \lim_{k\to\infty}\frac{\sigma_k(c)}{\{[\sigma_k(a)]^{n/k}+[\sigma_k(b)]^{n/k}\}^{k/n}} = 1.$$

Solution by J. V. Whittaker, University of California at Los Angeles. From the well-known inequalities

$$a^n < \sigma_n(a) < a^n\zeta(n),\qquad \frac{2^n-1}{2^n+1} < \zeta(n) < \frac{2^n}{2^n-2},$$

it follows that

$$\frac{1}{\zeta(n)} < \frac{\sigma_n(c)}{\sigma_n(a)+\sigma_n(b)} < \zeta(n)$$

and, hence, that

$$\left|1-\frac{\sigma_n(c)}{\sigma_n(a)+\sigma_n(b)}\right| < \zeta(n)-1 < \frac{1}{2^{n-1}-1} < \frac{2n-1}{n(n-1)}\,.$$

Moreover,

$$\frac{1}{\zeta(k)} < \frac{\sigma_k(c)}{\{[\sigma_k(a)]^{n/k}+[\sigma_k(b)]^{n/k}\}^{k/n}} < \zeta(k).$$

Since, as $k\to\infty$, $\zeta(k)\to 1$, the middle member of the inequality does the same.

Also solved by R. R. Phelps and the Proposer.

A Permutation Matrix

4534 [1953, 268]. *Proposed by Frank Harary, University of Michigan*

If n is an odd prime and M is a symmetric n by n matrix each of whose rows is a permutation of $1, \cdots, n$, then the main diagonal is also such a permutation.

Solution by T. S. Motzkin, National Bureau of Standards, University of California, Los Angeles. Every $s=1, \cdots, n$ appears n times; of these, because of symmetry, an even number are outside the diagonal. Hence for odd n, s must appear in the diagonal and the conclusion follows.

Similarly, if a_{ijk} is an n by n by n array with permutations of $1, \cdots, n$ in all rows, and if $a_{ijk}=a_{i'j'k'}$ for any permutation $i'j'k'$ of ijk, then for n not divisible by 3 the main diagonal is also a permutation of $1, \cdots, n$.

More generally, if $a_{i_1 \cdots i_r}$ is an n^r array of permutations and "symmetric" in the same sense, and if r is a power p^t of a prime p and n not divisible by p, the main diagonal is again a permutation, by virtue of the well known fact that every multinomial coefficient $r!/\prod \alpha_p!$, where $r=\sum \alpha_p$, except $r!/r!$, is divisible by p.

Also solved by T. A. Brown, W. E. Deskins, J. E. Freund, Harry Furstenberg, Katherine Gould, S. L. Jamison, J. B. Kelly, T. C. Littlejohn and R. J. Driscoll (jointly), D. C. B. Marsh, J. H. McKay, D. A. Norton, and E. M. Wright.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to E. P. Vance, Oberlin College, Oberlin, Ohio.

An Introduction to the History of Mathematics. By Howard Eves. New York, Rinehart and Company, Inc., 1953. xv+422 pages. $6.00.

There has long been a need for a book on the history of mathematics which was suitable for use as a text with undergraduates at the junior-senior level. There were a few books, but they were in turn, too advanced, too elementary, too long, too concise. Eves' book was written expressly for this purpose and this group. It succeeds remarkably well. The reviewer passes judgment not only on the basis of having read the book, but also on that of having used it as a text in a three credit hour, one semester course.

A unique and usable feature of the book is the sets of "Problem Studies" at the end of each chapter. The author contends that even in a history of mathematics class students should *do* some mathematics, not merely talk about, or, still worse, listen to others talk about it. His "Problem Studies" are not routine

problems; many outline small investigations or developments. Suggestions and answers for them are included in a twenty-five page appendix. They are designed to help the student to learn more of both mathematics and its history while having fun and accumulating enrichment material, which will, in part, be useful in secondary school teaching.

The reviewer made little use of these Problem Studies this past semester mainly because he had his own program of problems and references, but he thinks the plan excellent and well worked out. In some places the problems might be more closely related to the historical material, and perhaps more detailed and specific bibliographical material in both problems and the text would assist students. However, many of the problems do also contain significant historical material or add to the text the mathematical details needed for students to really appreciate what was done.

The book sets a high level of accuracy, both in respect to typographical errors and historical facts. Historically, the few sins are mostly of omission; for example, the sector compasses discussed on page 266 probably antedate Galileo who merely improved on the invention of Guido Ubaldo del Monte according to Gino Loria and others; the rolling wheel paradox attributed to Galileo in Problem Study 9.7 is to be found in Sir T. L. Heath's *The Mathematics in Aristotle* published posthumously in 1949. There are places, of course, where there could be honest differences of opinion or interpretation, such as the statement (page 331) that "· · · the so-called fundamental theorem of integral calculus is explicitly stated and proved in Barrow's *Lectiones*."

This last is, in the reviewer's opinion, a "hindsight" error. That is, it attributes to Barrow an insight, perception, or generalization which we can now read into his work, especially if we rewrite it in modern notation, which, however, it's doubtful that Barrow really had. Barrow's chief translator and interpreter, J. M. Child, agrees with Eves, however.

In such a task as writing this book the author must make many choices as to time intervals and topics to be covered, details and persons to be included, *etc.* One can not dispute tastes, but the reviewer feels that the selection here has been well made even though he, personally, would prefer such changes as: more emphasis on a topical approach (for example, a thorough treatment of the entire story of trigonometry or of irrationals in one place at the sacrifice of interrupting the chronological progress of the book); more discussion of some more recent topics (*e.g.*, transfinite numbers, modern algebra, topology—the book essentially ends with the development of the calculus although the last ten pages are titled "Transition to the Twentieth Century"); and some treatment of mathematics in America.

The format of the book is excellent. The type is clear and well spaced, the diagrams well chosen and legible. Line drawings only have been used. There are no pictures of persons, instruments, or original works, there are no exact, and only a few approximate, excerpts from original sources. Space restrictions may not allow excursions into original sources, but by their omission there is

lost to the reader the pleasure and insight which one can derive from occasionally thinking his way into and through the processes of an original genius as he takes an important step.

In spite of these minor disagreements and suggestions, I would like to reiterate that this book accomplishes its purpose in an excellent fashion. It is not a mere recital of names and dates, but does well in an attempt to stress the growth of ideas and interrelationships between them for readers who are not too advanced or mature mathematically.

P. S. Jones
University of Michigan

Analytic Geometry and Calculus. By L. L. Smail. Appleton-Century-Crofts, Inc., New York, 1953. 84+644 pages. $5.50.

Calculus and Analytic Geometry. By G. B. Thomas, Jr. Addison-Wesley Publishing Company, Inc., Cambridge, 1953. 11+731 pages. $7.50.

Smail's text is designed for liberal arts students and for technical students. Thomas's intention in this regard is not stated explicitly, but his text appears to be designed for the latter. It may be suitable, with certain omissions, also for the former group.

Each introduces integral calculus early, before such topics as conic sections. Each contains the standard topics expected in such a book. Thomas's text, which is the larger, includes also line integrals, differentiation of vectors, scalar and vector product, and a chapter on determinants and linear equations. It contains also introductions to a number of advanced topics, notably uniform continuity, Fourier series, and functions of a complex variable. Both books conclude with a chapter on differential equations. Each has generous lists of problems. In Smail, most of the problems are arranged in pairs, and answers are supplied for the odd-numbered ones. In Thomas, all answers, including those requiring sketches, are given in an unusual forty-nine page answer section. Smail includes a table of integrals and numerical tables. Thomas does not, evidently expecting that students will purchase a set of tables. Each book is attractively printed and almost entirely free of typographical errors. Smail makes good use of bold face. Each has good three dimensional figures.

An important question in any calculus text is how the foundations are treated. In Smail's book, although there is a definition of limit given, the approach may be fairly described as intuitive. In Thomas's book, the initial approach is intuitive, and some calculus is done. Then the author proceeds to a careful development of limits, another chapter of calculus, and then to a thorough discussion of continuity. These notions are explained at length, with examples, and there are problems. The author means business. Both authors, however, made the same slip in this area. Each used without proof the continuity of linear functions. The slip is more serious in Thomas, because he was engaged at the time in proving the continuity of certain other functions. Thomas has taken the unusual step of including a discussion of uniform continuity, and an inter-

esting pedagogical device introduces the possibility that elementary calculus students will understand it. However, this work is to some extent wasted when, later, proofs of two statements about definite integrals are complete except for the unstated invocation of uniform continuity, even though the appropriate theorem (continuity in a closed interval implying uniform continuity) has been stated in the earlier section.

Smail's book is evidently the work of a skillful teacher. A number of topics which are often obscure are treated very lucidly here. Some subjects are treated with more care and precision than is customary. Careful, well-motivated definitions of work, liquid pressure, and the like are instances of this; derivations of equations of loci are others. There are some unusually good sections on infinite series. Since this is a calculus text, the reviewer would like to have seen the integral test for convergence included, but this is a matter of taste. An occasional lack of precise language was noted. The notion of something being true "generally" or "in general" changes in different contexts. At other points students may be led to believe that there is such a thing as an arbitrarily small (or large) number. A number of errors, of two kinds, were found. First there are those which occur in rather difficult topics which very few students will attempt to read thoroughly; an example is the existence proof for the derivative of the inverse function. Errors of the second kind are more serious, since they are at such a level that students may discover or be confused by them; an example of this is the second sentence of Article 236, in which the fact that the quotient of the hyperbola ordinate and a line ordinate is nearly unity is said to imply the approximate equality of the ordinates. The preceding error is not used, a correct discussion of the asymptote being supplied on the next page.

The reviewer believes that while this book has some shortcomings with respect to self-study, its many excellent qualities may make it suitable as a classroom text.

Thomas's book is a real contribution to a field in which textbooks with no essential differences are published every year. The subject matter is standard, with the additions noted above. The treatment is careful, precise, illuminating, and lively. A number of different topics are explained or illustrated in refreshingly original ways. In applications of calculus to such problems as finding the area of a surface of revolution, the question of which approximating sums are close enough to lead to the correct integral is often slighted or ignored. Here it is treated at length in a readable way which can hardly fail to add to the student's confidence when he applies calculus in new situations. The book is sprinkled with interesting and informative remarks. The problem of cutting a length of wire into two pieces, to be bent to form a circle and a square of maximum total area, is one which will give an educational jolt to cocksure students who attack such problems with a minimum of thought.

The treatment of formal integration leans heavily on hyperbolic functions and their inverses. The only form of the answer offered for $\int \sec \theta \, d\theta$ is $\sinh^{-1}(\tan \theta)+C$. In order for the student to be satisfied with this, he will need

the thorough workout with hyperbolic functions given in the preceding chapter. Early in the book, a number of problems give rise to integrals which the student is not prepared to evaluate; possibly this was intentional.

There is a tendency to use terms and symbols which are defined only subsequently, if at all; none of these omissions is serious. Three errors were noted. The first evaluation of the slope at the origin (top of page 220) involves division by zero. The three infinite expansions following (9) on page 584 are in general not valid under the stated hypotheses. The proof of l'Hospital's Theorem has a gap too difficult for a student to fill. At certain other points it was felt that a few more words of proof or discussion were necessary. The reviewer regretted that the precision which characterized the author's presentation of the foundations of elementary calculus was not retained in the discussion of vector and complex calculus.

In general, the standards of both rigor and readability were found to be very high. The reviewer's feeling is that the publication of this book represents a splendid effort to teach calculus more carefully, while simultaneously retaining, and even enhancing, its intuitive appeal.

J. H. BLAU
Antioch College

Ordinary Differential Equations. By R. E. Langer. New York, John Wiley and Sons, Inc., 1954. xii+249 pp. $4.50.

This book gives a direct and concise presentation of the topics usually treated in a beginning course on differential equations. The material on first and second order equations is presented completely, while procedures for equations of higher order are only briefly considered. A summary of special functions, such as the gamma function, Bessel functions, *etc.*, is given in the last chapter. No material on partial differential equations is included. For the most part, the applications of differential equations are presented and discussed in separate chapters, thereby allowing their exclusion for a shorter course of study.

The author has endeavored to unify the material on differential equations in as simple and clear a manner as possible. In order to do this, he has emphasized a few procedures which are not ordinarily stressed in a beginning text. This variation in some of the basic methods will be to most teachers very refreshing. Although the primary effort has been directed toward solving differential equations, the author has carefully defined all terms, and has given rigorous proofs for the existence and uniqueness of solutions to certain types of equations. His treatment of solutions by power series is especially well done.

The format of the book is decidedly excellent, and very few typographical errors were noted. The book contains some 800 problems and the answers are given for the odd-numbered problems. The book is easily adaptable to either a short or long course of study.

T. S. PETERSON
Portland State Extension Center

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

M.I.T. SPECIAL SUMMER PROGRAMS

A two-week special summer program on Mathematical Problems of Communication Theory will be presented at the Massachusetts Institute of Technology from Monday, July 12, through Friday, July 16, 1954.

The course will be under the supervision of the Department of Mathematics. Topics will include time series, Brownian motion, spectra, filtering and prediction theory, computing machines, statistical nature of communication, message and entropy, and coding. Dr. Norbert Wiener, Professor of Mathematics, and Dr. R. M. Fano and Dr. Y. W. Lee, Associate Professors in the Department of Electrical Engineering, will be the principal lecturers.

Also there will be a program in Advanced Coding Techniques for Digital Computers from August 2 to August 6 under the direction of Professor Charles W. Adams. Following this session there will be a program in the Business Applications of Digital Computers from August 16 to August 27 under the direction of Professor Adams.

SUMMER CONFERENCES FOR TEACHERS

The Second Annual Mathematics Workshop will be held by the University of Arkansas during the period June 28 to July 2, 1954. Persons interested in this institute should write to the Director, Professor Davis P. Richardson, Department of Mathematics, University of Arkansas, Fayetteville, Arkansas.

A California Conference for Teachers of Mathematics offered by University of California Extension on the Los Angeles campus of the University is set for July 6 to 16. The fourth annual event of its kind, the conference is sponsored by the University's Department of Mathematics, School of Education, and Mathematics Extension, in cooperation with the California Mathematics Council and the National Council of Teachers of Mathematics. A complete program of the conference will be mailed to those interested on request to the Department of Conferences, University of California Extension, Los Angeles 24.

SUMMER SESSIONS

The following institutions announced advanced courses in mathematics for the summer of 1954:

Columbia University. July 6 to August 13: Professor Harish-Chandra, introduction to higher algebra, algebraic topology; Professor Taylor, differential equations, higher algebra; Professor Murray, probability, theory of functions

of a complex variable; Professor Kolchin, fundamental concepts of mathematics, introduction to differential algebra; Professor Levi, topics in geometry.

The University of Chicago. June 21 to August 28: Professor Schilling, algebra IV (theory of groups, commutative rings); Professor Kaplansky, point set topology, topology I; Professor Stone, introduction to partial differential equations, Boolean algebras; Professor Halmos, topological algebra; Professor Spanier, differentiable manifolds; Dr. Auslander, cohomology theory of groups.

PERSONAL ITEMS

Associate Professor Marguerite Lehr of Bryn Mawr College and Professor I. J. Schoenberg of the University of Pennsylvania were the representatives of the Association at the Annual Meeting of the American Academy of Political and Social Science which was held in Philadelphia, Pennsylvania, on April 2–3, 1954.

The University of California announces the following: Professors Jerzy Neyman and Alfred Tarski have been invited to deliver addresses at the International Congress in Amsterdam in September, 1954; Professor C. B. Morrey, Jr. will be on sabbatical leave for the academic year 1954–55 and plans to spend this year at the Institute for Advanced Study; Professor A. L. Foster will be on sabbatical leave for the first semester of the year 1954–55; Professor E. W. Barankin and Professor M. M. Loève will also be on sabbatical leave during the year 1954–55; Professor Loève plans to spend this year in Europe.

Associate Professor A. C. Eringen of the Illinois Institute of Technology has been appointed to an associate professorship at Purdue University.

Mr. R. R. Hare, Jr. of the Air Force Missile Test Center, Patrick Air Force Base, Florida, has accepted a position as Operations Analyst with the Operations Research Office, The Johns Hopkins University, Chevy Chase, Maryland.

Miss Jane C. Ingersoll, previously a research assistant at the Los Alamos Scientific Laboratory, New Mexico, has accepted a position as Operations Analyst with the Operations Research Office, Chevy Chase, Maryland.

Mrs. Louise C. Lim, who has a Ford Foundation Fellowship for the year 1953–54, is at Radcliffe College during the second semester of this academic year.

Mr. William McKay has been appointed to an instructorship at Drexel Institute of Technology.

Professor W. R. Mann of the University of North Carolina, has been promoted to an associate professorship.

Dr. N. M. Martin, formerly of the University of California, Los Angeles, has a position as Research Associate at the Willow Run Research Center, University of Michigan.

Dr. O. B. Moan of the I.B.M. Corporation has accepted a position as quality control engineer with the Hughes Aircraft Company, Culver City, California.

Dr. J. E. Morton has been appointed Consultant on Industrial Research to

the National Science Foundation; Dr. Morton is on leave of absence from his position as Professor of Statistics at Cornell University.

Dr. R. S. Novosad of Tulane University has been appointed to an assistant professorship at Pennsylvania State College.

Dr. Rufus Oldenburger has been appointed Director of Research at the Woodward Governor Company, Rockford, Illinois.

Mr. P. B. Richards, previously an analytical engineer at the Babcock and Wilcox Research Center, Alliance, Ohio, is now a research associate at Case Institute of Technology.

Mr. Jerome Sherman, formerly an engineer at the Babcock and Wilcox Research Center, Alliance, Ohio, has accepted a position as Senior Engineer with Westinghouse Electric Corporation, Atomic Power Division, Pittsburgh, Pennsylvania.

Mr. D. E. Thoro, recently a graduate assistant at the University of Florida, is employed as a mathematician by the R.C.A. Service Company, Cocoa, Florida.

Reverend H. J. Vandort, previously curate of Grace Episcopal Church, Grand Rapids, Michigan, is now Assistant to the Bishop, Episcopal Diocese of Erie, Erie, Pennsylvania.

Dr. R. E. Zink has been appointed to an instructorship at Purdue University.

Professor Emeritus R. A. Johnson of Brooklyn College died on February 9, 1954. He was a charter member of the Association and formerly an associate editor of this MONTHLY.

Associate Professor R. F. Smith, who had retired from his position at City College of the City of New York, died on January 31, 1954.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

A VISITING LECTURESHIP PROGRAM

With the financial support of the National Science Foundation a visiting lectureship program will be administered by the Mathematical Association of America during the academic year 1954–1955. The object of the program is, by increasing the introduction of modern ideas into under-graduate mathematics,

to assist teachers in colleges and universities to improve the quality and quantity of undergraduate students in mathematics—not only those students who will eventually become research mathematicians but those who will teach it in the secondary schools, apply this subject to other fields, or merely become part of the general informed public.

With this end in view, the lecturers will be prepared not only to give formal lectures but to confer with students and faculty singly and in groups. They will be glad to advise students on future opportunities in study and employment and to discuss with members of the staff teaching problems and curriculum and throw what light they can on practices in comparable institutions. In short, the lecturers will cooperate with the departments in all ways possible toward the furtherance of the aims of the program.

Those wishing further information about the program may write to one of the following members of the committee: Professor G. B. Huff, University of Georgia, Athens, Georgia; Professor Donald E. Richmond, Williams College, Williamstown, Massachusetts; or Professor Burton W. Jones, University of Colorado, Boulder, Colorado.

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 68 persons have been elected to membership by the Board of Governors on applications duly certified.

ISRAEL ABRAMS, M.A. (Pennsylvania) Teacher, Overbrook High School, Philadelphia, Pa.

S. O. ALBERT, Student, American University.

J. E. ASHE, B.S. (Hartwick) Research Engineer, Scintilla, Sidney, N. Y.

R. W. BAGLEY, M.S. (Tulane) Instr., University of Florida.

R. A. BARNETT, M.A. (Southern California) Teaching Assistant, University of Southern California.

S. R. BEYMA, M.S. (M.I.T.) Instr., Hampton Institute.

D. W. BLAKESLEE, M. A. (California) Instr., San Francisco State College.

MRS. LOIS W. BOLAND, B.S. (Stetson) Mathematician, Patrick Air Force Base, Fla.

C. A. BROWN, M.S. (South Carolina) Asst. Professor, The Citadel.

C. J. COHEN, Ph.D. (Johns Hopkins) Mathematician, Naval Proving Ground, Dahlgren, Va.

HELEN F. CULLEN, Ph.D. (Michigan) Asst. Professor, University of Massachusetts.

R. E. DOUTT, B.S. (Grove City) Asso. Professor, South Dakota School of Mines and Technology.

G. B. GLOVER, B.S. (U.S. Military Academy) Instr., Rutgers College of South Jersey.

BASIL GORDON, M.A. (Johns Hopkins) Instr., Johns Hopkins University.

MRS. ELEANOR P. GUYER, B.A. (Hanover) Head of Department, Southport High School, Indianapolis, Ind.

A. J. HALL, Ed.D. (Stanford) Asso. Professor, San Francisco State College.

J. J. HARRIS, B.A. (Augustana) Manhattan, Kan.

PAUL HEINS, Ph.G. (Mass. Coll. of Pharmacy) Pharmacist, Thrifty Drug Co., Los Angeles, Calif.

YVONNE G. HENRION, Student, University of British Columbia.

K. L. HILLAM, B.S. (Utah) Grad. Assistant, University of Utah.

S. S. HOLLAND, JR., M.S. (Chicago) Mathematician, Army Chemical Center, Md.

J. P. HOYT, Ph.D. (George Washington) Asso. Professor, United States Military Academy.

A. ROBERTA KIEWIT, M.S. (Omaha) Chair-

man of Department, Benson High School, Omaha, Neb.

H. S. KRAMER, B.S. (U. of Washington) Analytical Statistician, United States Air Force Headquarters, Washington, D. C.

HAROLD W. KUHN, Ph.D. (Princeton) Asst. Professor, Bryn Mawr College.

H. N. LADEN, Ph.D. (Pennsylvania) Research Officer, Chesapeake & Ohio Railway Co., Cleveland, Ohio.

R. A. LAIRD, B.S.C.E. (Georgia) Lt. Col., United States Army (Retired), New Orleans, La.

LORRAINE D. LAVALLEE, B.A. (Mount Holyoke) Instr., University of Massachusetts.

R. J. LAWTHER, B.A. (Penn. State) Asso. Engineer, Douglas Aircraft Co., Los Angeles, Calif.

J. A. LECHNER, Student, Carnegie Institute of Technology.

A. B. LEHMAN, B.S. (Ohio U.) Tulane University.

C. H. LEWIS, M.S. (C.I.T.) Chairman, Division of Science and Mathematics, Orange Coast College.

H. M. LINNETTE, M.S. (Michigan) Asso. Professor, Virginia State College.

C. P. LUEHR, B.S. (Oregon S.C.) Grad. Student, Oregon State College.

MRS. DOROTHY C. MARTIN, M.A. (Peabody) Instr., Wood Junior College.

N. M. MARTIN, Ph.D. (U.C.L.A.) Research Associate, Willow Run Research Center, University of Michigan.

R. M. MASON, M.A. (Toledo) Mathematician, Naval Research Laboratory, Washington, D. C.

T. B. MATTSON, Student, Carnegie Institute of Technology.

JOHN MCCARTHY, Ph.D. (Princeton) Acting Asst. Professor, Stanford University.

J. D. MCKNIGHT, JR., Ph.D. (Purdue) Senior Aerophysics Engineer, Consolidated Vultee Aircraft Corp., Fort Worth, Tex.

L. I. MISHOE, Ph.D. (N.Y.U.) Asso. Professor of Physics, Morgan State College.

O. B. MOAN, Ph.D. (Purdue) Quality Control Staff Engineer, Hughes Aircraft Co., Culver City, Calif.

MICHAEL MONTALBANO, B.A. (George Washington) Chief, Programming Section, Computer Control Co., Point Mugu, Calif.

R. A. MORELAND, JR., B.S. (Texas Tech.) Teaching Fellow, Texas Technical College.

D. B. MUMFORD, Student, Harvard University.

S. A. NILES, B.S. (St. Francis) Instr., St. Francis College.

R. H. OWENS, Ph.D. (C.I.T.) Physical Science Coordinator, Office of Naval Research, Pasadena, Calif.

J. A. PAINTER, B.S. (Pittsburgh) Grad. Assistant, University of Pittsburgh.

E. A. PETERS, B.S. (Ohio State) 2nd Lt., United States Army.

W. W. PROCTOR, M.A. (Pennsylvania) Asst. Professor, Morgan State College.

MARGUERITE B. ROSTON, M.A. (Radcliffe) Asst. Professor, Hood College.

ELLEN A. SANDERS, M.A. (Louisville) Teacher, Taylor County High School, Campbellsville, Ky.

O. P. SANDERS, M.S. (Oklahoma A. & M.) Asst. Professor, Southeastern State College.

W. M. SANDERS, M.S. (Arkansas) Instr., Mississippi Southern College.

D. E. SANDERSON, Ph.D. (Wisconsin) Instr., Iowa State College.

SISTER M. BIBIANA, B.A. (Michigan) Instr., College of St. Teresa.

GERALDINE D. SMITH, B.A. (Bates) Editor of Mathematics Texts, Ginn and Co., Boston, Mass.

L. C. SMITH, Student, Hofstra College.

W. A. SOPER, JR., M.A. (Boston U.) Asso. Engineer, Westinghouse Electric Corp., Baltimore, Md.

G. L. SPENCER, II, Ph.D. (Michigan) Asst. Professor, University of Maryland.

GEORGE STEPHENSON, B.A. (Princeton) Research Associate, George Washington University.

J. G. SUTTON, M.A. (Harvard) Science Editor, The Macmillan Co., New York, N. Y.

REV. B. A. TONNAR, S.J., M.A. (Catholic) Asst. Prof., Loyola University, New Orleans, La.

G. D. WALDMAN, Student, Trinity College.

W. A. WHITTLESEY, II, Student, University of Oklahoma.

IZAAK WIRSZUP, M.A. (Wilno) Asst. Professor, University of Chicago.

E. J. ZINDEL, Student, Seton Hall University.

R. E. ZINK, Ph.D. (Minnesota) Instr., Purdue University.

THE JANUARY MEETING OF THE NORTHERN CALIFORNIA SECTION

The sixteenth annual meeting of the Northern California Section of the Mathematical Association of America was held at the University of San Francisco, on January 16, 1954. Professor Roy Dubisch, Chairman of the Section, presided at both the morning and afternoon sessions.

There were eighty-five persons present, including the following fifty-four members of the Association:

H. L. Alder, H. M. Bacon, G. A. Baker, T. J. Bass, Jr., E. M. Beesley, Alice K. Bell, M. T. Bird, R. L. Blair, W. E. Bleick, A. C. Burdette, L. P. Burton, J. R. Byrne, R. C. Campbell, Randolph Church, Roy Dubisch, Hazel E. Eggett, F. D. Faulkner, Ruth A. Fish, Harley Flanders, S. A. Francis, C. M. Fulton, L. C. Graue, W. H. Gregory, C. A. Hayes, Jr., J. G. Herriot, Marjorie L. Hoffman, V. E. Hoggatt, Jr., Vern James, Free Jamison, Walter Jennings, R. M. Lakness, Milton Lees, B. J. Lockhart, A. R. Lovaglia, Sophia L. McDonald, R. B. Merkel, A. B. Mewborn, H. S. Moredock, Jr., F. R. Morris, W. H. Myers, C. D. Olds, C. L. Perry, Jr., J. P. Pierce, F. M. Pulliam, E. P. Rahn, C. H. Rawlins, Jr., R. M. Robinson, E. B. Roessler, Mary V. Sunseri, Irving Sussman, Gabor Szegö, C. C. Torrance, H. G. Tucker, K. J. Waider.

At the business meeting the following officers were elected for the coming year: Chairman, Professor J. G. Herriot, Stanford University; Vice-Chairman, Professor C. C. Torrance, United States Naval Postgraduate School, Monterey; Secretary-Treasurer, Professor C. D. Olds, San Jose State College.

Following a recommendation of the Board of Governors, action was taken to make the Sectional Governor a member of the Executive Committee.

By invitation of the Section, Professor G. E. Latta of Stanford University delivered an address at the morning session entitled *Singular Problems for Differential Equations.* Abstract of this address follows:

A singular problem for differential equations is defined as a boundary value problem for a differential system, whose solution depends non-uniformly on a small parameter. Such a problem occurs, for example, when the parameter is the coefficient of the highest order derivatives appearing in the equation. The problem of approximating the solution of the system in terms of the lower order system obtained by neglecting the small parameter is discussed, and asymptotic expansions are obtained in a number of cases, valid for sufficiently small parameter values. Examples are drawn from hydrodynamics, electric circuit theory, and eigenvalue problems, illustrating a heuristic approach; shock waves and boundary layer theory are mentioned as special cases.

The following papers were presented:

1. *Some remarks on the Lindberg case,* by Professor Free Jamison, San Jose State College.

Some extension of the results of the Lindberg trisections (this MONTHLY, vol. 61, pp. 334–336) were given. The method can be used for the approximate division of an angle by any positive integer greater than 3. In particular, the error in approximating an angle of one degree by the use of only 6 lines, 11 arcs, and 2 compass settings is less than two one-hundredths of one second. Remarks included comment on the wide appeal of the problem, the unsophisticated approach, desirability of study of impossible conditions at different school levels, and applicable principles of learning and counseling.

2. *Differentiation of logarithms,* by Professor C. M. Fulton, University of California, Davis.

Two methods are currently used to present the logarithmic functions in calculus. The first one introduces e as a limit. The second approach defines the natural logarithm by means of an integral. Instead of taking the existence of that limit for granted, one can make the equivalent assumption that a logarithm is differentiable. Then well-known algebraic properties of logarithms make it possible to find their derivatives.

3. *Some models and demonstrators for engineering mathematics*, by Professor A. B. Mewborn, United States Naval Postgraduate School, Monterey.

Models developed by the Staff of the Naval Postgraduate School in collaboration with the Special Devices Center of the Office of Naval Research, and built in the shops of the latter activity were shown and explained briefly. Among those discussed were: a $3\times$ linear enlarged Amsler polar planimeter, plastic models illustrating 3-dimensional coordinate systems and vector relationships involving dot and cross products, a wire and plastic model for illustrating the proof and properties of Stokes' theorem, and 3-dimensional models showing the relationship of skew lines in space.

4. *A simple minimum problem*, by Professor F. R. Morris, Fresno State College.

One or more circles are in a plane. They may be tangent but do not overlap. The diameter of each circle is 1 unit. The perimeter is defined as the length of the line composed of tangents and arcs of circles which surround the group of circles. The problem is to arrange the circles in order to have a minimum perimeter and to find its value.

5. *An undergraduate mathematics seminar*, by Professor E. J. Farrell, University of San Francisco, introduced by the Secretary.

A brief report on a seminar offered to upper division mathematics majors, and others with necessary prerequisites, by the University of San Francisco. The objectives were stated, contents of typical seminars shown, and the various types of students enrolling in the course were discussed.

6. *An introduction to determinants*, by Professor C. H. Rawlins, Jr., United States Naval Postgraduate School, Monterey.

Professor Rawlins discussed the traditional approach to the subject of determinants, as found in most of the text and reference books, and suggested that the determinant be introduced in a way that requires no mention of permutations and inversions. He proposed that it be defined as equal to its expansion by minors according to its first row. With this start he derived proofs for the standard elementary theorems. In the proofs, mathematical induction was employed somewhat less than in some previous treatments of the same subject.

7. *An isoperimetric inequality*, by Professor Robert Weinstock, Stanford University, introduced by the Secretary.

Let C be a simple closed plane curve having continuous curvature and bounding a convex domain D. Let A be the area of D, L the length of C, and J the polar moment of inertia of C with respect to its centroid. (For the computation of J it is assumed that a uniform distribution of total mass L is associated with C.) With the use of tangential coordinates it is shown that the inequality $LA \leqq \pi J$—not generally valid for nonconvex domains—holds; equality obtains only if C is a circle.

C. D. OLDS, *Secretary*

6. *Compact segmental determinants*, by Mr. C. W. Trigg, Los Angeles City College.

From a column of a rectangular array a p, u, n-segment is chosen by taking every uth element, beginning with the element in the pth row, until n have been selected. An array, A, with $a_{1,j}=a_j$, $a_{i,1}=a_1k^{i-1}$, $a_{i,j+1}=ma_{i,j}+ka_{i-1,j+1}$ is a generalized Pascal triangle. The p, u, n-segments from the first n columns of A form a compact equipatterned segmented determinant. The value of this determinant,

$$D = a_1^n(um)^{n(n-1)/2}k^x, \qquad \text{where} \qquad x = un(n-1)/2 - n + \sum_{j=1}^{n} p_j,$$

is independent of the a_j, $j>1$. For $k=\pm 1$, $|D|$ is independent of the p_j. Among the other special cases, $a_j=m=k=1$ gives the Pascal arithmetical triangle for which $|D|=1$ when $u=1$.

7. *Report on a conference for teachers of collegiate mathematics*, by Professor May M. Beenken, Immaculate Heart College.

Professor Beenken summarized the work of the eight weeks conference on collegiate mathematics held in Boulder, Colorado last summer under the sponsorship of the National Science Foundation. She included suggestions for improving the undergraduate mathematics major and reported on what some colleges are doing to modernize mathematics on the freshman level through the introduction of modern postulational concepts and methods in pre-calculus courses.

P. H. DAUS, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirty-fifth Summer Meeting, University of Wyoming, Laramie, Wyoming, August 30–31, 1954.

Thirty-eighth Annual Meeting, University of Pittsburgh, Pittsburgh, Pennsylvania, December 30, 1954.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS
INDIANA
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA
NORTHERN CALIFORNIA
OHIO
OKLAHOMA, Oklahoma City University, October 29, 1954.
PACIFIC NORTHWEST, Reed College, Portland, Oregon, June 18, 1954.
PHILADELPHIA, Princeton University, Princeton, New Jersey, November 27, 1954.
ROCKY MOUNTAIN
SOUTHEASTERN, Tennessee Polytechnic Institute, Cookeville, March 11–12, 1955.
SOUTHERN CALIFORNIA, Santa Monica City College, March 12, 1955.
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE
WISCONSIN

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 61 NUMBER 7

PART I

CONTENTS

AUGUST–SEPTEMBER 1954

The AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

EDITORIAL CORRESPONDENCE should be addressed to the Editor, C. B. ALLENDOERFER, Department of Mathematics, University of Washington, Seattle 5, Washington.

ADVERTISING CORRESPONDENCE should be addressed to J. F. RANDOLPH, Morey Hall, University of Rochester, Rochester 3, N. Y.

NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the MONTHLY should be sent to the Secretary-Treasurer, H. M. GEHMAN, University of Buffalo, Buffalo 14, N. Y.

THIS IS THE OFFICIAL JOURNAL OF THE

MATHEMATICAL ASSOCIATION OF AMERICA, INC.

(Devoted to the Interests of Collegiate Mathematics)

Entered as second class matter at the post office at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

SUBSCRIPTION PRICE: To Members, $4 a Year, To Others, $5 a Year.

PUBLISHED BY THE ASSOCIATION at Menasha, Wisconsin, and Seattle, Wash. during the months of January, February, March, April, May, June–July, August–September, October, November, December.

PRINTED IN THE UNITED STATES OF AMERICA

RECENT APPLICATIONS OF CONVEX FUNCTIONS*

J. W. GREEN, University of California at Los Angeles

1. Introduction. A comprehensive account of the history, properties, and applications of convex functions up to 1946 has been given by E. F. Beckenbach [1]. However, even in the few years since that article appeared, convex functions have been so consistently of value in analysis, geometry, and other branches of mathematics, notably mathematical economics, that a considerable wealth of material exists for a supplementary article. In the following I will report on some of this material. Coverage of the field will not be exhaustive; the choice of subject matter was determined by my own fields of interest; namely, convex functions *per se*, and their generalizations, geometry of convex figures, function theory, and (through various logistics and programming projects) linear and non-linear programming.

2. Convex functions and generalizations. We recall that a convex function f is one which satisfies the inequality

$$f(\lambda \mathbf{x} + (1 - \lambda)\mathbf{y}) \leqq \lambda f(\mathbf{x}) + (1 - \lambda)f(\mathbf{y})$$

for all λ such that $0 \leqq \lambda \leqq 1$. The variables $\mathbf{x}$ and $\mathbf{y}$ may be regarded as vectors in any linear vector space, which in this article will be taken to be Euclidean. This inequality states that between any two points of the graph of f, the graph lies nowhere above the chord joining those two points. For elementary properties of convex functions, see [1] and [2].

Now, as far as the study of properties of convex functions, regarded as functions of one or more real variables, is concerned, there has not been a great deal of activity of late. This is due perhaps in one dimension to a shortage of unsolved problems and in more dimensions to a shortage of solvable ones. Here we shall describe several generalizations of convex functions which seem interesting or useful.

First, let us stay in one dimension and suppose that there is a family $\mathcal{J}$ of continuous functions F such that through any two points (x_1, y_1) and (x_2, y_2) with $x_1 \neq x_2$ we may interpolate exactly one function F of $\mathcal{J}$. A given function f is said to be convex relative to $\mathcal{J}$, or sub-F, provided it agrees with F at x_1 and x_2, it lies nowhere above F for x between x_1 and x_2. It is clear that the functions F play the role which the linear functions play in the ordinary convexity. The sub-F functions were studied extensively by Beckenbach and Bing [3] and were reported on in [1]. However in the recent past, a number of papers [4, 5, 6] have appeared concerning these functions, particularly in connection with

* This article was originally prepared to be presented at the Christmas, 1952, meeting of the Association in St. Louis. However, an overly convex appendix on Christmas eve made its presentation in person impossible.

their smoothness and convergence properties. For example if the F are sufficiently well-behaved, the sub-F functions have very nearly the same differentiability properties as the convex functions—unilateral derivatives exist everywhere, derivatives everywhere with countable exceptions, second derivatives almost everywhere. A convergent sequence of sub-F functions converges uniformly, and the sequence of derivatives converges boundedly.

Now the sub-F functions, instead of being only generalizations for generalization's sake, have proved to be applicable to a number of problems. In fact, a particular case of them, the sub-trigonometric functions for which $\mathcal{J}$ consists of the functions $A \cos x+B \sin x$, were introduced by Phragmén and Lindelöf and used by Pólya in connection with problems in function theory. Pólya [7] proved that a function $p(\theta)$ of period 2π is the support function of a convex area in the plane if and only if it is sub-trigonometric. The present author rediscovered this fact [8] and made use of it in several geometrical problems (see §3 below). In [8], proper acknowledgement is not made to Pólya, and the author wishes to make it here. In [6], by use of the differentiability and convergence properties mentioned above, a simple proof is given of the known but apparently not easily accessible fact that the area of the general convex curve is given by $\frac{1}{2}\int_0^{2\pi} (p^2-p'^2)d\theta$ without any smoothness assumptions.

Another and novel generalization of convex functions has been given by Hyers and Ulam [9]. They call a function approximately convex, or ϵ-convex provided it satisfies the inequality

$$f(\lambda \mathbf{x} + (1 - \lambda)\mathbf{y}) \leqq \lambda f(\mathbf{x}) + (1 - \lambda)f(\mathbf{y}) + \epsilon,$$

where $\epsilon>0$ and $\mathbf{x}$, $\mathbf{y}$ are vectors in n-space. The chief theorem about an ϵ-convex function is that it is approximately equal to a convex function. More precisely, there is a convex function g such that $g\leqq f\leqq g+k_n\epsilon$, where k_n depends only on the dimension n. The best value for k_n has not been determined for all n. For $n=1$, it is 1, and it is easily seen that in one dimension, the most general ϵ-convex function is obtained by adding to an arbitrary convex function an arbitrary function whose values lie between 0 and ϵ. In [9], the value $k_n=(n^2+3n)/(2n+2)$ is given, which is the best value for $n=1$, 2, but not beyond. The present author [10] has obtained an improved value of k_n asymptotic to $\log_2 n$; this value is the best possible through $n=3$ but beyond that the question is open.

A third type of function akin to convex and studied of late is the sub-additive function, satisfying the inequality $f(x+y)\leqq f(x)+f(y)$. Rosenbaum [11] has studied these functions systematically. If f is sub-additive and homogeneous, it is convex; if f is convex for $x\geqq 0$ and $f(0)\geqq 0$, it is sub-additive. These functions have many properties analogous to those of convex functions, but behave quite differently as far as continuity properties are concerned. For example, there exist measurable everywhere discontinuous sub-additive functions.

3. Geometry. Each year dozens of papers on the geometry of convex bodies are written, and these will usually involve convex functions, directly or indirectly. However here are a few problems where the analysis of functions is used to produce geometric results.

In [8] the following problem was considered: Let K be a convex curve lying interior to a circle C. What can be said about K if it subtends at each point of C the same angle α? It was shown that if $\pi-\alpha$ is an irrational multiple of π or is of the form $(m/n)\pi$ with m even and prime to n, then K must be a circle concentric to C; in other cases—for example $\alpha=\pi/2$—there is a considerable variety of curves K.* The author used the sub-trigonometric characterization of the supporting function $p(\theta)$ of K to solve several external problems involving perimeter, area, diameter, and width of K for those α where there is any problem. Since, for example, the length is given by $\int_0^{2\pi} p(\theta)d\theta$, it is not surprising that this was possible. A typical result is that the maximum length of K is attained by a figure consisting of n elliptical arcs pieced together with n corners of angle π/n.

The same characterization also proved useful in connection with another geometrical problem. Let K be a convex curve, and let the plane be subjected to an area-preserving affine transformation T so as to minimize the length of TK. In [12] the present author showed that K minimizes the length if and only if the Fourier coefficients a_2 and b_2 of $p(\theta)$ vanish. The problem was then posed of finding the curve of area 1 which when transformed so as to minimize the length gives the greatest length. The partial result that the extreme figure must be a polygon of five or fewer sides was obtained by using variations of $p(\theta)$ consistent with its sub-trigonometric nature and with the fact that $a_2=b_2=0$. The fact that it must be a triangle was subsequently proved by Gustin [13].

It is interesting that new proofs of geometric theorems have been obtained by methods originally devised in connection with mathematical economics. Karlin and Shapley [14] have employed the following theorem, due to Bohnenblust and themselves [15]: Let $\phi_\alpha(\mathbf{x})$ be a family of continuous convex functions in a compact convex set Δ in n-space with $\inf_{x\in\Delta}\sup_\alpha \phi_\alpha(\mathbf{x})>0$. Then there exists a linear combination of $n+1$ of the ϕ_α with non-negative coefficients which is positive in Δ. Using this fact, Karlin and Shapley gave a new proof of Helly's theorem and also of the theorem that if a spherical surface in n-space is covered by a compact set of hemispherical sub-surfaces, then it is covered by a certain $n+1$ of these hemispheres.

4. Function theory. E. F. Beckenbach [16; references to related problems may be found therein] has recently obtained an interesting result relating the mean values of the modulus of a function analytic within a unit circle along radii and along concentric circles. Let

* It is curious that Santaló [20] has shown that if the very same problem is considered on the surface of a sphere instead of in the plane, one is led to the conclusion that K is in all cases a circle, no matter what α is.

$$\nu(\phi) = \int_0^1 |f(re^{i\phi})|\, dr, \qquad \mu(r) = \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\phi})|\, d\phi.$$

Then if $\nu(\phi) \leqq 1$, (or even if the average of ν is $\leqq 1$) then $\mu(r) \leqq |f(0)| + 2r(1-|f(0)|)$ for $r \leqq \frac{1}{2}$ and $\mu(r) \leqq |f(0)| + (1-|f(0)|)/2(1-r)$ for $\frac{1}{2} \leqq r \leqq 1$. This result followed from the fact that $\mu(r)$ is a convex function and $\int_0^{2\pi} \mu(r)dr \leqq 1$. These two facts, without further reference to analytic function theory, imply the stated inequalities on μ. It seems quite remarkable that inequalities such as these can be achieved so directly by consideration of simple geometric properties of convex functions.

5. **Programming.** The typical problem of linear programming is that of minimizing a linear function $g(\mathbf{x}) = \sum_1^n c_i x_i$ of n variables $x_1, x_2, \cdots, x_n$, subject to a certain set of m linear constraints, $f_j(\mathbf{x}) \equiv b_j - \sum_{i=1}^n a_{ji}x_i \geqq 0$, $j = 1, 2, \cdots, m$ ($\mathbf{x}$ means the vector with components $x_1, \cdots, x_n$) and also the n special linear constraints $x_i \geqq 0$. A problem readily cast in this form is that of minimizing the cost g of a diet composed of n foodstuffs consumed in quantities $x_1, \cdots, x_n$ and containing m basic dietary ingredients. The inequalities $f_j \geqq 0$ correspond to stating minimum requirements of the various ingredients. The interpretation of $x_i \geqq 0$ is clear.

The above problem is that of determining the maximum of a linear (and thus convex) function in a convex polyhedron. In fact, it amounts to finding the distance from the origin to the supporting plane to this polyhedron normal to the vector $(c_1, c_2, \cdots, c_n)$. It may be stated in the form of a minimax or saddle value problem by the introduction of Lagrangian multipliers as follows [**17**]. First define $\phi(\mathbf{x}, \boldsymbol{\lambda}) = g(\mathbf{x}) + \sum_1^m \lambda_j f_j(\mathbf{x})$. A vector $\mathbf{x}^0$ yields a solution of the maximum problem if and only if there exists a vector $\boldsymbol{\lambda}^0$ with nonnegative components such that

$$\phi(\mathbf{x}, \boldsymbol{\lambda}^0) \leqq \phi(\mathbf{x}^0, \boldsymbol{\lambda}^0) \leqq \phi(\mathbf{x}^0, \boldsymbol{\lambda})$$

for $\lambda_j \geqq 0$, $x_i \geqq 0$; that is, if $(\mathbf{x}^0, \boldsymbol{\lambda}^0)$ is a saddle point of the "surface" $z = \phi(\mathbf{x}, \boldsymbol{\lambda})$.

Related to the above problem is the dual problem; namely that of determining the minimum of $\sum_1^m b_j\lambda_j$ subject to the constraints $\lambda_j \geqq 0$, $\sum_1^m a_{ji}\lambda_j \geqq c_i$. One sees immediately that this problem leads to identically the same saddle problem as before, and indeed the two extremum problems are equivalent; if one has a solution, the other does, and the one minimum equals the other maximum.

Fenchel [**18, 19**], dropping linearity entirely, has made a very broad generalization of these notions, based upon a certain interesting conjugate relation between convex functions. This conjugate relation depends on the pole-polar relation with respect to the quadric

$$2z = x_1^2 + x_2^2 + \cdots + x_n^2 = \mathbf{x}\cdot\mathbf{x}$$

in $(n+1)$ space.

Let f be a convex function of $\mathbf{x}=(x_1, \cdots, x_n)$ in a convex set D. The polar plane of a point $(\mathbf{x}, z)$ is given by

$$z + \zeta = \mathbf{x}\cdot\xi,$$

where $(\xi, \zeta)=((\xi_1, \cdots, \xi_n), \zeta)$ are coordinates in another $(n+1)$ space in which it is convenient to imagine the polar plane. Then the function $\phi(\xi)$ conjugate to $f(\mathbf{x})$ is determined as the upper envelope of these polar planes as $(\mathbf{x}, z)$ varies over the graph $z=f(\mathbf{x})$; that is,

$$\phi(\xi) = \sup_{\mathbf{x}\in D} (\mathbf{x}\cdot\xi - f(\mathbf{x})).$$

The domain Δ of definition of ϕ is those points where the sup is finite. Now Fenchel shows (under certain continuity conditions on f) that Δ is a convex set and ϕ a convex function; furthermore the conjugate of ϕ is f.

Now let f be convex in C and g concave in D, where C and D are convex sets. Let ϕ in Γ and ψ in Δ be the conjugate functions of f and g respectively ($-\psi$ is conjugate to $-g$). Then the following two extremal problems are equivalent: (a) maximize $g(x)-f(x)$ in $C\cap D$; (b) minimize $\phi(\xi)-\psi(\xi)$ in $\Gamma\cap\Delta$. Either both or neither of the problems have solutions, and the one maximum equals the other minimum. In case $g(\mathbf{x})=\sum c_i x_i$ in the positive orthant D: $x_i\geqq 0$, and $f(\mathbf{x})=0$ in the convex region C determined by the inequalities $\sum a_{ji}x_i\geqq b_j$, this pair of problems reduces to the dual linear programming problems mentioned earlier.

It would appear that this duality should be capable of further exploitation. Incidentally, Fenchel [**19**] shows how it can be used to prove many of the classical inequalities—Hölder's, *etc.*

References

1. E. F. Beckenbach, Convex functions, Bull. Amer. Math. Soc., vol. 54, 1948, pp. 439–460.

2. T. Bonnesen and W. Fenchel, Theorie der konvexen Körper, Berlin, Springer, 1934.

3. E. F. Beckenbach and R. H. Bing, On generalized convex functions, Trans. Amer. Math. Soc., vol. 58, 1945, pp. 220–230.

4. M. Matos-Peixoto, On the existence of derivatives of generalized convex functions, Summa Brasil Math., vol. 2, no. 3, 1948, pp. 35–42.

5. F. F. Bonsall, The characterization of generalized convex functions, Quart. J. Math., Oxford Ser. 2, vol. 1, 1950, pp. 100–111.

6. J. W. Green, Support, differentiability, and convergence properties of generalized convex functions. Proc. Amer. Math. Soc., vol. 4, 1953, pp. 391–396.

7. G. Pólya, Untersuchungen über Lücken und Singularitaten von Potenzreihen. Math. Zeit., vol. 29, 1929, pp. 549–560.

8. J. W. Green, Sets subtending a constant angle on a circle, Duke Math. J., vol. 17, 1950, pp. 263–267.

9. D. H. Hyers and S. N. Ulam, Approximately convex functions, Proc. Amer. Math. Soc., vol. 3, 1952, pp. 821–828.

10. J. W. Green, Approximately convex functions, Duke Math. J., vol. 19, 1952, pp. 499–504.

11. R. A. Rosenbaum, Sub-additive functions, Duke Math. J., vol. 17, 1950, pp. 227–247.

12. J. W. Green, Length and area of a convex curve under area-preserving affine transformation, Pac. J. Math., vol. 3, 1953, pp. 393–402.

13. W. S. Gustin, An isoperimetric minimax, Pac. J. Math., vol. 3, 1953, pp. 403–405.
14. S. Karlin and L. S. Shapley, Some applications of a theorem on convex functions, Ann. of Math., 2, vol. 52, 1950, pp. 148–153.
15. H. F. Bohnenblust, S. Karlin, and L. S. Shapley, Games with continuous, convex pay-off, Annals of Mathematics Studies 24, 1950, pp. 181–192. Princeton University Press.
16. E. F. Beckenbach, A property of mean values of an analytic function, Rend. Circ. Mat. Palermo II, vol. 1, 1952, pp. 1–7.
17. H. W. Kuhn and A. W. Tucker, Nonlinear programming, Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, pp. 481–492. University of California Press, 1951.
18. W. Fenchel, On conjugate convex functions, Canadian J. Math., vol. 1, 1949, pp. 73–77.
19. W. Fenchel, Convex Cones, Sets, and Functions, Logistics Project Report (multilith), Department of Mathematics, Princeton University, 1953.
20. L. A. Santaló, Two characteristic properties of circles on a spherical surface, Math. Notae, vol. 11, 1951, pp. 73–78.

INVERSE AND COMPLEMENTARY SEQUENCES OF NATURAL NUMBERS

J. LAMBEK, McGill University, and L. MOSER, University of Alberta*

1. In 1926, S. Beatty** proposed the following problem: If x is a positive irrational number, the sequences $m(1+x)$ and $n(1+x^{-1})$ jointly contain one and only one term between each pair of consecutive positive integers. (Here and elsewhere in this paper, m and n range over the set of positive integers.) This was proved by A. Ostrowski and A. C. Aitken; the latter obtained a similar result for rational x.

We shall call two sets of positive integers *complementary* if they have no common elements and together exhaust all positive integers. As usual, let $[x]$ denote the greatest integer in x. Then another way of stating the above result is to say that the sets of positive integers $[xm]+m$ and $[x^{-1}n]+n$ are complementary.

This problem has re-appeared in the literature at various times‡, in particular its connection with Wythoff's game§ has been noted. It seems however that no really wide generalization has been given, in spite of the fact that other such pairs of complementary sequences are easily constructed. Thus $[e^m]+m$

* This paper was written jointly at the Summer Research Institute of the Canadian Mathematical Congress.

** This MONTHLY, problem 3173, vol. 33, 1926, p. 159.

‡ J. V. Uspensky and M. A. Heaslet, Elementary Number Theory, New York, 1939, page 98. See also this MONTHLY, problems 4270, vol. 54, 1947, p. 549, and 4399, vol. 57, 1950, p. 343.

§ W. W. Rouse Ball and H. S. M. Coxeter, Mathematical Recreations and Essays, London, 1944, pp. 38–39. See also D. B. Sawyer, A Property of the Golden Section, Eureka, March, 1948.

and $[\log n]+n$ are complementary, and so are p_m+m and $\pi(n-1)+n$, where p_m denotes the m-th prime, and $\pi(n)$ the number of primes not exceeding n.

2. In what follows it will be convenient to use the term *number* to denote a non-negative integer or infinity. If $f=\{f(1), f(2), \cdots\}$ is a non-decreasing sequence of numbers, it must be of one of three types:

(A) $f(n)$ is finite for a finite number of n only,

(B) $f(n)$ is finite for all n and constant from a certain n on,

(C) $f(n)$ is finite for all n but tends to infinity with n.

We then define

$$f^\dagger(n) = \text{number of } m \text{ such that } f(m) < n. \tag{1}$$

It is easily verified that f is of type A, B, or C if and only if $f^\dagger$ is of type B, A, or C respectively.

Two sequences f and g of numbers will be called *inverses* provided, for all pairs of positive integers m, n,

$$f(m) < n \text{ or } g(n) < m, \text{ but not both.} \tag{2}$$

Note that the inequality $f(m)<n$ is false when $f(m)$ is infinite. We shall prove

THEOREM 1. *A sequence of numbers has an inverse if and only if it is non-decreasing. The inverse of f is unique and coincides with $f^\dagger$.*

Proof. Assume f has an inverse g. If g' were another inverse, (2) would imply that $g(n)<m$ if and only if $g'(n)<m$, for all positive integers m, n. If $g(n)$ is finite, take $m=g(n)+1$. Then $g'(n)<g(n)+1$ so that $g'(n)\leqq g(n)$ is also finite. Similarly, if $g'(n)$ is finite, then $g(n)\leqq g'(n)$. Hence, in any case, $g(n)=g'(n)$. Thus g is unique.

Next we wish to show that $f(m+1)\nless f(m)$. Without loss in generality we may assume that $f(m+1)$ is finite, say $f(m+1)=n-1<n$. Then, by (2), $g(n)\nless m+1$, and *a fortiori* $g(n)\nless m$. Again by (2), $f(m)<n$, so that $f(m)\leqq n-1 =f(m+1)$. Thus f is non-decreasing.

Conversely let f be non-decreasing, so that $f^\dagger$ is defined by (1). Consider the inequality $f^\dagger(n)<m$. According to (1), this means that the inequality $f(x)<n$ has fewer than m solutions for x. Since f is non-decreasing, $x=m$ is a solution if and only if all positive integers $x\leqq m$ are solutions, that is, if and only if the number of solutions is m or more. Thus $f^\dagger(n)<m$ if and only if $f(m)\nless n$. In view of (2), $f^\dagger$ is inverse to f.

This completes the proof of theorem 1. We may remark in passing that $(f^\dagger)^\dagger=f$, as follows immediately from theorem 1 and the symmetry of definition (2). We shall consider some examples of pairs of inverse sequences.

Example 1. $f(m)=p_m$, $f^\dagger(n)=\pi(n-1)$. In general, if $f(m)=m$-th largest positive integer with property P, or $f(m)=$infinity if fewer than m positive integers have property P, then $f^\dagger(n)=$number of positive integers less than n with property P.

Example 2. $f(m)=[e^m]$, $f^\dagger(n)=[\log n]$; $f(m)=m^2$, $f^\dagger(n)=[\sqrt{n-1}]$. In general, let $\phi(x)\geqq 0$ be any real-valued strictly increasing function defined for all real $x\geqq 0$, $f(m)=[\phi(m)]$, then $f^\dagger(n)=$number of positive integers less than $\phi^{-1}(n)$. (If $\phi^{-1}(n)$ is undefined, then $f^\dagger(n)=0$.)

Example 3. $\pi(m^2)$ and $[\sqrt{p_n}]$ are inverses. In general, if f and g are non-decreasing sequences of positive integers, then $f^\dagger(g(m))$ and $g^\dagger(f(n)+1)$ are inverses.

3. If F is a set of positive integers, let $F(m)$ denote the m-th largest member of F, as long as m does not exceed the number of elements of F, otherwise let $F(m)$ denote infinity. The equation $F(m)=f(m)+m$ establishes a one to one correspondence between all non-decreasing sequences f of numbers and all sets F of positive integers, as is easily verified. (It is understood that infinity plus or minus a finite number is infinity.) This correspondence will be exploited in the following

THEOREM 2. *Two non-decreasing sequences f and g of numbers are inverses if and only if the corresponding sets F and G of positive integers, defined by $F(m)=f(m)+m$ and $G(n)=g(n)+n$, are complementary.*

Proof. Let f and g be inverses. By (2), the inequalities $f(m)<n$ and $g(n)<m$ are contradictories, in the sense that one and only one of them is true. Hence also the inequalities $F(m)<m+n$ and $G(n)<m+n$ are contradictories, and we conclude that $F(m)$ is never equal to $G(n)$. In particular, the set of all $F(m)$ with $m\leqq a$ and the set of all $G(n)$ with $n\leqq f(a)$ have no elements in common. Let $f(a)$ be finite, then none of the elements in either of these sets exceeds $a+f(a)$; for we infer from (2) that $g(f(a))<a$ since $f(a)\nless f(a)$. Thus the two sets just considered contain together the first $a+f(a)$ positive integers. If f is not of type A, this argument holds for any a, hence the sets F and G are complementary. If f is of type A, then g is of type B, and the same argument can be used with f and g interchanged.

Conversely, let F and G be complementary sets. Then f is a non-decreasing sequence of numbers, hence possesses an inverse $f^\dagger$, by theorem 1. By the above, $f^\dagger(n)+n$ defines the complementary set of F, hence coincides with $g(n)+n$. Thus $g=f^\dagger$ is the inverse of f.

This completes the proof of theorem 2.

Example 4. We wish to calculate the n-th non-square. Let $F(m)=m^2$ be the m-th square. Then $f(m)=m^2-m$ and $f^\dagger(n)=$number of m such that $m^2-m<n$, which is the same as the largest integer m satisfying this inequality. Since both sides of the inequality are integers, we may just as well compute the largest integer m satisfying $m^2-m+1/4<n$, that is $m-1/2<\sqrt{n}$. Thus $f^\dagger(n)=[1/2+\sqrt{n}]=\{\sqrt{n}\}$, where $\{x\}$ denotes the closest integer to x. The n-th non-square is therefore $n+\{\sqrt{n}\}$.

Example 5. In the same way we find that the n-th non-triangular number is $n+\{\sqrt{2n}\}$.

4. If F is a set of positive integers, let CF denote the complementary set. We have shown in theorem 2 that $CF(n)=n+f^{\dagger}(n)$, where $f(m)=F(m)-m$. The set F may be identified with the sequence $\{F(1), F(2), \cdots\}$, which is a non-decreasing sequence of numbers, and therefore possesses an inverse $F^{\dagger}$. The question arises: Can CF be expressed in terms of $F^{\dagger}$ rather than $f^{\dagger}$? It turns out to be more convenient, if instead of $F^{\dagger}$ we introduce F^* defined as follows:

$$F^*(n) = F^{\dagger}(n+1) = \text{number of elements of } F \text{ not exceeding } n. \tag{2'}$$

This may be supplemented by defining $F^*(\infty)=\infty$. We shall need the following

LEMMA 1. $n+F^*(CF(n))=CF(n)$.

Proof. When $CF(n)$ is infinite, this is clear. Otherwise $F^*(CF(n))$ is the number of elements of F not exceeding $CF(n)$. Now the number of all positive integers not exceeding $CF(n)$ is $CF(n)$, and the number of elements of CF not exceeding $CF(n)$ is n. The result follows.

We shall define recursively

$$F_0(n) = n; \qquad F_k(n) = n + F^*(F_{k-1}(n)) \qquad \text{for } k > 0. \tag{3}$$

Then we obtain the following

THEOREM 3. $CF(n)=\lim_{k\to\infty}F_k(n)$.

Proof. We first show that $F_{k+1}(n)\geqq F_k(n)$. Clearly this is true for $k=0$. Suppose $k>0$ and $F_k(n)\geqq F_{k-1}(n)$; then

$$F_{k+1}(n) = n + F^*(F_k(n)) \geqq n + F^*(F_{k-1}(n)) = F_k(n),$$

by (3), induction hypothesis, and the fact that F^* is non-decreasing.

We next show that $F_k(n)\leqq CF(n)$. Without loss in generality we may assume that $CF(n)$ is finite. Suppose $F_{k-1}(n)\leqq CF(n)$, then

$$F_k(n) = n + F^*(F_{k-1}(n)) \leqq n + F^*(CF(n)) = CF(n),$$

by (3), induction hypothesis, monotoneness of F^*, and lemma 1.

Thus we may write $G(n)=\lim_{k\to\infty}F_k(n)\leqq CF(n)$. Without loss in generality we may assume that $G(n)$ is finite. Then for sufficiently large k,

$$G(n) = F_{k-1}(n) = F_k(n) = n + F^*(F_{k-1}(n)) = n + F^*(G(n)),$$

so that $n=G(n)-F^*(G(n))$. By definition, this is the number of elements of CF not exceeding $G(n)$. Thus n is the number of m such that $CF(m)\leqq G(n)$, whence $CF(n)\leqq G(n)$. Since we have already shown the converse inequality, we have equality, as was to be proved.

Example 5. The n-th non-prime is the limit of the sequence: n, $n+\pi(n)$, $n+\pi(n+\pi(n))$, $\cdots$.

5. For given n, the sequence $F_k(n)$, $(k=0, 1, \cdots)$, is a sequence of integers, hence attains its limit $CF(n)$ in a finite number of steps, provided this

limit is finite. In fact, it follows from theorem 3 that we need not go beyond $k=CF(n)-n$. It may happen that the number of steps need not depend on n. We shall see that in many cases two steps are sufficient.

THEOREM 4. *If* $F(m+1)-F(m)\geqq m$, *then* $CF(n)=F_2(n)$.

Proof. In view of lemma 1 and (3), it suffices to prove that $F^*(CF(n)) = F^*(F_1(n))$. We know from the proof of theorem 3 that $F_1(n)\leqq CF(n)$, hence we need only show that $F^*(CF(n))\leqq F^*(F_1(n))$, which means that any element of F not exceeding $CF(n)$ does not exceed $F_1(n)$. Assuming $F(m)\leqq CF(n)$, we shall prove that $F(m)\leqq F_1(n)$.

Since F and CF have no elements in common, we even have $F(m)<CF(n)$, so that $(CF)^*(F(m))\leqq n-1$, hence by lemma 1,

$$F(m) = m + (CF)^*(F(m)) \leqq m + n - 1. \tag{4}$$

If $m=1$, we deduce immediately that $F(1)\leqq n\leqq F_1(n)$, by (3). There remains the case $m>1$.

If $m>1$, by hypothesis and (4), $F(m-1)+m-1\leqq F(m)\leqq m+n-1$, so that $F(m-1)\leqq n$, whence $m-1\leqq F^*(n)$. Hence, by (4) again,

$$F(m) \leqq n + m - 1 \leqq n + F^*(n) = F_1(n),$$

as was to be proved.

Example 6. The n-th positive integer which is not a perfect k-th power $(k\geqq 2)$ is $n+[(n+[n^{1/k}])^{1/k}]$. For $k=2$ this gives a new formula for the n-th non-square.

Example 7. The n-th positive integer not of the form $[e^m]$ with $m\geqq 1$ is $n+[\log (n+1+[\log (n+1)])]$.

Postscript. It has come to the authors' attention that Viggo Brun [*Rechenregel zur Bildung der n-ten Primzahl*, Norsk. Mat. Tidsskr., vol. 13, 1931, pp. 73–79] proved the following: If $n_k=n-\pi(n+n_1+\cdots+n_{k-1})$ for $k>0$, *then* $n+n_1+\cdots+n_k\rightarrow p_n$ *as* $k\rightarrow\infty$. This result is an immediate consequence of Theorem 3. For, letting $CF(n)=p_n$, one obtains $F^*(n)=n-\pi(n)$ and $n+n_1+\cdots+n_k=F_k(n)\rightarrow p_n$. D. H. Lehmer [*An inversive algorithm*, Bull. Amer. Math. Soc., vol. 38, 1932, pp. 693–694] generalized Brun's result from p_n to any increasing sequence of positive integers, thus obtaining a result essentially equivalent to Theorem 3.

CARNOT AND THE CONCEPT OF DEVIATION

C. B. BOYER, Brooklyn College

In this MONTHLY for October, 1952 (p. 535), one reads that "The concept of aberrancy . . . was originally introduced by Transon (using the name *deviation*)" [1]. It is the intention here to point out that this notion is older still, for it appeared in 1803 in the *Géométrie de position* of Lazare Nicholas Marguerite Carnot, published thirty-eight years before Transon wrote. The reputation of this book has been well-known, yet references to it leave the impression that it is devoted exclusively to synthetic geometry. It is indeed true that in it the author contributed significantly to so-called "pure" geometry; but the last of the five main sections of the work is concerned primarily with the use of analytic methods. In fact, the point of view toward coordinates here presented is the broadest of any mathematician since Newton, another who traditionally is placed among the synthesists. Carnot may well have been unaware of the suggestion of polar, bipolar, and other coordinate systems in Newton's *Methodus fluxionum*, for this has been overlooked by other mathematicians and historians [2] just as has a similar contribution in the *Géométrie de position*. Under the heading, "*De la détermination d'un point dans l'espace, et du changement de ses coordonnées*," Carnot proposed a wide variety of systems: polar, bipolar, biangular, and various triangular types. In such cases the equations of a curve depend on the elements used as a frame of reference; and so Carnot looked for intrinsic coordinates. He wrote,

> For any curve there exists at each point a certain line which does not depend on any particular hypothesis or any basis of comparison taken in absolute space; this is the radius of curvature . . . I believe nevertheless that it is possible to find various variables which shall have the condition demanded.

Carnot considered using arc length as a second coordinate, along with the radius of curvature; but he rejected this because it depends upon the choice of a fixed point from which the distance is measured. Instead he introduced what now is known as the angle of deviation—or, strictly speaking, its complement:

> I propose, for example, the angle which is formed, at the point describing the curve, by the tangent and the line which bisects the infinitesimally small secants drawn through the curve parallel to this tangent.

In illustrating this idea, he replaced the tangent by the normal, thus bringing his angle into conformity with the modern aberrancy. If M is a point on a curve and K is the center of curvature for this point, Carnot drew perpendicular to MK a variable secant mm' with midpoint n. His angle of deviation was then the angle between MK and the limiting position of the line Mn as m and m' approach M along the curve. As an example of the use of the angle as a coordinate, he cites the parabola $y^2 = px$, for which one finds $r = \frac{1}{2}p \csc^3 z$, where r is the radius of curvature and z the angle of deviation. This latter equation, in

curvature and aberrancy as coordinates, is one form of intrinsic equation of the parabola. Carnot also derived a general formula for cot z, pointing out that for a circle this expression vanishes; *i.e.*, in modern terminology, the deviation of a circle at every point is zero [3].

In 1841 Transon gave a formula for tan z, introducing the phrase "axis of deviation" and pointing out that the deviation is a geometric representation of the third derivative. He also called attention to some remarkable properties of the deviation of conics, including the relation tan $z=r'/r$, where r and r' are the radii of curvature of the conic and its evolute; and he introduced the center of deviation [4]. Walker recently has added striking relations between the deviation of a conic and its differential equation [5]. The concept which Carnot introduced so unobtrusively thus has led to developments comparable to those connected with the idea of curvature; yet the adumbration in the *Géométrie de position* has been so completely overlooked that there seems to have been no reference to it by mathematicians or historians. Even Transon himself, *ancien élève* of the École Polytechnique which Carnot helped to establish, made no mention of it.

The year 1953 marks the bicentennial of the birth of Lazare Carnot (as well as the sesquicentennial of his *Géométrie de position*), and hence brief attention may appropriately be called to the extraordinary career of this versatile mathematician. He was born at Nolay (Côte-d'Or), on May 13, 1753, one of eighteen children, several of whom came to occupy positions of prominence. Lazare in 1771 entered the École de Mézières, where he studied under Monge. His fellow military students regarded him as *un original* because he preferred the library to the cafe. His early published works were both literary and scientific, for he wrote considerable poetry and took a particular interest in balloons, fortifications, and politics. When the Revolution opened, Carnot, then a captain, accepted enthusiastically the views of the National Assembly; but he took no active role until in 1791 he was elected to the Legislative Assembly. Subsequently designated to the National Convention, he was among those who voted for the execution of Louis XVI on grounds of "justice and policy alike." Becoming more and more deeply involved in politics, he was chosen a member of the Committee of Public Safety in 1793. Charged with the organization and operation of the revolutionary army, his genius achieved such striking success against the invading forces that he was honored spontaneously as "The Organizer of Victory." Carnot did not hesitate to seize a musket and lead his troops against the enemy. Robespierre was bitterly jealous of his military ability and St. Just accused him of moderantism, but he survived the Terror through his indispensability to the defence of France. However, he was proscribed in 1797 as the result of false charges of implication in a royalist plot. His name was stricken from the rolls of the *Institut*, and his chair as geometer was filled, *mirabile dictu*, by General Bonaparte. Monge, the great geometer, was so mesmerized by Napoleon that even he joined in this unanimously approved outrage against mathematical honesty.

When Bonaparte scored his *coup d'Etat* of 1799, Carnot returned to France to become minister of war and was reelected to the *Institut.* He was elected a member of the Tribunate, and, being an inflexible republican, he exerted all his influence against Napoleon's imperial designs. In the midst of extraordinary factional turmoil, Carnot remained aloof from political parties, opposing tyranny of every description. A man of courage and conviction, in 1804 he delivered a brilliant speech against the conferring of the Imperial Dignity upon Napoleon [6], but his was the lone negative vote. Yet, when in 1814 Carnot saw France in jeopardy, he offered Napoleon the services of his "weak sexagenarian arm." As commander at Antwerp he held out against the enemy with notorious stubbornness, withdrawing only after the abdication of the emperor. Carnot did not conspire to recall Napoleon; but during the Hundred Days he again served his country, this time as Minister of the Interior, and he struggled in vain against the second abdication of Napoleon. He became a member of the provisional government, but he was exiled on the return of Louis XVIII and died at Magdeburg on Aug. 2, 1823 [7]. Carnot's influence, however, was felt long afterward: through his memory, in his descendants, and through his published works. His eldest son was the famous physicist, Sadi Carnot, author of *Réflexions sur la puissance motrice du feu* (1824); and a grandson, Marie François Sadi Carnot, became the fourth president of the Republic.

Notwithstanding a life of intense military and political activity, Carnot contributed regularly to mathematics. The work which enjoyed greatest popularity was his *Réflexions sur la métaphysique du calcul infinitésimal* of 1797. In this he sought to reconcile the various approaches to the calculus—the method of exhaustion, Leibnizian differentials, Newtonian fluxions, the limits of D'Alembert, the series of Lagrange, and others—concluding, unfortunately, that the true unifying basis was to be found in the principles of the compensation of errors. The *Réflexions* appeared in half a dozen French editions, as well as in many other languages, including English [8]; but as a permanent contribution to mathematics it falls far below the *Géométrie de position* of 1803.

Monge and Carnot may be looked upon as the two founders of modern synthetic geometry, the former through his influence as a great teacher at the École Polytechnique, the latter through his published works, especially the *Géométrie de position.* This contains numerous theorems of pure geometry, a striking example of which may be cited here: If M, N, P, Q are the areas of the four faces of a tetrahedron, and if m, n, p are the angles between the faces N and P, M and P, and M and N respectively, then $Q^2=M^2+N^2+P^2-2MN \cos p-2NP \cos m-2PM \cos n$ [9]. Yet the best known of Carnot's contributions to pure geometry comes not from the *Géométrie de position*, but from his *Essai sur le théorie des transversales* of 1806. This contains the celebrated "theorem of Carnot"—Given any algebraic curve of order n which cuts a triangle ABC, let A_1 be the product of the n distances, real or imaginary, from A to the n points of intersection of the curve with the side AB, and let B_1 and C_1 be defined similarly for the sides BC and CA; and let A_2, C_2, and B_2 be the simi-

lar products corresponding to the sides AC, CB, and BA respectively. Then $A_1B_1C_1 = A_2B_2C_2$. If the curve is a straight line, one has a well-known theorem of antiquity; if the curve is a cubic, it follows from Carnot's theorem that the three points of inflection lie on a straight line, a familiar result of the eighteenth century.

Carnot was aware that one of the advantages of analytic geometry was its generality; and hence he had sought, in *De la corrélation des figures de géométrie* (1801), to establish for synthetic geometry a comparable universality. Here he proposed a unifying idea somewhat akin to the controversial "principle of continuity" of Poncelet. This idea was further developed in the *Géométrie de position.*

Among Carnot's earliest works was the *Essai sur les machines en général* (1783), which contained the Carnot theorem on the loss of energy due to abrupt changes in velocity. This work, revised, appeared again in 1803 as *Principes fondamentaux de l'équilibre.*

There have been greater scientists and geometers than Carnot; there may have been greater political and military figures (although none more upright and courageous). But as a brilliant example of a man of thought and of action, it is difficult indeed to find a mathematician who outshines Carnot the elder—"The Great Carnot."

References

1. This ascription is found in other sources also: George Salmon, Higher Plane Curves, 3rd ed., London, 1879, pp. 368–369; E. J. Wilczynski, Some remarks on the historical development and the future prospects of the differential geometry of plane curves, Bull. Amer. Math. Soc. vol. 22, 1915–1916 pp. 320–324 (especially p. 321); Encyklopädie der Mathematischen Wissenschaften, III, vol. 3, p. 40.

2. See Newton as an originator of polar coordinates, this MONTHLY, vol. 56, 1949, pp. 73–78.

3. Géométrie de position, Paris, 1803, pp. 475 f.

4. Abel Transon, Recherches sur la courbure des lignes et des surfaces, Journal de mathématiques pures et appliquées, vol. 6, 1841 pp. 191–208.

5. This MONTHLY, vol. 59, 1952, pp. 531–539.

6. For English translations of some of Carnot's famous speeches, see Lewis Goldsmith, A sketch of M. Carnot's life, together with some remarkable speeches which he made, London, c. 1814, and An exposition of the political conduct of Lieut. General Carnot, since the first of July, 1814. By himself. To which is added Carnot's speech in the Tribunate in 1804, trans. by Henry Wheaton, New York, 1815).

7. There is an English translation of Arago's standard scientific biography. See D. F. J. Arago, Biographies of distinguished scientific men (2nd series, Boston, 1859). A brief note is also found in D. E. Smith, Lazare Nicholas Marguerite Carnot, *Scien. Monthly*, vol. 37, 1933, pp. 188–189. For a full-length biography in English, but largely from the political point of view, see Huntly Dupre, Lazare Carnot. Republican patriot, Oxford, Ohio, 1940. There are several recent French biographies, including Henri Carré, Le grand Carnot, vol. I, Paris, c. 1950, as well as an older German account, K. Fink, Lazare Nicolas Marguerite Carnot, sein Leben und seine Werke, Tübingen, 1894. A chapter on Carnot, with particular reference to this work on fortifications, is included in E. M. Lloyd, Vauban, Montalembert, Carnot: Engineer Studies, London, 1887, pp. 154–199.

8. For editions and further analysis of the work see my Concepts of the Calculus, New York, 1939, pp. 257 f.

9. See Niels Nielsen, Géomètres français sous la revolution, Copenhagen, 1929, for further details on his life and work. Cf. also Maximilien Marie, Histoire des sciences mathématiques et physiques, 12 vols., Paris, 1883–1888, X, pp. 157–168.

MATHEMATICAL NOTES

EDITED BY F. A. FICKEN, University of Tennessee

Material for this department should be sent to F. A. Ficken, University of Tennessee, Knoxville 16, Tenn.

INEQUALITIES CONNECTED WITH DEFINITE HERMITIAN FORMS, II

A. OPPENHEIM, University of Malaya, Singapore

1. Let $A=(a_{ik})$, $B=(b_{ik})$ be two non-negative definite Hermitian matrices of order n. Then by considering the corresponding non-negative definite quadratic forms we see plainly that the matrix

$$C = (c_{ik}), \qquad c_{ik} = a_{ik} + b_{ik} \tag{1}$$

is also non-negative definite Hermitian.

THEOREM 1. *Suppose that the latent roots of A, B, C are the (non-negative) numbers arranged in increasing order:*

$$\alpha_1 \leqq \alpha_2 \leqq \cdots \leqq \alpha_n, \qquad \beta_1 \leqq \beta_2 \leqq \cdots \leqq \beta_n, \qquad \gamma_1 \leqq \gamma_2 \leqq \cdots \leqq \gamma_n. \tag{2}$$

Then

$$(\gamma_1\gamma_2 \cdots \gamma_i)^{1/i} \geqq (\alpha_1 \cdots \alpha_i)^{1/i} + (\beta_1 \cdots \beta_i)^{1/i}, \qquad (i = 1, 2, \cdots, n). \tag{3}$$

If we take $i=n$ this inequality becomes (if $a=\det A$, *etc.*).

$$c^{1/n} \geqq a^{1/n} + b^{1/n}, \tag{4}$$

an inequality due (for real forms) to Minkowski [2] from which as I have shown [3] inequalities of Hadamard and Fischer are deducible. (See also Hardy, Littlewood, and Pólya, *Inequalities.*)

Suppose that λ and μ are arbitrary non-negative numbers whose sum is unity. Then from (4) we obtain

$$\det(\lambda a_{ik} + \mu b_{ik}) \geqq (\lambda a^{1/n} + \gamma b^{1/n})^n, \tag{5}$$

and so by the inequality of the arithmetic mean and geometric mean,

$$\det(\lambda a_{ik} + \mu b_{ik}) \geqq (\det(a_{ik}))^{\lambda}(\det(b_{ik}))^{\mu}. \tag{6}$$

This inequality was given recently by Ky Fan [1]. Plainly (5) is stronger than (6).

2. The proof of Theorem 1 depends on an interesting algebraic inequality which is worth stating as a theorem.

THEOREM 2. *Suppose that the numbers e_i, f_i $(i=1, \cdots, n)$ are non-negative. Suppose that e_i^*, f_i^* are the same sets arranged in non-decreasing order. Then if*

$$\prod{}^* = (e_1^* + f_n^*)(e_2^* + f_{n-1}^*) \cdots (e_n^* + f_1^*), \tag{7}$$

and

$$\prod{}_* = (e_1^* + f_1^*)(e_2^* + f_2^*) \cdots (e_n^* + f_n^*), \tag{8}$$

$$\prod{}^* \geqq \prod_{i=1}^{n} (e_i + f_i) \geqq \prod{}_*. \tag{9}$$

Equality holds throughout (9) *if one of the sets consists of equal numbers. Equality holds on the right of* (9) *if one of the permutations which carries (e_i) into (e_i^*) carries (f_i) into (f_i^*). Equality holds on the left if a permutation carrying (e_i) into (e_i^*) carries (f_i) into the reverse set to (f_i^*).*

The second inequality in (9) is a special case of an inequality of Ruderman [4].

We may suppose without loss of generality that $e_i^* = e_i$ for each i. This is merely a renumbering.

For $n=1$ the theorem is trivial. For $n=2$ it follows from the obvious identity

$$(x + y')(x' + y) - (x + y)(x' + y') = (x' - x)(y' - v)$$

when we take

$$0 \leqq x \leqq x', \qquad 0 \leqq y \leqq y'.$$

Consider the product

$$\prod = (e_1 + f_1)(e_2 + f_2) \cdots (e_n + f_n), \qquad (n \geqq 3).$$

If in any pair (e_i+f_i) $(e_{i+1}+f_{i+1})$ we have unequal f_i, f_{i+1}, then by the case $n=2$ we get the minimum if $f_i<f_{i+1}$ and the maximum if $f_i>f_{i+1}$. Thus after a finite number of steps we reduce $\prod$ to $\prod_*$ by interchanges which do not increase $\prod$ and likewise we can change $\prod$ to $\prod^*$ by interchanges which do not decrease $\prod$. The maximum number of steps needed will be $\frac{1}{2}n(n-1)$. It is clear that Theorem 2 follows at once.

3. Suppose that A is positive definite. There is a non-singular transformation which carries A and B into diagonal form with elements α_i (in some order) and β_i (in some order). This transformation carries C into the diagonal form γ_i (in some order) so that

$$\gamma_i = \lambda_i + \mu_i$$

where the λ_i are a permutation of the α_i and the μ_i are a permutation of the β_i.

By Theorem 2

$$\gamma_1\gamma_2 \cdots \gamma_i \geqq \prod_{j=1}^{i} (\lambda_j^* + \mu_j^*) \geqq \prod_{j=1}^{i} (\alpha_j + \beta_j),$$

since $\lambda_1^*, \cdots, \lambda_i^*$ are a selection (in non-decreasing order of magnitude) from the non-decreasing set $\alpha_1, \cdots, \alpha_n$ so that

$$\lambda_j^* \geqq \alpha_j, \quad \text{and likewise} \quad \mu_j^* \geqq \beta_j.$$

Now apply Hölder's inequality to obtain

$$(\gamma_1\gamma_2 \cdots \gamma_i)^{1/i} \geqq \left(\prod_1^i \alpha_j \right)^{1/i} + \left(\prod_1^i \beta_j \right)^{1/i}.$$

Theorem 1 follows therefore when A is positive definite, B non-negative definite. But if A and B are both non-negative and neither is positive definite, there is nothing to prove.

If we apply (3) to the matrix $\lambda A + \mu B$ where λ and μ are positive numbers with sum unity and then use the inequality of the arithmetic mean and the geometric mean we obtain

THEOREM 3. *Let $\gamma_1 \leqq \gamma_2 \leqq \cdots \leqq \gamma_n$ be the latent roots of the matrix $C = \lambda A + \mu B$ $\lambda \geqq 0$, $\mu \geqq 0$, $\lambda + \mu = 1$, where A and B are non-negative Hermitian matrices with latent roots*

$$\alpha_1 \leqq \alpha_2 \leqq \cdots \leqq \alpha_n, \qquad \beta_1 \leqq \beta_2 \leqq \cdots \leqq \beta_n;$$

then, for $i = 1, 2, \cdots, n$,

$$\gamma_1\gamma_2 \cdots \gamma_i \geqq (\alpha_1\alpha_2 \cdots \alpha_i)^{\lambda}(\beta_1\beta_2 \cdots \beta_i)^{\mu}.$$

Theorem 3 is due to Ky Fan (Problem 4430, this MONTHLY, proposed in vol. 58, 1951, p. 194; solution in vol. 60, 1953, p. 50). The inequality is weaker than that of Theorem 1.

4. I am indebted to the referee for giving an alternative proof of Theorem 1 which employs another result of Ky Fan (Problem 4429, *ibid.*, proposed in vol. 58, 1951, p. 194; solution in vol. 60, 1953, p. 48).

"Let A, B be two non-negative Hermitian matrices of order n. Let the eigenvalues of A, B, $A+B$ be

$$\begin{aligned} A&: \ \alpha_1 \leqq \alpha_2 \leqq \cdots \leqq \alpha_n, \\ B&: \ \beta_1 \leqq \beta_2 \leqq \cdots \leqq \beta_n, \\ A + B&: \ \gamma_1 \leqq \gamma_2 \leqq \cdots \leqq \gamma_n, \end{aligned}$$

respectively. Let $x_1, \cdots, x_n$ be n orthonormal eigenvectors of $A+B$ such that

$$(A + B)x_j = \gamma_j x_j \qquad (1 \leqq j \leqq n).$$

According to the theorem in Problem 4429 we have

$$\alpha_1\alpha_2 \cdots \alpha_i \leqq \prod_{j=1}^{i} (Ax_j, x_j), \qquad \beta_1\beta_2 \cdots \beta_i \leqq \prod_{j=1}^{i} (Bx_j, x_j),$$

and therefore

$$(\alpha_1\alpha_2 \cdots \alpha_i)^{1/i} + (\beta_1\beta_2 \cdots \beta_i)^{1/i} \leqq \left[\prod_{j=1}^{i} (Ax_j, x_j)\right]^{1/i} + \left[\prod_{j=1}^{i} (Bx_j, x_j)\right]^{1/i}.$$

By Hölder's inequality, the right side is not greater than

$$\left[\prod_{j=1}^{i} ((A + B)x_j, x_j)\right]^{1/i} = (\gamma_1\gamma_2 \cdots \gamma_i)^{1/i}.$$

Hence

$$(\alpha_1\alpha_2 \cdots \alpha_i)^{1/i} + (\beta_1\beta_2 \cdots \beta_i)^{1/i} \leqq (\gamma_1\gamma_2 \cdots \gamma_i)^{1/i}."$$

But the proof of Theorem 1 given in Section 2 does not use the theorem of Problem 4429.

References

1. Ky Fan, On a theorem of Weyl concerning eigenvalues of linear transformations, I, II, Proc. Nat. Acad. Sci., vol. 35, 1949, pp. 652–655, vol. 36, 1950, pp. 31–34. (I have not been able to get access to these papers.)

2. H. Minkowski, Diskontinuitätsberich für arithmetisch Äquivalenz, Journ. für Math., vol. 129, 1905, pp. 220–274.

3. A. Oppenheim, Inequalities connected with definite Hermitian forms, Journ. London Math. Soc., vol. 5, 1930, pp. 114–119.

4. H. D. Ruderman, Two new inequalities, this MONTHLY, vol. 59, 1952, pp. 29–32.

CLASSROOM NOTES

EDITED BY G. B. THOMAS, Massachusetts Institute of Technology

All material for this department should be sent to G. B. Thomas, Department of Mathematics, Massachusetts Institute of Technology, Cambridge 39, Mass.

REMARK ON A GRAPHICAL PROCEDURE

W. T. GUY, JR., The University of Texas

In a recent note, Russell [1] extended the graphical method of Liénard [2, 3] (used in plotting phase trajectories in non-linear problems) to some differential equations of the type

$$\frac{dy}{dx} = \frac{\phi(y) + x}{\psi(x) + y}. \tag{1}$$

Russell's construction is based on the curves Γ_1: $[x=-\phi(y)]$ and Γ_2: $[y=-\psi(x)]$, where $\phi(y)$ and $\psi(x)$ denote continuous functions.

The purpose of this remark is to point out that his construction, with a slight modification, is also valid for certain differential equations of the type

$$\frac{dy}{dx}=\frac{g(x, y)}{h(x, y)}. \tag{2}$$

We assume $g(x, y)=0$ defines x as a continuous (single-valued) function of y [say, $x=-\phi(y)$] and $h(x, y)=0$ defines y as a continuous (single-valued) function of x [say, $y=-\psi(x)$] over the required domains. Hence, $g(x, y)=[x+\phi(y)]\cdot c(x, y)$, $h(x, y)=[y+\psi(x)]k(x, y)$, and equation (2) becomes

$$\frac{dy}{dx}=\frac{\phi(y)+x}{\psi(x)+y}\,\frac{c(x, y)}{k(x, y)}=\frac{\phi(y)+x}{\psi(x)+y}R(x, y). \tag{3}$$

Here $R(x, y)$ might be called a rotation factor. The construction of Russell based on his curves Γ_1 and Γ_2 is repeated except now the line PD must be replaced by a line PE whose slope is equal to that of PD when multiplied by the value of $R(x, y)$ at point P.

The justification and limitations of the method are clear. If $R(x, y)=1$, the Russell construction obtains. If $R(x, y)=$ a constant, the indicated modification is easily carried out. A protractor graduated with numbers indicating slopes will be of help in measuring the slope of PD and in drawing the line PE, especially for the case $R(x, y)\neq$ a constant. This procedure has some merit over the usual directional field scheme, even in this latter case, if Γ_1 and Γ_2 can be easily determined and if $R(x, y)$ is simpler to use than $g(x, y)/h(x, y)$.

References

1. J. P. Russell, A graphical procedure, this MONTHLY, vol. 60, 1953, p. 710.

2. A. Liénard, Etude des oscillations entretenues, Rev. Gén. d'Elect., 1928.

3. J. J. Stoker, Non-linear Vibrations in Mechanical and Electrical Systems, Interscience, New York, 1950, pp. 31–36.

COMMENTS ON THE POLAR PLANIMETER

L. I. LOWELL, Northrop Aircraft

Figure 1 shows a popular design of planimeter used in engineering offices for measuring areas of various shapes. No doubt the question has occurred to many operators: "Is the total angle generated by the indicating roller, as the point traces a closed curve, in direct proportion to the area encompassed by the curve?"

The following verification of the planimeter would seem worthwhile, first, because it is more straightforward and complete than the brief statement in Calculus books that "the polar planimeter is an instrument which makes use of

Guldin's Formula for measuring areas." Secondly, it provides a valuable example of a curvilinear coordinate system and the teacher of Advanced Calculus is frequently hard put for examples when covering that topic.

For purposes of analysis the planimeter will be thought of as a coordinate system. The arms have arbitrary lengths, a and b, which are constant, but can be assumed long enough for coverage of any arbitrary area. The variables, α and β, may be related to the cartesian variables, x and y, by means of the relations:

$$x = a\cos\alpha + b\cos\beta \tag{1}$$

$$y = a\sin\alpha + b\sin\beta. \tag{2}$$

The differential element of area is:

$$dA = |J|\, d\alpha d\beta \tag{3}$$

in which

$$J = \begin{vmatrix} \dfrac{\partial x}{\partial\alpha} & \dfrac{\partial y}{\partial\alpha} \\ \dfrac{\partial x}{\partial\beta} & \dfrac{\partial y}{\partial\beta} \end{vmatrix}.$$

Computing the partial derivatives from the basic equations, we find that the Jacobian becomes:

$$J = -\,ab\sin(\alpha - \beta). \tag{4}$$

The element of area is thus found to be:

$$dA = -\,ab\sin(\alpha - \beta)d\alpha d\beta. \tag{5}$$

Now let the planimeter trace an element of area, ΔA, and an accounting will be made of the motion of the indicating wheel. We see from the drawing that while the point is moving from 1 to 2, α is being increased by $\Delta\alpha$ but there is no relative motion between the two arms. From 2 to 3, β is being increased by $\Delta\beta$ while α is held fixed. The remaining two sides of the curvilinear quadrilateral are traced by motions which are the exact reverse of those which moved the point from 1 to 2 and 2 to 3, respectively. The wheel movement resulting from the point's moving from 2 to 3 is equal and in the opposite direction to that which occurs as the point moves from 4 to 1. Hence, these movements cancel each other and will be neglected.

As the point moves from 1 to 2, the total wheel rotation will be some multiple of

$$a\cdot\Delta\alpha\cos(\alpha - \beta); \tag{6}$$

from 3 to 4 it will be the same multiple of

$$a\cdot\Delta\alpha\cos(\alpha - \{\beta + \Delta\beta\}). \tag{7}$$

Subtracting (7) from (6) and rearranging will yield:

$$a\cdot\Delta\alpha[\cos(\alpha-\beta)(1-\cos\Delta\beta)-\sin\Delta\beta\sin(\alpha-\beta)]. \tag{8}$$

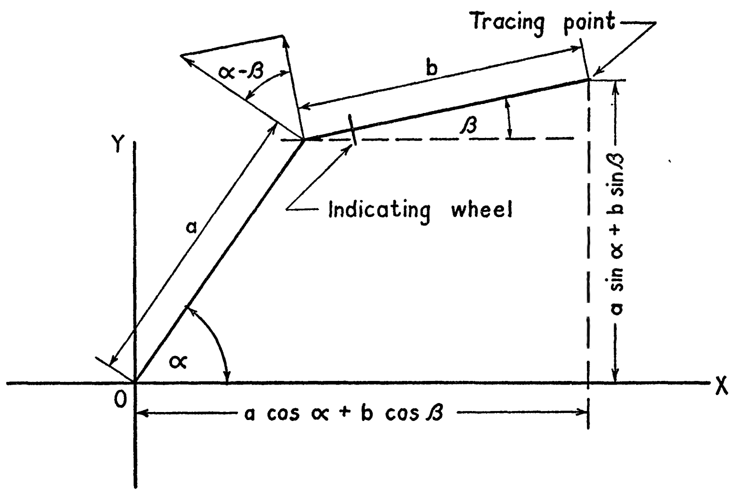


FIG. 1. The Polar Planimeter as a Coordinate System.

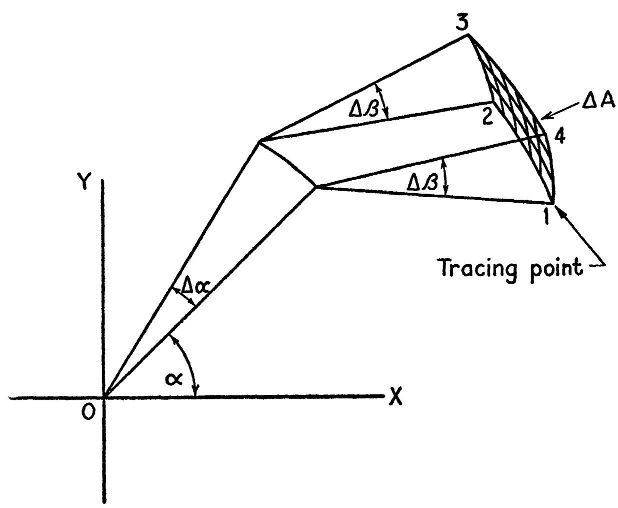


FIG. 2. Element of Area.

If we take $\Delta\alpha=d\alpha$ and $\Delta\beta=d\beta$ and neglect all powers of $\Delta\beta$ higher than the first power, we may replace (8) by

$$-a\sin(\alpha-\beta)(d\alpha d\beta)$$

which is a multiple of the right hand side of equation (5).

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1126. *Proposed by J. V. Pennington, Houston, Texas*

Smith: "Down in Todd County, which is a 23-mile square, I have a ranch—oblong in shape—measuring a whole number of miles each way."

Jones: "Hold on a minute. I happen to know the area of your ranch; let me see if I can figure its length." He figures furiously. "I need more information. Is the width more than half the length?"

Smith answers the question.

Jones: "I now know the length."

Brown: "I too know the area and, although I did not hear your answer to Jones' question, I too can tell you the length."

Green: "I did not know the area, but I can now tell you the length."

How many miles long is the ranch?

E 1127. *Proposed by Edmund DeWan, Harpur College*

Enumerate the total number of squares appearing on an $m\times n$ checker board, $m\geqq n$. Solve the analogous problem for rectangular parallelopipeds.

E 1128. *Proposed by C. S. Ogilvy, Hamilton College*

Show that the point P at eye level from which a picture high on a vertical wall subtends the maximum angle can be found as follows. With eye-level line as directrix and bottom of picture as focus, draw a parabola; from the intersection of this parabola with a line through the center of the picture parallel to the directrix, a line perpendicular to the directrix meets it at P.

E 1129. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn*

Find an integral arithmetic progression with an arbitrarily large number of terms such that no term is a perfect rth power for $r=2, 3, \cdots$, or N. Is this still possible if $N=\infty$?

E 1130. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Let the perpendicular bisector of the median BB' of triangle ABC and the tangent at B to the circumcircle of triangle ABC cut the line AC in points M and N respectively. Show that triangle ABC is isosceles with vertex at A if and only if $AM/AN=3/4$.

SOLUTIONS

E 1051 [1953, 551]. *Correction.* The condition given in the solution is sufficient but not necessary.

E 1088 [1954, 347]. *Correction.* In line 3 of the solution, the statement "$P_1=1$" should read "$P_1=P$."

Differential Equation of All Conics of Given Eccentricity

E 1070 [1953, 332; 1954, 50]. *Proposed by A. W. Walker, University of Toronto*

Find the lowest order differential equation satisfied by all conics with a given eccentricity e.

Solution by Alexander Oppenheim, University of Malaya. The required differential equation is

$$(8u_2)^{1/2} = k[(1 + y_1^2)(2uu_2 - u_1^2) - 4y_1u_1u^{-1/2} - 4u^{-1}], \tag{1}$$

where

$$u = y_2^{-2/3}, \qquad k = (e^2 - 1)^{1/2}/(2 - e^2), \tag{2}$$

and suffixes denote differentiations with respect to x.

Note that

$$2k = \tan\phi, \qquad e = \sec\phi/2, \tag{3}$$

where ϕ is the angle between the asymptotes of the conic.

To obtain (1) it is best to use the general equation of the conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \tag{4}$$

and the relation

$$(a + b)\tan\phi = 2(h^2 - ab)^{1/2}. \tag{5}$$

From (4)

$$by = -hx - f + X^{1/2}, \qquad X = -Cx^2 - 2Gx - A, \tag{6}$$

where, as usual, $C=ab-h^2$, $G=hf-bg, \cdots$, are the cofactors of $c, g, \cdots$, in the determinant Δ of (4). Differentiate (6). Then

$$by_1 = -h - (Cx + G)X^{-1/2}, \tag{7}$$

$$by_2 = b\Delta X^{-3/2}, \qquad \Delta^{2/3}u = X, \tag{8}$$

by (2). Differentiate (8) twice and obtain

$$\Delta^{2/3}u_1 = -2(Cx + G), \tag{9}$$

$$\Delta^{2/3}u_2 = -2C, \tag{10}$$

whence, since $AC-G^2=b\Delta$,

$$\Delta^{1/3}(2uu_2 - u_1^2) = 4b. \tag{11}$$

We now employ (5) in the form

$$b(-C)^{1/2} = k(C + h^2 + b^2), \tag{12}$$

since $C=ab-h^2$. From (7), (8), (9)

$$h = -(1/4)\Delta^{1/3}(2uu_2 - u_1^2)y_1 + (1/2)\Delta^{1/3}u_1u^{-1/2}, \tag{13}$$

so that, by (10),

$$16\Delta^{-2/3}(C + h^2) = 4b\Delta^{-1/3}[(2uu_2 - u_1^2)y_1^2 - 4u_1u^{-1/2}y_1 - 4u^{-1}] \tag{14}$$

after a little reduction and by (11). From (10), (12) it follows that

$$(8u_2)^{1/2} = k[(2uu_2 - u_1^2)(1 + y_1^2) - 4u_1u^{-1/2}y_1 - 4u^{-1}], \tag{15}$$

since we may ignore $b=0$. Equation (15), of order four, is the differential equation of all conics of given eccentricity e where k and e are related by the second equation in (2).

As a check take $e=1$ or $\phi=0$. Then (15) gives $u_2=0$, the well known differential equation of all parabolas. If we put $\phi=\pi/2$, we see that the differential equation of all rectangular hyperbolas is

$$(2uu_2 - u_1^2)(1 + y_1^2) - 4u_1u^{-1/2}y_1 - 4u^{-1} = 0. \tag{16}$$

The derivative of the left hand side of (16) is $2uu_3(1+y_1^2)$, so that it is easy to verify that elimination of k from (15) leads to $u_3=0$, the differential equation for all conics. It is worth adding that (10) gives the differential equation of all conics with given area or with given product for semi-axes, since, if α, β are the semi-axes, $\alpha^2\beta^2C^3=\Delta^2$.

The Counterfeiters of Lower Slobbovia

E 1096 [1954, 46]. *Proposed by L. R. Ford, Jr., University of Illinois*

As is well known, Lower Slobbovia is too poor a country to afford its own mint. There are N coiners engaged in making Rasbuckniks, the local currency, to government specifications. However it is suspected that some of them may be counterfeiting by introducing some base metal into the alloy. Any pair of counterfeits will weigh the same, although slightly different from the weight of a good coin. Each coiner produces either all good coins or all counterfeits. With one guaranteed good coin, a set of infinitely refinable weights, a beam balance, and as many coins from each coiner as may be needed, determine in three weighings whether any of the coiners is dishonest, and which ones.

Solution by Julian Braun, China Lake, Calif. First, determine the weight of the good coin, W_g. Second, put one coin from each coiner on one side of the balance and determine their total weight, T. The discrepancy is $D=T-NW_g$. If $D=0$, all the coiners are honest. Thirdly, if $D\neq 0$, put 2^{i-1} coins from the ith coiner, $i=1, 2, \cdots, N$, on one side of the balance and determine the total weight of these coins, T'. The discrepancy in this case is $D'=T'(2^N-1)W_g$.

Find the integer S such that $D'/D=S/\beta(S)$, where $\beta(S)$ is the number of ones in the binary representation of S. Let $S=\sum_{i=1}^{N}B_i 2^{i-1}$, where $B_i=0$ or 1. Then $\beta(S)$ is the number of dishonest coiners and the ith coiner is honest or dishonest according as $B_i=0$ or 1.

Also solved by G. B. Charlesworth, Octave Levenspiel, P. B. Maggs and D. B. Mumford (jointly), R. H. Marquis, D. C. B. Marsh, L. V. Mead, C. S. Ogilvy, C. R. Perisho, H. W. Resnikoff, L. A. Ringenberg, Thomas Saaty, A. J. Tingley, and the proposer. These solutions showed interesting variations from one another.

A 3:4:5 Right Triangle

E 1097 [1954, 46]. *Proposed by Leon Bankoff, Los Angeles, Calif.*

Arcs AB and CD are quadrants of circles tangent externally at their midpoints, E, and such that AC and BD when extended meet perpendicularly in F. A circle is inscribed in the mixtilinear triangle EDB, touching ED in M. G is the projection of M upon EF. Show that triangle MGF is a 3:4:5 right triangle.

Solution by W. J. Cherry, North Riverside, Illinois. Let K be the center of arc AB. Taking BF and BK as x and y axes, denote the incenter of the mixtilinear triangle by P: (x, y). Let N and H be the projections of P on BF and BK. Then, by applying the Pythagorean Theorem to triangles KHP and NPF, we find

$$x^2 = 4r_1 y \quad \text{and} \quad (x - r_1)^2 = r_2(2y + r_2),$$

where r_1 and r_2 denote the radii of arcs AB and CD respectively. Taking $r_1=1$, arbitrarily, and $r_2=\sqrt{2}-1$, and solving the equations of the two parabolas simultaneously, we get

$$x = 2(2\sqrt{2} - 1)/7, \qquad y = (9 - 4\sqrt{2})/49,$$

whence

$$\sphericalangle PFN = \text{arc tan } y/(1 - x) = \text{arc tan } 1/7,$$

and

$$\sphericalangle MFG = \pi/4 - \text{arc tan } 1/7 = \text{arc tan } 3/4,$$

and the conclusion is established.

Also solved by Hüseyin Demir, E. I. Gale, A. R. Hyde, Beckham Martin, L. A. Ringenberg, David Rothman, C. W. Trigg, Roscoe Woods, and the proposer.

A Set of Finite Geometries

E 1098 [1954, 47]. *Proposed by L. A. Ringenberg, Eastern Illinois State College*

Construct all geometries satisfying the following axioms:

Axiom I. Space S is a set of n points, n a positive integer.

Axiom II. A line is a non-null subset of S.

Axiom III. Any two distinct lines have exactly one common point.

Axiom IV. Every point lies in exactly two distinct lines.

Solution by the Proposer. Since there is at least one point, there are at least two distinct lines (I, IV). Two such distinct lines, say a and b, have a point A in common; to be distinct, at least one of them, say a, contains a point other than A, say B. Then B must lie on some third line c (III, IV) distinct from a and b. It follows that the number m of lines in the geometry is an integer exceeding 2.

Two distinct pairs of distinct lines cannot have the same common point (IV). Hence $n=C(m, 2)=m(m-1)/2$. Thus a geometry with m lines has $n=m(m-1)/2$ points, each line consisting of $m-1$ points and each point lying on two lines.

Following is a diagram showing how to construct the geometry with $m=5$, $n=10$, the extension to an arbitrary $m\geqq 3$ is obvious.

line a:	A	B	C	D
b:	A	E	F	G
c:	B	E	H	I
d:	C	F	H	J
e:	D	G	I	J

Also solved by W. E. Briggs, Myles McConnon, D. C. B. Marsh, G. H. Meisters, J. D. Miller, D. B. Mumford, and D. R. Sudborough.

Editorial Note. The postulates form an interesting but simple set for classroom purposes. The student should examine the set for consistence, independence, categoricalness, and fertility.

A Gambling Table

E 1099 [1954, 47]. *Proposed by W. R. Van Voorhis, Fenn College*

On a certain gambling table there are N squares marked "2 to 1," "3 to 1," . . . , "$N+1$ to 1." A sum of money is placed on each square and one of the

squares is selected as a winner, paying the player at the odds marked on that square. The player loses the amounts placed on the other squares. What is the maximum N for which it is possible for a player to place money so that he can never suffer a net loss?

Solution by C. F. Pinzka, Princeton, N. J. Let S_k be the sum placed on the kth square and T the total amount bet. In order that there be no net loss, it is necessary and sufficient that $(k+2)S_k \geqq T$ for $k=1, 2, \cdots, N$. Writing this as $1/(k+2) \leqq S_k/T$, and summing both sides from $k=1$ to $k=N$, gives $\sum_{k=1}^{N} 1/(k+2) \leqq 1$. It is easily verified that 4 is the maximum N.

Also solved by W. E. Briggs, S. H. Eisman, A. L. Epstein, J. M. Howell, P. G. Kirmser, M. S. Klamkin, Bart Park, F. D. Parker, L. A. Ringenberg, D. C. Russell, N. Shklov, Hans Ury, and R. Z. Vause.

Ringenberg pointed out that not only can the player insure a net gain for $N=1, 2, 3, 4$, but that he can so bet that the net gain is independent of the square selected. If $N=2$, $S_1=4$, $S_2=3$, the gain is 5; if $N=3$, $S_1=20$, $S_2=15$, $S_3=12$, the gain is 13; if $N=4$, $S_1=20$, $S_2=15$, $S_3=12$, $S_4=10$, the gain is 3. Russell generalized the problem so that the stated odds are p_1 to 1, p_2 to 1, $\cdots$, p_n to 1.

An Inequality

E 1100 [1954, 47]. *Proposed by L. E. Ward, Sr., Naval Ordnance Test Station, China Lake, Calif., and L. E. Ward, Jr., University of Nevada*

For $x \geqq 0$ and $t \geqq 2$, prove that $(1+x^t)^{1/t} - (1+x^t)^{-1/t} \leqq x$.

Solution by J. V. Whittaker, U.C.L.A. Setting $1+x^t = y^t$, raising each side to the tth power, and dividing through by y^t, we obtain the equivalent inequality

$$(1 - 1/y^2)^t \leqq 1 - 1/y^t,$$

for $y \geqq 1$, $t \geqq 2$. But now

$$(1 - 1/y^2)^t \leqq (1 - 1/y^t)^t \leqq 1 - 1/y^t.$$

Equality is attained only when $y=1$. Hence in the given inequality, equality occurs only when $x=0$.

Also solved by Tom Black, W. E. Briggs, A. E. Danese, H. M. Feldman, Norman Greenspan, A. R. Hyde, M. S. Klamkin, Viktors Linis, A. E. Livingston, Lee Lorch, L. L .Pennisi, B. E. Rhoades, L. A. Ringenberg, R. Z. Vause, Chih-yi Wang, L. E. Ward, Jr., Albert Wilansky, R. E. Wild, and J. R. Winkelman.

Wilansky established the improved inequality

$$x - 1/x < L \leqq x^2(1 + x^2)^{-1/2},$$

where L is the original left hand side.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4598. *Proposed by I. J. Schoenberg, University of Pennsylvania*

Let the Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n, \qquad (r < |z| < r^{-1}),$$

converge in an open ring containing the circle $|z| = 1$. Show that a sequence $\{b_n\}$ exists satisfying the conditions

(i) $\quad |b_n| < K, \qquad$ for some K, for all $n = 0, \pm 1, \pm 2, \cdots$,

(ii) $\quad \sum_{n=-\infty}^{\infty} a_n b_{m-n} = 0 \qquad$ for all m,

(iii) $\quad b_n \neq 0 \qquad$ for some n,

if and only if $f(z)$ vanishes somewhere on the circle $|z| = 1$.

4599. *Proposed by Leonard Carlitz, Duke University*

Let p be a prime >2, $a \not\equiv 0 \pmod{p}$. Show that the number of solutions of the congruence

$$(x + y + z)^2 \equiv 2axyz \pmod{p}$$

is p^2+1.

4600. *Proposed by Leon Bankoff, Los Angeles, California*

Vertices A–C and B–D of square $ABCD$ are joined by quadrants of circles (B) and (C). A semi-circle (O_1) is described internally on the diameter BC and a circle (O_2) is drawn tangent to the three arcs. Another circle (O_3) is drawn tangent to circle (O_2) and to arcs AC and BC, and a right triangle is formed by joining O_3 to O_1 and dropping a perpendicular from O_3 upon BC. Successively tangent circles are drawn in the same manner (with (O_n) tangent to (O_{n-1}) and to arcs AC and BC) and right triangles are formed (with O_1O_n for hypothenuse).

Show that the infinitude of triangles so constructed are Pythagorean.

4601. *Proposed by Harry Goheen, Iowa State College, Ames, Iowa*

What is the necessary and sufficient condition that as $i \to \infty$ the limit of n_i exists, if n_i is defined by the recurrence relation

$$n_{i+1} = bn_i^2 + c,$$

n_0 being given?

4602. *Proposed by C. M. Ablow and D. L. Johnson, Seattle, Washington*

Show that

$$\sum_{i=1}^{n} A_i \cos (B_i t + C_i)$$

changes sign as t varies from zero to positive infinity. The A_i, B_i, C_i are real constants; $A_i B_i \neq 0$.

SOLUTIONS

Squarefull Integers

4459 [1951, 636]. *Proposed by D. J. Newman, Republic Aviation Corporation, Farmingdale, N. Y.*

Find an asymptotic expression for the number of integers, not exceeding x, each of which has the property that each of its prime divisors divides it to the second power at least.

Editorial Note. As has been pointed out by P. T. Bateman and others, the evaluation of the integral (1) of an earlier solution [1953, 719] is more than a direct application of Cauchy's theorem on residues and its complete justification requires tools far from simple.

Actually a rather good result can be obtained in this problem by a relatively elementary argument, as follows:

2. *Solution by P. T. Bateman, University of Illinois.* If c_n is 1 or 0 according as n does or does not have the property in question and if $A(x)$ is the number of integers not exceeding x having this property, then

$$A(x) = \sum_{n \leq x} c_n \tag{1}$$

and

$$\sum_{n=1}^{\infty} c_n n^{-s} = \zeta(2s)\zeta(3s)/\zeta(6s),$$

as in the former solution. Let b_n be the number of ways in which the positive integer n can be expressed in the form $m^2 n^3$, that is, as a product of a square and a cube. Thus

$$\sum_{j=1}^{\infty} b_j j^{-s} = \sum_{m=1}^{\infty} m^{-2s} \sum_{n=1}^{\infty} n^{-3s} = \zeta(2s)\zeta(3s)$$

for $R(s) > \frac{1}{2}$. We consider the summatory function

$$B(x) = \sum_{j \leqq x} b_j = \sum_{m^2 n^3 \leqq x} 1.$$

If m and n are any positive integers such that $m^2 n^3 \leqq x$, then either $m \leqq x^{1/5}$ or $n \leqq x^{1/5}$, possibly both. Thus

$$\begin{aligned} B(x) &= \sum_{m \leqq x^{1/5}} \sum_{n \leqq x^{1/3} m^{-2/3}} 1 + \sum_{m \leqq x^{1/5}} \sum_{m \leqq x^{1/2} n^{-3/2}} 1 - \sum_{m \leqq x^{1/5}} \sum_{n \leqq x^{1/5}} 1 \\ &= \sum_{m \leqq x^{1/5}} \left\{ x^{1/3} m^{-2/3} + O(1) \right\} + \sum_{n \leqq x^{1/5}} \left\{ x^{1/2} n^{-3/2} + O(1) \right\} - \left\{ x^{1/5} + O(1) \right\}^2 \\ &= x^{1/3} \sum_{m \leqq x^{1/5}} m^{-2/3} + x^{1/2} \sum_{n \leqq x^{1/5}} n^{-3/2} - x^{2/5} + O(x^{1/5}). \end{aligned}$$

Now we use the asymptotic formula

$$\sum_{n \leqq y} n^{-r} = y^{1-r}/(1-r) + \zeta(r) + O(y^{-r}),$$

where r is some fixed positive number other than 1, and $y \geqq 1$. (This formula is an easy consequence of the Euler-Maclaurin sum-formula of the first order; *cf.* Titchmarsh, *The Theory of the Riemann Zeta-Function*, Oxford, 1951, p. 14.) Using this in (2) we obtain

$$\begin{aligned} B(x) &= x^{1/3}\left\{ 3x^{1/15} + \zeta(2/3) + O(x^{-2/15}) \right\} \\ &\quad + x^{1/2}\left\{ \zeta(3/2) - 2x^{-1/10} + O(x^{-3/10}) \right\} - x^{2/5} + O(x^{1/5}) \\ &= \zeta(3/2) x^{1/2} + \zeta(2/3) x^{1/3} + O(x^{1/5}). \end{aligned} \tag{3}$$

Now if $R(s) > 1$, we have

$$1/\zeta(s) = \sum_{m=1}^{\infty} \mu(m) m^{-s},$$

where μ denotes the Möbius function. Accordingly

$$\sum_{n=1}^{\infty} c_n n^{-s} = \left\{ 1/\zeta(6s) \right\} \sum_{j=1}^{\infty} b_j j^{-s} = \sum_{m=1}^{\infty} \mu(m) m^{-6s} \sum_{j=1}^{\infty} b_j j^{-s}$$

(for $R(s) > \frac{1}{2}$) and thus

$$c_n = \sum_{m^6 j = n} \mu(m) b_j.$$

Hence by (3)

$$\begin{aligned} A(x) &= \sum_{m^6 j \leqq x} \mu(m) b_j = \sum_{m \leqq x^{1/6}} \mu(m) B(x/m^6) \\ &= \sum_{m \leqq x^{1/6}} \mu(m) \{ \zeta(3/2) x^{1/2} m^{-3} + \zeta(2/3) x^{1/3} m^{-2} + O(x^{1/5} m^{-6/5}) \} \\ &= \zeta(3/2) x^{1/2} \left\{ \sum_{m=1}^{\infty} \mu(m) m^{-3} + O\left(\sum_{m > x^{1/6}} m^{-3} \right) \right\} \\ &\quad + \zeta(2/3) x^{1/3} \left\{ \sum_{m=1}^{\infty} \mu(m) m^{-2} + O\left(\sum_{m > x^{1/6}} m^{-2} \right) \right\} \\ &\quad + O\left\{ x^{1/5} \sum_{m \leqq x^{1/6}} m^{-6/5} \right\}. \end{aligned} \tag{4}$$

Now if r is a fixed positive number greater than 1, and y is positive, $\sum_{n>y} n^{-r} = O(y^{1-r})$. Thus (4) gives

$$\begin{aligned} A(x) &= x^{1/2}\zeta(3/2)/\zeta(3) + O(x^{1/6}) + x^{1/3}\zeta(2/3)/\zeta(2) + O(x^{1/6}) + O(x^{1/5}) \\ &= x^{1/2}\zeta(3/2)/\zeta(3) + x^{1/3}\zeta(2/3)/\zeta(2) + O(x^{1/5}). \end{aligned} \tag{5}$$

This seems to be the best result readily obtainable by elementary methods. By more delicate methods, however, it is possible to sharpen the error term in (5) somewhat, *e.g.*, $O(x^{1/6} \log^2 x)$.

Inequality Involving Coefficients of Star-Like Functions

4535 [1953, 268]. *Proposed by M. S. Robertson, Rutgers University*

Let the function $f(z) = \sum_{n=1}^{\infty} a_n z^n$, $a_1 \neq 0$, be regular for $|z| \leqq 1$ and let it map $|z| \leqq 1$ onto a simply-connected domain D, star-like when viewed from the origin. Show that

$$2\left(\sum_{n=1}^{\infty} n \mid a_n |^2 \right)^2 \leqq \sum_{n=1}^{\infty} (n+1) \mid a_1 a_n + a_2 a_{n-1} + \cdots + a_n a_1 |^2,$$

with equality only for $f(z) = a_1 z$.

Solution by W. C. Royster, Alabama Polytechnic Institute. Let $f(z) = Re^{i\phi}$ and denote by C_1 the map of $|z| = 1$ effected by $w = f(z)$, and by C_2 the map of $|z| = 1$ effected by $w = [f(z)]^2$. Let A_1 be the area enclosed by C_1 and A_2 the area enclosed by C_2, the area on separate sheets counted separately. Then

$$A_1 = \frac{1}{2} \int_{C_1} R^2 d\phi = \frac{1}{2} \int_0^{2\pi} R^2 \frac{\partial \phi}{\partial \theta} d\theta,$$

$$A_2 = \int_{C_2} R^4 d\phi = \int_0^{2\pi} R^4 \frac{\partial \phi}{\partial \theta} d\theta,$$

where $z = re^{i\theta}$. Since $f(z)$ is starlike with respect to the origin, $\partial\phi/\partial\theta \geqq 0$ and $\int_0^{2\pi}(\partial\phi/\partial\theta)d\theta = 2\pi$. By the Schwartz integral inequality one has

$$\left(\int_0^{2\pi} R^2 \frac{\partial\phi}{\partial\theta} d\theta\right)^2 \leqq \int_0^{2\pi} R^4 \frac{\partial\phi}{\partial\theta} d\theta \int_0^{2\pi} \frac{\partial\phi}{\partial\theta} d\theta = 2\pi \int_0^{2\pi} R^4 \frac{\partial\phi}{\partial\theta} d\theta.$$

Thus $A_1^2 \leqq (\pi/2)A_2$, equality occurring only when R is constant. From this the required inequality follows since

$$A_1 = \sum_{n=1}^{\infty} n \mid a_n \mid^2, \qquad A_2 = \sum_{n=1}^{\infty} (n+1) \mid a_1a_n + \cdots + a_na_1 \mid^2,$$

with equality holding only for $f(z) = a_1z$.

It is interesting to note that if $f(z)$ is p-valently starlike with respect to the origin then the right hand member of the given inequality is multiplied by p. This happens since $\int_0^{2\pi}(\partial\phi/\partial\theta)d\theta = 2\pi p$ for functions of this class. In this case equality holds only for $f(z) = a_pz^p$.

Also solved by the Proposer.

Uniform Convergence of Trigonometric Sums

4536 [1953, 268]. *Proposed by O. E. Stanaitis, St. Olaf College, Northfield, Minn.*

Prove that

$$\sum_{n=1}^{\infty} \frac{\sin n^2\theta \sin n\theta}{n}, \qquad \sum_{n=1}^{\infty} \frac{\cos n^2\theta \sin n\theta}{n}$$

are uniformly convergent in any interval, and that

$$\sum_{n=1}^{\infty} \frac{\sin n^2\theta \cos n\theta}{n}$$

is divergent.

Solution by E. M. Wright, University of Aberdeen, Scotland. The first series is uniformly convergent. We have

$$\begin{aligned} 2\sum_{n=1}^{N} \frac{\sin n^2\theta \sin n\theta}{n} &= \sum_{n=1}^{N} \frac{\cos n(n-1)\theta - \cos n(n+1)\theta}{n} \\ &= 1 - \frac{\cos N(N+1)\theta}{N} - \sum_{n=1}^{N-1} \frac{\cos n(n+1)\theta}{n(n+1)} \end{aligned}$$

which, as $N \to \infty$, becomes

$$1 - \sum_{n=1}^{\infty} \frac{\cos n(n+1)\theta}{n(n+1)}$$

uniformly in θ, and the last series is uniformly convergent. Similarly for the second given series, since

$$2\cos n^2\theta \sin n\theta = \sin n(n-1)\theta - \sin n(n+1)\theta.$$

The third series converges for some values of θ and diverges for others. Let us put $\theta = p\pi/q$, where p, q are relatively prime positive integers and

$$S(N) = \sum_{n=1}^{N} \sin\frac{n^2 p\pi}{q} \cos\frac{np\pi}{q}.$$

Since $\sin(n^2 p\pi/q)\cos(np\pi/q)$ has period q, we have $S(rq) = rS(q)$ and

$$S(N) = \left[\frac{N}{q}\right]S(q) + S\left\{N - q\left[\frac{N}{q}\right]\right\} = \frac{N}{q}S(q) + f(N).$$

Thus $f(N)$, as just defined, satisfies $|f(N)| \leqq 2q$. Hence, if $\theta = p\pi/q$,

$$\sum_{n=1}^{N} \frac{\sin n^2\theta \cos n\theta}{n} = \sum_{n=1}^{N} \frac{S(n) - S(n-1)}{n}$$
$$= \frac{S(q)}{q}\sum_{n=1}^{N}\frac{1}{n} + \sum_{n=1}^{N}\frac{f(n) - f(n-1)}{n}.$$

As $N \to \infty$ this last series is absolutely convergent by the Dirichlet test. Therefore the given series with $\theta = p\pi/q$ converges or diverges according as $S(q) = 0$ or $S(q) \neq 0$.

By combining the terms in $S(q)$ which correspond to $n = r$ and $n = q - r$, it is easy to see that $S(q) = 0$ for all even q (p is then odd). For $q = 3, 5$; $S(q) \neq 0$, evidently. It is somewhat more difficult to see that $S(q) \neq 0$ whenever q is odd; so that, in every interval, the given series converges for infinitely many values of θ and diverges for infinitely many others.

The problem is given in Bromwich, *Infinite Series*, 1926, p. 68, ex. 16, except for a misprint: the third series is printed whereas the second is intended.

Also solved by Joshua Barlaz, S. Parameswaran, and the Proposer.

Inversion of Order of Summation

4537 [1953, 268]. *Proposed by Albert Wilansky, Lehigh University*

Given that $\sum_n \sum_k a_{nk}x_k$ converges (as an iterated sum) whenever $\{x_k\}$ is a sequence such that $\sum |x_k| < \infty$. Show that $\sum_n \sum_k a_{nk}x_k = \sum_k \sum_n a_{nk}x_k$ for each such $\{x_k\}$.

Solution by R. P. Agnew, Cornell University. The required result and, moreover, necessary and sufficient conditions that a matrix a_{nk} have the property in question, follow immediately from theories of linear transformations originated by Silverman and Toeplitz in 1913 and greatly extended by Hans Hahn (Über Folgen linearer Operationen, *Monatshefte für Math. und Physik*, vol. 32 (1922), pp. 3–88) and others.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to E. P. Vance, Oberlin College, Oberlin, Ohio.

Calculus, A Modern Approach, Second Enlarged Edition. By Karl Menger, The Bookstore, Illinois Institute of Technology, 1953.

Of books on calculus there is no end. Yet seldom, if ever, does one of them make a new and significant contribution to the eternal problem of presenting the fundamental ideas correctly and systematically and "at a level of clarity and simplicity comprehensible to college students of mathematics, science and engineering." This book is exceptional; it is really new. And to some experienced teachers its newness may be disturbing, especially to those who until now have labored under the impression that existing textbooks are all right as they are.

Professor Menger explains in the preface the circumstances which prompted him to write the book: "Most mathematicians consider calculus as a domain about which they know everything. Twenty years ago the author, all in all, concurred in this view. Later, teaching the subject repeatedly and striving for greater clarity of the conceptual framework, he began to realize that he had underestimated the difficulties. In the late 1940's he ventured on a project to rescue calculus from one of the greatest dangers that may befall a discipline—the danger of petrification. A situation which rules out any essential progress can easily arise when specialists abstain from a critical study of books on a subject and at best skim the pages with an eye toward their possible use as texts for sophomore classes. In 1952, even after publishing the first edition of this book, the author felt that he should study calculus further."

The book is an outgrowth of a sequence of works by Professor Menger which have appeared since 1944, beginning with his *Algebra of Analysis*, and which have had as their aim a systematization and clarification of the foundations of analysis. These earlier works were not primarily intended for beginners. The present work is a textbook for beginners and for their instructors. In fact it is through its effect on this latter class that mathematical instruction will benefit most from the ideas which are here presented for the first time in a relatively simple manner and in textbook form.

It is not surprising that a reform of this kind should present striking changes. Among these the most striking at first are those of notation, and the author states in justification that " . . . the principal objective . . . is rather a clarification of the basic ideas and of their applications. In pursuing this aim the author found (to his regret) that changes in the notation were inevitable. In a certain sense these changes are only minor." It becomes clear after reading the book that the invention of the new notation was an essential step toward the clarification of the basic ideas and their applications and is thus amply justified. However, it is likely that these changes will be regarded by many readers as by no means minor ones.

There is so much in the book that is new that it is difficult to give a fair and comprehensive review of its content in a limited space. At the risk of injustice it may be said, briefly, that the main objectives of the book are the following:

(1) To clarify, in a manner which accords with modern points of view, the ideas underlying the basic terms: *number, class, function, variable quantity* (herein designated by V.Q.), *domain of a function, range of a function, domain of a V.Q., range of a V.Q., the derivative* (function) *of a function, anti-derivative* (function) *of a function, integral* (function) *of a function, inverse* (function) *of a function.*

(2) To study the operations performed on functions; the *algebraic* operations and that of *substitution*, and the basic operations of calculus—differentiation, integration—and their reciprocity laws.

(3) To present a new and adequate notation, free from ambiguities and inconsistencies, with which to represent the various concepts enumerated in (1) and (2).

(4) To provide an introduction to the application of these concepts to the physical sciences.

It goes without saying that all of the elementary functions are adequately presented.

Never before, in the reviewer's opinion, have these objectives been pursued in an elementary book with such painstaking care or with as much enthusiasm and vigor. As the author implies, this is no book to be skimmed over. Nearly every page must be read with scrutiny. The careful reader will detect several misprints not included in the list of errata, but these are almost all of the harmless variety or of the kind that will puzzle the reader to a point where he will be rewarded by a better understanding.

In Chapter I the author attacks the problems of differentiation and integration from the geometrical point of view. He defines a *simple curve* (graph) C as one which is intersected in at most one point by any line perpendicular to the X-axis. There is no Y-axis. The two problems are those of finding the slope of the curve C and the area under C. New notation is introduced at the outset:

$$\int_a^b C \text{ (read: the integral of } C \text{ from } a \text{ to } b) = \text{the area under } C \text{ from } a \text{ to } b.$$

$$\mathcal{D}Cx \text{ (read: the derivative of } C \text{ at } x) = \text{the slope of } C \text{ at } x.$$

The horizontal line at the altitude c is denoted by $\underline{c}$. Thus

$$\int_a^b \underline{c} = c(b-a), \qquad \mathcal{D}\underline{c}x = 0.$$

The letter I denotes the line which, for any point x on X, has the altitude x above the point x. Thus

$$Ix = x, \qquad \mathcal{D}Ix = 1.$$

If L is any non-vertical line, there are two numbers c and c' such that the altitude of L above any point x on X is $xc+c'$. If this line is denoted by $cI+c'$ then

$$\mathcal{D}(cI + c')x = \underline{c}$$

for every x on X.

If $\underline{c}$ is the slope line of L, then L is an area line of $\underline{c}$,

$$L \text{ is an } \int \underline{c} \text{ if and only if } \underline{c} = \mathcal{D}L.$$

A line whose slope line or derivative is the horizontal line $\underline{c}$ is called an antiderivative of $\underline{c}$ or $\mathcal{D}^{-1}\underline{c}$.

$$L \text{ is a } \mathcal{D}^{-1}\underline{c} \text{ if and only if } \mathcal{D}L = \underline{c}.$$

Thus every $\int \underline{c}$ is a $\mathcal{D}^{-1}\underline{c}$, and vice versa.

The word function is not discussed in Chapter I, but the notation used for a simple curve (graph) is the same as that used later to represent the corresponding function.

In Chapter II the study of the slope and area problems is continued by first introducing *step lines* and *polygonal lines*, the former being slope curves of the latter, and the latter being area curves of the former. Then the approximate area under a curve is found by drawing a step line, which approximates the given curve, and constructing the polygonal line which is the area curve of the step line. Similarly, the slope curve of the given curve is found approximately by means of an inscribed polygon whose slope curve gives approximately the slope of the given curve.

In Chapter III the student is shown how to perform the calculations corresponding to the approximate graphical solutions found in Chapter II. The formula giving the approximate solution of the area problem is expressed in the form

$$\int_a^b Q = c_1(x_1 - a) + c_2(x_2 - x_1) + \cdots + c_n(b - x_{n-1})$$

where Q is a step line with steps at heights $c_1, c_2 \cdots$. This leads to the fundamental inequalities for estimating the accuracy of the approximate solution, namely

$$\int_a^b \underline{Q_n} \leqq = \int_a^b C \leqq \int_a^b \overline{Q_n}$$

where $\underline{Q}_n$, $\overline{Q}_n$, are step lines whose steps are at heights equal to those of the low-

est and highest points of C, respectively, in the n intervals. These ideas are illustrated by copious exercises and examples.

Chapter IV is entitled *The Idea and the Use of Functions.* This chapter could be read with profit by almost every teacher of elementary calculus. In it the author, with admirable skill, leads the reader toward a thorough understanding of the function concept. He limits himself to functions whose domains are classes of numbers and whose ranges are also classes of numbers. Thus he states, among many examples, "If, with each number, 16 times the square of the number is paired (*e.g.* with 0 the number 0, with 1 the number 16, with 3/2 the number 36), then mathematicians say that a function has been defined. More precisely, the class of the pairs of numbers so defined is called a function. Traditionally, these definitions are expressed by saying that *with each number x the number $16x^2$ is paired.* The function is described as the class of the pairs $(x, 16x^2)$ for all numbers x."

"The pairs (0, 0), (1, 16), (1.5, 36), $(-1.5, 36)$, $(\sqrt{2}, 32)$ belong to the function, the pairs (16, 1), (3, 1/3), $\cdots$ do not, since $16\cdot 16^2 \neq 1$, $16\cdot 3^3 \neq 1/3$, $\cdots$. Clearly the same concept can be defined by saying that with each number t the number $16t^2$ is paired, or by considering the class of pairs $(a, 16a^2)$ for all numbers $a \cdots$. In each pair belonging to the function the *first* element is the number *with* which a number is paired. The class of all these first numbers is called the *domain* of the function. The *second* element in each pair is the number *which* is paired with the first. Each such number is called a *value* of the function . . . . The class of all values of a function is the *range* of the function . . . ."

"Clearly each function is a class of pairs of numbers. Two functions are *equal* if each pair of numbers belonging to the first function also belongs to the second, and each pair belonging to the second function also belongs to the first. Thus the function of which the pairs are $(x, 16x^2)$ for every x is equal to the function whose pairs are $(t, (4t-1)(4t+1)+1)$ for every t. . . . "

"But not each class of pairs of numbers is a function. In order that a class of pairs of numbers be a function it is necessary and sufficient that in no two different pairs of the class are the first elements equal. . . . "

"Can a function be equal to a number? Would anyone confuse a party of married couples with a spinster?"

In this long chapter the basic properties of the elementary functions are studied with unusual care: the constant functions, the functions I, I^n, polynomial functions, the exponential and logarithmic functions, trigonometric functions and their inverses. A great deal of space is devoted to the difficult general concept of the inverse, in which the author makes a clear distinction between pairs of functions f, f^* which are mutually inverse and those in which f^* is inverse to f but f is not inverse to f^*. This involves careful study, mainly geometrical, of the domains and ranges of functions, and is worth while. But the reader will be puzzled at first by the proposal, at the very end of the chapter, to define the class of elementary functions by means of the categories

I. (a) The constant functions $\underline{c}$;
(b) $\exp_{10}$, $\log_{10}$;
(c) arc $\tau = 2$ arc tan, $\tau = \tan\ (I/2)$;
(d) $I^{1/(2n+1)}$, $(n = 1, 2, 3, \cdots)$.

II. Functions which can be obtained from those in (I) by the fundamental operations: addition, multiplication, division, and substitution.

This is elegant but somewhat sophisticated for the beginner. He will have to read on in order to discover that I^c can be proved to be an elementary function in the domain $[>0]$, for every c, and will have to use this result to prove for himself that the function arc sin is elementary:

$$\text{arc sin} = \text{arc}\ \tau\left(\frac{I}{1 + (1 - I^2)^{1/2}}\right).$$

In fact the author gets the wrong answer.

Chapter V, *On Limits*, gives an easy introduction to the theory of limits. The author illustrates, in a common sense manner, all of the essential ideas without actually using deltas and epsilons. He explains precisely what is meant by writing for example

$$\text{if}\quad y \underset{\text{suf}}{\sim} 2 \quad\text{and}\quad z \underset{\text{suf}}{\sim} 6, \quad\text{then}\quad y + z \underset{\text{arb}}{\sim} 8;$$

which is read: "if y is sufficiently close to 2 and z is sufficiently close to 6, then $y+z$ is arbitrarily close to 8." Using this notation he makes it easy for the student to understand what is meant by

$$\lim_{x\to 2} (x^2 - 1) = 3, \qquad \lim_{z\to 2} (z^2 - 1) = 3,$$

or, in his notation,

$$\lim_{2} (I^2 - 1) = 3,$$

and, in general,

$$\lim_{a} f = b.$$

The application of these ideas to the study of continuity is well done and the author takes the opportunity to identify "simple curves" with the graphs of continuous functions.

The author's intention here, as in other chapters, is to make the concepts clear and to state the facts correctly without supplying always the details of a complete proof. For example the number e is investigated by careful computations of the values of the expression $h = (1+x)^{1/x}$ for small values of x. From which, with the use of *tables of logarithms*, it is then made clear that $\log_{10} e$ lies between .434 and .435.

Chapter VI, *The Basic Concepts of Calculus*, deals with derivatives, antiderivatives, and integrals, by utilizing the analysis developed in Chapter V. Here the fundamental theorems of integral calculus are explained and made plausible. Applications involving elementary functions are plentiful and the exercises are good. The chapter ends with a fairly thorough discussion of the Reciprocity Laws:

$$\text{I.}\quad \mathcal{D}\int_a f = f \qquad\qquad (f \text{ continuous}),$$

$$\text{II.}\quad \int_a \mathcal{D}f = f - f_a.$$

Law I is proved. Law II is not proved at this point because the Theorem of the Mean, needed in the proof, does not appear until Chapter X. The reviewer believes that the postponement of this basic theorem is unwise as it is essential to the study of the increase and decrease of functions and of problems in maxima and minima, a subject which has important applications, and which, on account of its appeal to the student, should come as early as possible in the course. It is noticeable that the author introduces Law II with a misleading statement to the effect that it will be proved in a later chapter *for any function f having a derivative* $\mathcal{D}f$. Of course the proof, in Chapter X, is based on the continuity of $\mathcal{D}f$.

In Chapter VII, *Applications to Physical Science*, the author treats the difficult concepts of mensuration, variable quantity, and function, as applied to physical sciences. It is not easy to give an adequate idea of the effectiveness with which these concepts are developed. By easy stages, starting with a class Σ of elements $\sigma, \sigma', \cdots$ of undetermined nature, which can be *compared with each other* by means of the usual order and equivalence relationships $\sigma \precsim \sigma'$, $\sigma \sim \sigma'$, there is established a pairing of the elements of Σ with numbers of a numerical scale so that with each element σ of Σ a number $n\sigma$ is paired—just as in the definition of function—but such that if $\sigma \precsim \sigma'$ then $n\sigma \leqq n\sigma'$, and $n\sigma = n\sigma'$ if and only if $\sigma \sim \sigma'$. By postulating the additive property for the elements of Σ and the existence of a unit element σ_1 (an element with which the number 1 is paired), there is established a *mensuration scale*. Thus, if $\sigma \sim \sigma_1 + \sigma_1 + \sigma_1$, then with σ is paired the number 3 and one writes $\sigma = 3\sigma_1$; and if $\sigma + \sigma + \sigma = \sigma_1 + \sigma_1$, then with σ the number $\frac{2}{3}$ is paired and one writes $\sigma = \frac{2}{3}\sigma_1$. And by an obvious extension the statement $\sigma = \sqrt{2}\sigma_1$, is defined. The ideas are copiously illustrated of course. The notion of the unit element is illustrated, for example, by such common statements of mensuration as:

"The rod R is 7.5 ft.," "the weight P is $\frac{1}{2}$ lb.,"

which yield the corresponding equivalent formulae:

$$\text{(a)} \qquad R = 7.5 \text{ ft.,} \qquad P = \tfrac{1}{2} \text{ lb.}$$

Each such formula is an equality of an observable and a so-called denominate

number (a pure number followed by an observable comparable with the first). Menger then defines a preferred notation, which consists in replacing $\sigma' = k\sigma$ by $(\sigma' \text{ in } \sigma) = k$. So that, in the example, (a) above is replaced by:

$$\text{(b)} \qquad (R \text{ in ft.}) = 7.5, \qquad (P \text{ in lbs.}) = \tfrac{1}{2}.$$

By easy stages we then arrive at the *Fundamental Law of Mensuration*:

$$\begin{array}{ll} \textit{If} & (\sigma' \text{ in } \sigma) = a \\ \textit{and if} & (\sigma'' \text{ in } \sigma') = b, \\ \textit{then} & (\sigma'' \text{ in } \sigma) = ab. \end{array}$$

The author is now prepared to answer the question "What is a variable as studied in Science?" The answer is:

"If a number is paired with each element of a class Σ—the same number with any two equivalent elements—then we call the class of pairs (element, number paired with the element) a *variable*, more specifically, a variable with the domain Σ. For example, if Σ is additive and a unit σ_1 has been selected in Σ, the class of pairs

$$(\sigma, \sigma \text{ in } \sigma_1) \qquad \text{for all elements } \sigma \text{ of } \Sigma$$

is a variable with the domain Σ. More generally, every numerical scale in a class Σ of comparable elements gives rise to a variable, namely the class of pairs $(\sigma, n\sigma)$, for all σ, where $n\sigma$ is the label of σ in the numerical scale. The numbers paired with the elements are called the *values* of the variable, and the class of all values of the variable its *range*."

Thus, for example, if in the class of rods having length, a foot is selected as the unit, for each rod ρ we have the pairing, $(\rho, \rho \text{ in ft.})$, and the class of all such pairings, for all rods ρ, is an *observable variable*, called the length of rods. The variable is denoted by the letter l. Its values are $l\rho = (\rho \text{ in ft.})$.

Functions are the only variables whose domains are numbers. From this it follows that into a function a variable can be substituted provided that the range of the variable is a part of the domain of the function.

The chapter culminates in a penetrating exposition of the central question: When is a variable m a function of another variable n?

The fundamental idea comes from the definition of functionally related classes: Let Σ and T be two classes. A class of pairs (τ, σ) such that τ belongs to T and σ belongs to Σ is called a *pairing of* Σ *with* T. Σ is *functionally related to* T *with regard to the pairing* Π if any two elements of Σ which are paired with two equivalent elements of T are equivalent; in other words, if whenever (τ, σ) and (τ', σ') belong to Π and $\tau \sim \tau'$ then $\sigma \sim \sigma'$.

This gives the central theorem: If the class Σ is functionally related to the class T with regard to the pairing Π, and numerical scales are established, pairing labels $m\sigma$ and $n\tau$ with all elements of Σ and T respectively, then the variable m is a function of the variable n with regard to the pairing Π; that is to say,

$m=fn$ for some function f. [There is an unfortunate misprint in the text, not noted in the *Errata*; the result is given as $n=fm$.]

The rest of this long chapter is devoted to the study of derivatives and integrals of functions connecting variable quantities. To illustrate: Let s be the distance travelled by a particle (in units of length), t the time elapsed since the beginning of the observation of the motion (in units of time), s_t the function connecting s with t. If τ, τ', are the instants of time at which t has the values x, x', respectively, σ, σ', the positions corresponding to those instants of time, so that

$$s\sigma = s_t t\tau, \qquad s\sigma' = s_t t\tau',$$

then the difference quotient

$$\frac{s\sigma' - s\sigma}{t\tau' - t\tau} = \frac{s_t t\tau' - s_t t\tau}{t\tau' - t\tau} = \frac{s_t x' - s_t x}{x' - x}$$

is the average speed of the particle between x and x' (that is, between the instants τ and τ'). If this quotient has a limit as $x' \to x$ then this limit, which is $\mathcal{D}s_t x$, is called the *instantaneous speed* of the particle *at* x (that is, at the instant τ). In symbols,

$$\mathcal{D}s_t x = \lim_{x' \to x} \frac{s\sigma' - s\sigma}{t\tau' - t\tau} = \lim_{x' \to x} \frac{s_t x' - s_t x}{x' - x}$$

is the instantaneous speed at x. Using v to represent the variable quantity *instantaneous speed of the particle*, then

$$v_t x = \mathcal{D}s_t x, \qquad \text{for every number } x,$$

or

$$v_t = \mathcal{D}s_t.$$

Another way to describe this relation is to say that:

If $s_t = f$, then $v_t = \mathcal{D}f$. For example;

If $s_t = 16I^2$ then $v_t = 32I$.

These are functions. The traditional way of expressing these ideas would be: If f is the function connecting s with t then

$$s = ft, \qquad v = \mathcal{D}ft.$$

These quantities are variables. In particular:

$$\text{if} \quad s = 16t^2 \quad \text{then} \quad v = 32t.$$

To illustrate the use of integrals the author considers, among other examples, the work done by a force p acting on a particle (the rectilinear case). If p_s is the function connecting p with the distance travelled s, we have

$$_a w_s = \int_a p_s.$$

This is *the work function from a on.* The *variable* $_a w$ *is the work from a on.*

$$_a w = \left(\int_a p_s\right) s.$$

For example, if $p_s = -I$ then $\int_a(-I) = -\frac{1}{2}I^2 + \frac{1}{2}a^2$ and

$$_a w = -\tfrac{1}{2}s^2 + \tfrac{1}{2}a^2.$$

More generally, if u and v are two variables, v a function of u, then $_a w_u = \int_a v_u$ is called the *cumulation function of v with u from a on.* The variable $_a w$, *the cumulation* of v with *u from a on*, is connected with u by the cumulation function as follows:

$$_a w = \left(\int_a v_u\right) u.$$

The rest of the book need not be discussed in detail. In fact the most important parts are those, already discussed, in which the fresh point of view of the author is set forth. In the remaining chapters the ideas presented in the first seven chapters are applied to the techniques of calculus which, in turn, are applied to problems in analytic geometry and mechanics.

Chapter VIII, *The Calculus of Derivatives*, gives a thorough exposition of the techniques used in calculating the derivatives of functions expressed in terms of differentiable functions by means of the fundamental operations (addition, multiplication, taking the reciprocal, and substitution).

The operation of differentiation is represented by the symbol $\mathcal{D}$ when applied to a pure function. $\mathcal{D}f+g$ means $(\mathcal{D}f)+g$, $\mathcal{D}f\cdot g$ means $(\mathcal{D}f)\cdot g$, $\mathcal{D}fg$ means $(\mathcal{D}f)g$; the derivatives of $f+g, f\cdot g, fg$ are denoted by $\mathcal{D}(f+g)$, $\mathcal{D}(f\cdot g)$, $\mathcal{D}(fg)$.

One of the main features of this chapter is the illuminating discussion of the meaning of the various notations used in representing the derivative of a function and the derivative of one variable with respect to another (with which the first is functionally related).

Chapter IX, *The Calculus of Antiderivatives*, is a lucid exposition of methods of finding elementary antiderivatives. It affords thorough training in the basic techniques and in the use of the Menger notation.

Chapter X, *The Mean Value Theorem and Its Consequences*, begins with a common sense treatment of the approximate value of a function near a point a by means of its value at a and the value of the derivative at a. This is followed by a Mean Value Theorem, which is proved by means of Rolle's Theorem which, in turn, is rendered plausible by a clear geometrical argument. The chapter culminates in Taylor's Theorem with the Lagrange form of the remainder. The rest

of the chapter deals with the usual applications to computation with due attention to estimations of the remainder term.

Chapter XI, *Two-Place Functions or Functions of Two Variables.* This final chapter introduces new notation for the study of functions of several variables. Capital letters are used to denote two-place functions and all one-place functions are denoted by lower case letters. Here I is the two-place function such that $I(x, y)=x$ for every (x, y), J is the two-place function such that $J(x, y)=y$ for every (x, y). The one-place function denoted in the preceding chapters by I, such that $It=t$ for every t is now denoted by ι. Graphs (surfaces) of simple two-place functions are studied. The idea of substituting one-place and two-place functions in a two-place function is given careful treatment. This idea is applied to the differentiation of one-place functions defined implicitly. Partial differentiation of functions is expertly presented but only the simpler techniques are treated in the text proper. The differentiation of a one-place function obtained by substituting two one-place functions in a one-place function is the only case of the Substitution Rule considered. There is an interesting but brief Appendix devoted to the study of partial differentiation of n-place functions and a statement of the general Substitution Rule.

There is little doubt that this book deserves the most careful attention of teachers of calculus. It is written in the lively, vigorous style which is characteristic of the author and on this account it should appeal to any earnest student. Opinions among teachers may vary widely as to the seriousness of the defects of the traditional point of view of which the author complains. But there can be little difference of opinion as to the effectiveness of the remedies which he supplies. The fact is that even an experienced teacher who reads the book with care will probably find that he has acquired a deeper insight into the significance of the fundamental ideas and a better understanding of the art of presenting them to beginners. It must not be dismissed with the pointless remark that it is a book on function-theory, not calculus. The two are inseparable. But there is a debatable question: How deeply should one go into the foundations of analysis in writing a book for beginners? Some teachers prefer a book that skims over the surface of the subject because it affords them free opportunity to present the deeper ideas in the class room according to their own tastes. This is not that kind of book. But it still leaves opportunities for the teacher to introduce supplementary material, since much that is traditionally included in a calculus book is omitted; for example, drill exercises and problems of a so-called practical nature drawn from geometry, mechanics, *etc.*

Unfortunately the typography of the book is unattractive, but this fault will undoubtedly be remedied by an enterprising publisher who will print the book in the attractive format in which so many lesser books are now being offered.

H. E. BRAY
The Rice Institute

NEW BOOKS RECEIVED

History of the Theories of Aether and Electricity. By Sir Edmund Whittaker. New York, Philosophical Library, 1954. xi+319 pages. $8.75.

College Algebra. New Third Edition. By P. K. Rees and F. W. Sparks. New York, McGraw-Hill Book Company, Inc., 1954. xiii+460 pages. $4.25.

Analytic Geometry. Second Edition. By E. S. Smith, Meyer Salkover and H. K. Justice. New York, John Wiley and Sons, Inc., 1954. xiii+306 pages. $4.00.

Understanding College Algebra. By E. R. Smith, Samuel Selby and Murray Kleiman. New York, The Dryden Press, 1954. xvi+571 pages. $3.50.

Elements of Statistics. By H. C. Fryer. New York, John Wiley and Sons, Inc., 1954. vi+262 pages. $4.75.

An Introduction to the Calculus of Finite Differences. By C. H. Richardson. New York, D. Van Nostrand Company, Inc., 1954. v+142 pages.

Theory of Equations. By C. C. MacDuffee. New York, John Wiley and Sons, Inc., 1954. vii+120 pages. $3.75.

Tables of Integral Transforms. By A. Erdélyi. New York, McGraw-Hill Book Company, Inc., 1954. xx+391 pages. $7.50.

Matter Energy Mechanics. By Jakob Mandelker. New York, Philosophical Library, 1954. ix+73 pages. $3.75.

First Course in Calculus. By H. R. Cooley. New York, John Wiley and Sons. Inc., 1954. xii+643 pages. $6.00.

Introductory College Mathematics. By Adele Leonhardy. New York, John Wiley and Sons, Inc., 1954. ix+459 pages. $4.90.

Introductory College Mathematics. By C. G. Jaeger and M. M. Bacon. New York, Harper and Bros., 1954. xii+382 pages. $4.75.

The Mechanism of Economic Systems. By Arnold Tustin. Cambridge, Harvard University Press, 1954. xi+161 pages. $5.00.

Number. Fourth Edition. By Tobias Dantzig. New York, The Macmillan Company, 1954. ix+340 pages. $5.00.

Grundzuge der Ausgleichungsrechnung. By Walter Grossmann. Berlin, Germany, Springer-Verlag, 1953. 261 pages. $4.56.

The Japanese Abacus, its Use and Theory. By Takashi Kojima. Rutland, Vermont, Charles E. Tuttle Company, 1954. 102 pages. $1.25.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

SALE OF MATHEMATICAL LIBRARY

The library of the late Professor Otto Szász is available for sale. It contains over 100 mathematical books of importance, complete sequences of the *Bulletin of the American Mathematical Society*, the *Journal of the London Mathematical Society*, and the *Proceedings of the London Mathematical Society* from 1933–1952, and other useful periodicals. To be mentioned also are the Proceedings of various mathematical congresses from 1912 on. In addition, about 4500 reprints are available. Interested persons may contact Mrs. Brigitta Dobratz, 2911 Knoll Drive, Concord, California.

PERSONAL ITEMS

The following have received Guggenheim Fellowships for 1954–55 for the studies indicated:

Professor C. B. Boyer, Brooklyn College, studies in the history of the theory of the rainbow;

Associate Professor D. G. Chapman, University of Washington, studies of the size and dynamic structure of wild animal populations;

Professor S. S. Chern, University of Chicago, studies in the field of differential geometry;

Professor M. R. Hestenes, University of California, Los Angeles, study of the applications of the theory of linear spaces to the calculus of variations;

Associate Professor E. R. Kolchin, Columbia University, study of certain questions in the Galois theory of differential fields;

Professor C. C. Lin, Massachusetts Institute of Technology, studies in the field of theoretical fluid dynamics;

Assistant Professor F. I. Mautner, Johns Hopkins University, research on unitary representations of locally compact topological groups;

Professor Hans Rademacher, University of Pennsylvania, studies of generalized modular functions belonging to real quadratic fields;

Assistant Professor M. A. Rosenlicht, Northwestern University, study of non-complete group varieties;

Research Professor A. H. Taub, University of Illinois, studies of the gravitational field in relativity and of curved shocks in hydrodynamics;

Professor Alexander Weinstein, University of Maryland, study of initial and boundary value problems for singular hyperbolic equations and for partial differential equations of mixed type.

National Science Foundation Postdoctoral Fellowships have been awarded to the following: G. E. Baxter, University of Minnesota; R. C. Blanchfield,

Princeton University; D. A. Buchsbaum, Columbia University; C. W. Curtis, Yale University; Jacob Feldman, Columbia University; W. M. Huebsch, University of Notre Dame; A. P. Mattuck, Princeton University; J. C. Moore, Princeton University; Edgar Reich, University of California; M. F. Ruchte, University of Wisconsin; V. L. Shapiro, Institute for Advanced Study; Edwin Weiss, Massachusetts Institute of Technology.

Professors B. W. Jones of the University of Colorado and Einar Hille of Yale University represented the Mathematical Association of America at the International Congress of Mathematicians which was held in Amsterdam, The Netherlands, September 2–9, 1954.

Assistant Professor A. N. Aheart of West Virginia State College represented the Association at the inauguration of President W. J. L. Wallace of the College on April 9, 1954.

Professor H. J. Ettlinger of the University of Texas was the representative of the Association at the inauguration of President D. H. Morgan of Texas Agricultural and Mechanical College on May 20, 1954.

Professor Fritz John of the Institute for Mathematics and Mechanics, New York University, was appointed to represent the Association at the Second University Convocation of the Bicentennial Year of Columbia University which was held on June 1, 1954.

Brown University announces the following; Dr. John Wermer has been appointed to an assistant professorship; Assistant Professor Joanne Elliott of Mt. Holyoke College has been appointed Visiting Assistant Professor for the year 1954–55; Professor R. E. Gilman has retired with the title of Professor Emeritus. The University reports also that a program of instruction leading to the Bachelor of Science in Applied Mathematics will be offered beginning Fall, 1954; Professor E. H. Lee, chairman of the Division of Applied Mathematics, is in charge of the program.

At Montana State University: Dr. A. S. Merrill, dean of the faculty and professor of mathematics, has been appointed Vice-President of the University while retaining his other titles; Professor Harold Chatland, chairman of the Department of Mathematics, has been appointed Dean of the College of Arts and Sciences.

Randolph-Macon Woman's College announces: Professor Evelyn P. Wiggin is on leave during the academic year 1954–55 and is studying at the University of Chicago; Miss Sara L. Ripy has been appointed to an instructorship.

Rutgers University reports the following: Assistant Professor R. M. Cohn has been promoted to an associate professorship; Dr. B. H. McCandless, Dr. Solomon Leader, Dr. V. L. Shapiro, and Dr. K. G. Wolfson have been promoted to assistant professorships; Mr. N. R. Stanley is transferring from the New Brunswick Branch to the Newark Colleges of the University; Mr. A. G. Wootton has resigned.

University of Connecticut announces the following appointments: Dr. Allen Devinatz of the Institute for Advanced Study to an assistant professorship; Dr.

R. L. Ingraham of the Institute for Advanced Study and Dr. H. C. Griffith of the University of Tennessee to instructorships.

University of Illinois announces: Professor D. G. Bourgin has a Fulbright Grant to Rome for the academic year 1954–55; Professor S. S. Cairns has a Fulbright Grant to the University of Strasbourg in France and a sabbatical leave for the year 1954–55; Assistant Professor Irving Reiner is on sabbatical leave during 1954–55; Assistant Professor Lowell Schoenfeld is on sabbatical leave during 1954–55 and is dividing his time between the University of Pennsylvania and the Institute for Advanced Study; Dr. Alex Heller of Harvard University has been appointed to an assistant professorship.

At the University of Wisconsin: Associate Professor R. C. Buck has been promoted to a professorship; Assistant Professor W. F. Eberlein has been promoted to an associate professorship; Dr. R. C. T. Smith of New England University College, New South Wales, has been appointed Visiting Lecturer for the academic year 1954–55; Assistant Professor Elizabeth Sokolnikoff Hirschfelder has resigned.

University of Wichita reports the following: Assistant Professors Harold Huneke and Ferna Wrestler have been promoted to associate professorships; Miss Jeneva J. Brewer has been promoted to an assistant professorship; Miss Ann Klein has been appointed to an instructorship.

Wayne University announces the following appointments to instructorships: Mr. W. S. Bicknell, Mr. Charles Briggs, Dr. Shu-Teh C. Moy, and Dr. E. S. Northam.

Associate Professor B. H. Bissinger of Lebanon Valley College has been elected Chairman of the Department of Mathematics.

Mrs. Elizabeth B. Bockelman, formerly with the Acoustics Research Laboratory of Harvard University, has been appointed to an instructorship at Wellesley College.

Professor H. K. Brown of Northeastern University has been promoted to the position of Dean of the Graduate Division.

Associate Professor D. E. Christie has returned to Bowdoin College after a year's leave of absence as a visiting Fellow at Princeton University.

Mr. Albert Derin of Kansas State College is now in Military Service.

Assistant Professor D. G. Duncan of the University of Arizona has accepted an appointment at San Jose College.

Mr. J. J. Gehrig of the Ballistics Research Laboratories, Aberdeen Proving Ground, has accepted a position as research engineer with Northrop Aircraft, Incorporated, Hawthorne, California.

Mr. M. N. Haller, Jr., previously an engineer with Southern Bell Telephone and Telegraph Company, is now engaged as an engineer by Capehart-Farnsworth Company, Fort Wayne, Indiana.

Mr. P. W. M. John, who has been teaching at the Casady School, Oklahoma City, has been appointed to an instructorship at the University of Oklahoma.

Mr. H. T. LaBorde of the University of North Carolina has been appointed

to an assistant professorship at the University of the South.

Dr. H. N. Laden has accepted a position as senior methods research officer with the Chesapeake and Ohio Railway Company, Cleveland, Ohio.

Mr. P. M. Landry, formerly with the Vitro Corporation, Fort Walton, Florida, is now in Military Service.

Dr. H. M. MacNeille, who has resigned as Executive Director of the American Mathematical Society effective September 1, 1954, has been appointed Professor and Chairman of the Department of Mathematics of Washington University.

Associate Professor R. R. Otter has returned to the University of Notre Dame after a year as Visiting Professor at Princeton University.

Assistant Professor L. J. Paige, who spent the year 1953–54 at the Institute for Advanced Study, has returned to the University of California.

Associate Professor T. S. Peterson, Portland State Extension Center, has been promoted to a professorship.

Mr. S. L. Pollack of the National Bureau of Standards has accepted a position as mathematician with the Raytheon Manufacturing Company, Waltham, Massachusetts.

Dr. Edgar Reich of the Rand Corporation is spending the academic year 1954–55 at the Institute for Advanced Study as a National Science Foundation fellow.

Mr. Yomei Sawanobori, formerly at the Cornell Aeronautical Laboratory, Buffalo, New York, has a position as a mathematician with the I. B. M. Corporation, New York City.

Dr. Alice T. Schafer has been appointed to an assistant professorship at Connecticut College.

Assistant Professor M. R. Spiegel of Rensselaer Polytechnic Institute has been promoted to an associate professorship.

Dean G. W. Starcher of the College of Arts and Sciences of Ohio University has been appointed President of the University of North Dakota.

Assistant Professor B. B. Townsend of Louisiana State University has been promoted to an associate professorship.

Professor J. F. Wampler, previously professor of mathematics and registrar at York College, has been appointed to an associate professorship at Nebraska Wesleyan University.

Dr. R. C. Briant, project director at Oak Ridge National Laboratory, died on April 25, 1954. He had been a member of the Association for twenty-seven years.

Professor Emeritus H. S. Brown of Hamilton College died on March 27, 1954. He was a charter member of the Association.

Professor J. A. Daum of Texas Agricultural and Mechanical College died on March 23, 1954.

Professor Emeritus Arnold Dresden of Swarthmore College died on April 10,

1954. He had served as President of the Association during 1933–34, as Vice-President during 1931, and as a member of the Board of Governors for the periods 1935–40 and 1943–45.

Professor Emeritus William Findlay of McMaster University died on May 7, 1953. He was a charter member of the Association.

Professor D. L. Holl, head of the Department of Mathematics of Iowa State College, died on May 22, 1954. He was a member of the Iowa Section for nine years. He served as Section Chairman in 1950–51 and had been elected Governor of the Iowa Section for the period 1953–56.

Professor E. L. Post of City College of the City of New York died on April 21, 1954. He had been a member of the Association for thirty-one years.

Mr. J. E. Sanders, retired meteorologist of Knoxville, Tennessee, died on April 8, 1953. He was a charter member of the Association.

Professor J. M. Stetson, William and Mary College, who was a charter member of the Association, died on May 5, 1954.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW SECTIONAL GOVERNORS OF THE ASSOCIATION

The following have been elected Governors of the Association for a three-year term beginning July 1, 1954 by a mail vote of the membership of the Association in the Sections indicated:

Allegheny Mountain	Morris Ostrofsky, Westinghouse Electric Corp.
Indiana	Ralph Hull, Purdue University
Kentucky	H. H. Downing, University of Kentucky
Metropolitan New York	R. M. Foster, Polytechnic Institute of Brooklyn
Nebraska	M. A. Basoco, University of Nebraska
Northern California	W. H. Myers, San Jose State College
Oklahoma	L. W. Johnson, Oklahoma A. and M. College
Rocky Mountain	C. A. Hutchinson, University of Colorado
Wisconsin	R. C. Huffer, Beloit College

Over 50% of the members eligible to vote actually cast their votes in the Kentucky Section and the Wisconsin Section. In general, a high percentage of votes is cast in the elections of Sectional Governors, thus indicating the desire of members of the Association to have their Sections represented by outstanding candidates.

H. M. GEHMAN, *Secretary-Treasurer*

THE 1954 COMBINED MEMBERSHIP LIST

Each year the Mathematical Association of America and the American Mathematical Society issue a Combined Membership List. The 1954 edition of this List will be sent to members of the Society.

The office of the Association should be notified promptly of all changes in rank, position, and address which have not previously been reported. The final date for receipt of changes is October 15. Any errors in the 1953 Membership List should also be reported before October 15.

H. M. GEHMAN, *Secretary-Treasurer*

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 89 persons have been elected to membership by the Board of Governors on applications duly certified.

S. S. ABHYANKAR, A.M. (Harvard) Teaching Fellow, Harvard University.

A. G. ASHOOK, A.B. (Boston C.) Teacher, Oliver W. Holmes Jr. High School, Dorchester, Massachusetts.

SHELDON BALK, Student, Arizona State College, Tempe.

G. E. BARLOW, JR., B.S. (V.P.I.) Assistant County Agent, Gate City, Virginia.

G. C. BARTON, B.A. (U. of Washington) Teacher, Concrete Senior High School, Washington.

R. S. BEARD, M.S. (Kansas) Col. U.S.A. retired; Part-time Prof., Yeshiva Institute of Mathematics.

C. P. BENNER, M.S. (C.I.T.; Houston) Asst. Prof., University of Houston.

H. F. BENNETT, B.S. (Illinois Wesleyan) Patent Agent, Eastman Kodak, Rochester, N. Y.

RENE BLOCH, Student, University of Basel, Switzerland.

J. D. BUCKHOLTZ, Student, Little Rock Junior College.

REV. FR. FLORENT CARTUYVELS, Lic. in Math. (Louvain) Teacher, St. Xavier's College, Ranchi, India.

C. P. CEBULLA, M.Ed. (St. Thomas) Instr., Hermantown High School, Duluth, Minn.

J. R. COX, Student, Lebanon Valley College.

J. B. CRAWFORD, B.S. (U. S. Military Academy) Asst. Prof., Alabama Polytechnic Institute.

C. L. DAVIS, M.A. (Michigan) Instr., General Motors Institute, Flint, Michigan.

D. E. DEAN, B.S. (Akron) Technical Supervisor, Firestone Tire & Rubber, Akron, Ohio.

T. P. DENNEHY, M.S. (Notre Dame) Instr., John Carroll University.

R. S. DICK, Student, Queens College.

OLIVE A. DOUGHTY, B.S. (Northeast La. S.C.) Mathematician, David Taylor Model Basin.

B. M. DRUCKER, Ph.D. (North Carolina) Asst. Prof., Georgia Institute of Technology.

G. E. DUNCAN, Student, Georgia Institute of Technology.

G. A. EASTERLY, B.S. (Allen) Teacher, Morristown College High School, Tennessee.

E. J. FARRELL, M.A. (Stanford) Asst. Prof., University of San Francisco.

M. A. FELDSTEIN, Student, Arizona State College, Tempe.

C. T. FIKE, Student, University of the South.

L. R. FORD, JR., Ph.D. (Illinois) Research Instructor, Duke University.

L. R. FOY, M.S. (St. John's) Instr., Marian College, Poughkeepsie, N. Y.

J. D. GATES, A.B. (Hiram) Teacher, Maple Heights High School, Ohio.

J. B. GOEBEL, Student, University of Oregon.

S. W. GOLOMB, M.A. (Harvard) Graduate Student, Harvard University; Instr., Boston University.

T. E. HAGENSEE, M.S. (DePaul) Teacher, Crane Technical High School, Chicago, Ill.

D. I. HAMMER, M.A. (Columbia) Instr., Montclair State Teachers College.

W. G. HANKS, Student, University of Manitoba.

M. E. HARRIS, Student, Yale University.

H. L. HENDRICKS, B.A. (Pomona) Teacher, Pasadena City College.

PFC. R. A. HULTQUIST, M.S. (Purdue) Research Assistant, Fort Sill, Oklahoma.

REV. L. E. ISENECKER, S.J., Ph.B. (Loyola) Graduate Student, Catholic University.

DIANE M. JOHNSON, Student, University of Manitoba.

VIRGINIA L. JOHNSON, Student, Harpur College.

MRS. CATHERINE S. KAY, M.A. (Michigan) Instr., North Central College, Naperville, Illinois.

C. S. KIMBALL, Tech. Representative, Philco Corporation, Philadelphia, Pa.

J. E. KIST, M.S. (Purdue) Research Assistant, Purdue University.

S. M. KLEINMAN, Brooklyn, New York.

DIANE L. KOCHER, Student, Harpur College.

A. R. LAMONTAGNE, M.A. (New Hampshire) Graduate Assistant, University of New Hampshire.

ADELE LEONHARDY, Ed.D. (Missouri) Chairman of Dept., Stephens College.

K. R. LUCAS, B.S. (Washburn) Graduate Student, University of Kansas.

DALE MANESS, M.A. (Vanderbilt) Asst. Prof., Baylor University.

M. D. MARCUS, A.B. (California) Research Assistant, University of California at Berkeley.

H. J. MCBLAINE, JR., M.S. (Illinois) Training Instr., Meteorology, Dept. of Weather School, Chanute AFB, Ill.

WILLIAM MCKAY, B.S. (Drexel) Instr., Drexel Institute of Technology.

D. E. MCOWEN, M.A. (Michigan) Instr., General Motors Institute, Flint, Michigan.

B. T. MENDELSOHN, Student, City College of New York.

A. A. MULLIN, Student, Syracuse University.

E. R. MULLINS, JR., Ph.D. (Illinois) Asst. Prof., Swarthmore College.

W. E. NANCE, Student, University of the South.

J. H. ORTMAN, B.A. (Wisconsin) Mathematician, Douglas Aircraft Co., Santa Monica, California.

G. K. OVERHOLTZER, Ph.D. (Indiana) Asst. Prof., Georgia Institute of Technology.

J. L. PALLONE, A.B. (Fordham) Asst. Actuary, Woodward & Fondiller, New York City.

N. C. PERRY, Ph.D. (U.S.C.) Asst. Prof., Alabama Polytechnic Institute.

W. L. PHILLIPS, JR., Student, Yale University.

T. B. PITTS, B.A. (Texas) Instr., University of Texas.

S. D. RABUSHKA, Student, Washington University.

P. A. REICHLE, M.S. (Pennsylvania) M.A. (Columbia) Col., U.S.A., retired Asst. Prof., Lenoir Rhyne College.

MRS. SUSAN L. REID, Student, Queens College.

MRS. JOY B. RUSSEK, M.S. (New York Univ.) Instr., University of Buffalo.

N. B. SMITH, M.A. (Oregon S. C.) Graduate Student, Iowa State College.

LAWRENCE SOKOLOFF, B.A. (Brooklyn) Jr. Mathematician, Republic Aviation Corp.

J. M. STARK, S.M. (M.I.T.) Graduate Student, Massachusetts Institute of Technology.

R. F. STEINHART, M.A. (Columbia) Instr., Montclair State Teachers College.

W. A. STOCK, M.A. (Northwestern) Instr., Purdue University.

C. R. STRAIN, M.S. (Purdue) Teaching Fellow, University of Michigan.

C. M. STUART, M.A. (Duke) Asst. Prof., Clemson College.

A. C. SUGAR, Ph.D. (California) Consultant, Ryan Aeronautical Co., San Diego, Calif.

F. B. TAYLOR, A.M. (Columbia) Asst. Prof., Manhattan College.

L. C. TENG, M.A. (Michigan) D.I.C. Staff, Massachusetts Institute of Technology.

E. A. TRABANT, Ph.D. (C.I.T.) Assoc. Prof., Purdue University.

C. K. TSAO, Ph.D. (Oregon) Asst. Prof., Wayne University.

J. J. URBANCEK, M.A. (Northwestern) M.S. (DePaul) Chairman, Chicago Teachers College.

W. A. VEZEAU, Ph.D. (St. Louis) Assoc. Prof., St. Louis University.

G. E. VOYLES, B.A. (North Texas S. C.) Teacher, Decatur Baptist College.

E. A. WALKER, M.A. (Sam Houston) Graduate Student, University of Kansas.

T. C. WALLACE, Student, Arizona State College, Tempe.

IRENE S. WELNA, Student, St. Joseph College.
ROY WESTWICK, Student, University of British Columbia.
E. F. WHITTLESEY, A.B. (Princeton) Instr., Bates College.
OSWALD WYLER, Sc.D. (Swiss Federal Inst. of Tech.) Asst. Prof., University of New Mexico.
MONICA J. WYZALEK, Student, Harpur College.
H. J. ZASSENHAUS, Ph.D. (Hamburg) Prof., McGill University.

THE MARCH MEETING OF THE METROPOLITAN NEW YORK SECTION

The thirteenth annual meeting of the Metropolitan New York Section of the Mathematical Association of America was held at St. John's University, Brooklyn, New York, on March 27, 1954. Professor H. F. Fehr, Collegiate Vice-Chairman of the Section, presided at the morning session and Professor W. H. Fagerstrom, Chairman of the Section, presided at the afternoon session.

One hundred two persons attended the meeting, including the following seventy-five members of the Association:

R. G. Archibald, C. Y. Bartholomew, Jonas Beraru, Samuel Borofsky, C. B. Boyer, A. B. Brown, W. F. Cassidy, G. B. Charlesworth, K. P. Clancy, Charles Clos, P. J. Cocuzza, T. F. Cope, Demetrios Counes, W. H. H. Cowles, I. A. Dodes, J. N. Eastham, W. H. Fagerstrom, H. F. Fehr, J. M. Feld, A. B. Finkelstein, A. D. Fleshler, William Forman, R. M. Foster, G. C. Francis, D. H. Frank, E. T. Frankel, E. J. Germino, William Gonzalez, Bernard Greenspan, Harriet Griffin, W. T. Hamilton, C. M. Hebbert, G. C. Helme, E. Marie Hove, Aida Kalish, G. W. Kays, L. S. Kennison, H. S. Kieval, A. E. Kinney, M. S. Klamkin, Charles Koren, Edna Kramer-Lassar, C. H. Lehmann, D. R. Lintvedt, D. M. MacEwen, May H. Maria, Emanuel Mehr, F. H. Miller, Morris Morduchow, A. J. Mortola, Eugene Odin, Maria T. Pan, J. J. Quinn, J. K. Reckzeh, R. M. Reed, H. D. Ruderman, J. P. Russell, John Salerno, Charles Salkind, A. H. Sarno, Seymour Schuster, Abraham Schwartz, Aaron Shapiro, Edward Shapiro, James Singer, Geraldine D. Smith, Morris Smith, E. R. Stabler, Mildred M. Sullivan, R. L. Swain, P. M. Treuenfels, R. M. Walter, Alan Wayne, M. E. White, R. C. Yates.

The following officers were elected for the year 1954–55: Chairman, Professor H. F. Fehr, Teachers College, Columbia University; Collegiate Vice-Chairman, Professor A. B. Brown, Queens College; High School Vice-Chairman, Mr. D. H. Frank, Forest Hills High School; Secretary, Professor E. Marie Hove, Hofstra College; Treasurer, Mr. Aaron Shapiro, Midwood High School, Brooklyn.

At the business meeting, the following report on the activities of the Committee on Contests and Awards for the year 1952–53 was given by its chairman, Professor W. H. Fagerstrom:

The fourth annual contest was held on May 14, 1953. There were 476 schools registered for the contest. Of these, 291 were in the Metropolitan New York area, and the other 185 were distributed throughout 30 states and provinces. Among the schools in this latter group were those in the units operated by British Columbia, Oregon and Western N. Y. each conducting its own state wide contest using the contest questions of the Metropolitan New York Section.

The maximum score available for the school was 450 points, and the maximum for the individual was 150 points. The three highest ranking schools were:

James Madison High School, Brooklyn, N. Y., score 349; Phillips Academy, Andover, Mass., score 336; Talmudical Academy, New York, N. Y., score 325. The three highest ranking students were: E. Alan Phillips, Weston High School, Middlesex, Mass., score 130; Jerrold Rubin, James Madison High School, Brooklyn, N. Y., score 126; Isaac J. Sharon, Talmudical Academy, New York, N. Y., score 126. There were 28 recipients of Certificates of Merit. These certificates are awarded to the highest ranking student in each area, provided his score was 85 or more. Honor keys, given in recognition of the students having won the award for two consecutive years, were awarded to 13 students.

The Committee's receipts for the year were $1687.33. Its expenses were $1680.84, leaving a balance of $6.49.

Very Reverend John A. Flynn, C. M., S. T. D., President of St. John's University, welcomed the people at the meeting, and then the following papers were presented:

1. *Mathematics in communication*, by Dr. Brockway McMillan, Bell Telephone Laboratories, Murray Hill, New Jersey. (By invitation.)

The technology of communication is divided broadly into the fields of transmission (the actual conveying of messages) and switching (the setting up of transmission paths, when needed, out of equipment common to many possible paths). Most transmission is accomplished by devices governed by linear time-invariant differential equations. The relevant applicable mathematics is then centered around the Laplace and Fourier transforms. Switching devices are discrete and quantized; their relevant mathematics is that of logic, general algebra, and combinatorics. In both fields one distinguishes between the study of apparatus *per se* and the study of its performance in the environment of use. The latter study always invokes probability theory in some form. Almost all communication problems invoke linear graphs in their statement or study.

2. *Fundamental preparation in mathematics for college study in engineering science—What the colleges need and expect*, by Professor F. H. Miller, Cooper Union.

Topics, concepts and techniques needed by the engineering student for college work in mathematics were discussed. Some of the items are essential prerequisites; others are highly desirable if the requirements of physics and engineering departments are to be adequately met. Specific examples of various topics and applications of techniques were cited, reference being made to *A Report on Mathematics Preparation for Engineering Colleges*, by F. H. Miller and S. G. Roth, published in the Journal of Engineering Education, April, 1947, and to other reports bearing on the subject.

3. *Fundamental preparation in mathematics for college study in engineering science—What the high school can do*, by Mr. D. H. Frank, Forest Hills High School.

Difficulties in meeting the needs of the engineering school are: 1) Too little mathematics taken too long before entering college, 2) too many non-essentials taught, 3) lowering of standards, 4) requirements the same for all students, 5) small numbers entering engineering resulting in difficulty in separating them from the others, 6) interest in science and engineering not necessarily accompanied by interest in mathematics.

The ways to meet these problems are: 1) separation in classes or segregation within classes, 2) integration where possible, 3) rewriting of syllabus so that each student is exposed to all the

branches of mathematics for every term of the subject, 4) discontinuation of solid geometry as separate subject and distribution over all grades, 5) weeding out all non-essentials, 6) teaching elements of differential and integral calculus in last year with stress on all necessary concepts and techniques, 7) at all times teaching for the necessary disciplines so well exemplified in mathematics.

4. *On informal symbolic logic and its place in mathematical education*, by Professor E. R. Stabler, Hofstra College.

The speaker outlined some key topics for an informal treatment of symbolic logic. These included propositions, propositional functions, quantified propositions, and classes. Then he presented examples of the potential usefulness of these topics in promoting the objectives of high school and college mathematics courses. Finally, considering the possible role of informal logic in the curriculum, he proposed: 1) an experimental twelfth grade course in logic and logical aspects of mathematics; 2) incorporation of some informal symbolic logic in college mathematics courses for general students; 3) a basic college course including informal symbolic logic, and related foundational topics, for students majoring in mathematics.

E. MARIE HOVE, *Secretary*

THE MARCH MEETING OF THE MICHIGAN SECTION

The annual meeting of the Michigan Section of the Mathematical Association of America was held on March 27, 1954, at the University of Michigan, Ann Arbor, in conjunction with the meetings of the Michigan Academy of Science, Arts and Letters. Professor J. S. Frame of Michigan State University presided at both morning and afternoon sessions and at the luncheon and business meeting.

A total of one hundred thirty-two persons attended the meetings, including the following sixty-six members of the Association:

A. C. Andersen, K. J. Arnold, J. L. Bagg, J. W. Baldwin, R. C. F. Bartels, F. A. Beeler, J. H. Bell, W. S. Bicknell, J. W. Bradshaw, Fred Brafman, C. H. Butler, A. T. Butson, C. D. Calhoon, Y. W. Chen, R. V. Churchill, D. F. Coffey, S. D. Conte, J. W. Coy, C. C. Craig, D. A. Darling, P. S. Dwyer, C. M. Erikson, N. C. Fisk, K. W. Folley, J. S. Frame, J. W. Gaddum, Casper Goffman, G. W. Grotts, V. G. Grove, Frank Harary, Gerald Harrison, G. E. Hay, H. M. Heater, Fritz Herzog, T. H. Hildebrandt, E. F. Ingalls, L. G. Johnson, L. S. Johnston, P. S. Jones, Leo Katz, A. F. Lampen, J. F. Lanahan, Harry Langman, H. D. Larsen, K. B. Leisenring, J. F. Manogue, J. E. McLaughlin, L. E. Mehlenbacher, G. E. Meike, D. M. Mesner, E. E. Moise, W. K. Moore, H. W. Nace, A. L. Nelson, P. A. Nurnberger, Mary H. Payne, Emily C. Pixley, H. H. Pixley, J. H. Powell, P. H. Raker, H. F. Stelson, Brother Andrew Stephen, B. M. Stewart, J. G. Sowul, Leonard Tornheim, R. L. Wilder.

At the business meeting the nominating committee consisting of Professors H. D. Larsen, B. M. Stewart, and L. E. Mehlenbacher proposed Professor R. V. Churchill, University of Michigan, for chairman and Professor S. D. Conte, Wayne University, for Secretary-Treasurer for the year 1954–55. The slate was elected unanimously.

A motion was made by Professor R. V. Churchill and seconded by Professor V. G. Grove that the By-Laws of the Section be changed, if necessary, to allow the Sectional Governor to be a member of the Executive Committee of the Section. The motion was passed.

It was voted to empower the Executive Committee of the Section to make such resolutions as are necessary to effect the deposit and withdrawal of funds from banks.

The following papers were presented at the morning and afternoon sessions:

1. *The enumeration of connected linear graphs: an analogy to rooted trees*, by Professor Frank Harary, University of Michigan.

Using a combination of Pólya's counting theorem and a functional identity involving exponentials and the cycle indexes of the symmetric groups, it is easy to show that the method of Cayley for enumerating rooted trees is identical in form with a derivation of the number of connected linear graphs in terms of the known number of all (connected or disconnected) linear graphs.

2. *Distribution of binomial coefficients modulo p*, by Mr. D. M. Mesner, Michigan State College.

The set of $(n^2+n)/2$ binomial coefficients for exponents $0, 1, \cdots, n-1$ is considered. For p a prime, the distribution of this set into residue classes modulo p is obtained. If n has the form p^t the distribution is quite simple; in particular, the combined frequency of non-zero residues is $[(p^2+p)/2]^t$. If $f_n(i)$ denotes the frequency of numbers $\equiv i \pmod{p}$, then

$$\lim_{n\to\infty} f_n(i)/f_n(0) = 0; \tag{1}$$

and

$$\lim_{n\to\infty} f_n(i)/f_n(j) = 1; \qquad 0 \neq i \neq j \neq 0,$$

It is shown that (1) holds even when p is replaced by an arbitrary modulus m.

3. *An elementary proof of an old number theorem*, by Professor Harry Langman, Detroit Institute of Technology.

The theorem states that it is impossible to have four different integral squares in arithmetic progression. The proof is entirely elementary and is based on the argument of "infinite descent." It is shown that if a set of four integers exists whose squares in order have a common difference, another set exists conforming to the same conditions and such that the largest of these is less than the largest of the assumed set.

4. *Tschebyscheff polynomial approximations in high speed computing*, Dr. Leo Razgunas, University of Michigan, Willow Run Research Center.

Presented by title.

5. *The angle sums in a simplex*, by Dr. J. W. Gaddum, Michigan State College.

In vol. 59, no. 6 of this MONTHLY, bounds were found by the speaker for the angle sums (dihedral and trihedral) in a tetrahedron. In this paper, similar bounds are discussed for the dihedral angle sum in a simplex. The bounds on a trihedral angle in terms of the surrounding dihedral angles are also discussed.

6. *Teaching of mathematics at Russian universities*, by Professor G. G. Lorentz, Wayne University, introduced by the Secretary.

Programs, teaching methods and the organization of Russian Universities were discussed.

7. *Some mathematical problems in industry*, by Dr. Morris Ostrofsky, Westinghouse Electric Corporation, Pittsburgh. (By invitation.)

Several industrial problems in mathematics were discussed, two of them in some detail. The first of these involved the numerical solution of a heat conduction equation $\partial u/\partial t = D\nabla^2 u +$heat source where the heat source was non-linear in both time and space coordinates. It was pointed out that numerical solutions can be used as an experimental tool to crystallize physical concepts.

The second problem involved the solution of the integral equation known as the transport equation. In course of that solution numerical integrations in a form convenient for digital computers were necessary. A useful and simple formula for calculating the coefficient for numerical integration was presented. (In conclusion, the opportunities for mathematicians in industry were discussed.)

8. *Solving problems on Wayne's UDEC (Unitized digital electronic computer)*, by Dr. E. P. Little, Wayne University, introduced by the Secretary.

The Wayne University UDEC (Unitized Digital Electronic Calculator built by the Burroughs Corporation) has magnetic drum storage and five hole teletype input and output. Pulse duration is 0.1 sec. and pulse frequency is 120KC/sec. The drum stores 5300 words of 9 decimal digits plus sign and check number. Information within the machine is expressed in excess-three binary coded decimal notation. Numerical data and instructions are both stored on the drum. Random access is possible by a decision code but instructions are normally stored in sequence as far as possible. The tabulation of a simple linear function is described.

9. *On a probability distribution arising in group organization theory*, by Mr. J. H. Powell, Michigan State College.

In the very simplest model of a group organization we consider an organization of n individuals in which connections in either direction exist or do not exist between pairs. Such an organization may be represented by an incidence matrix with 0's on the principal diagonal. The immediate problem confronting us is that of the chance distribution of the number of mutual connections, *i.e.* pairs of symmetrically placed 1's in the incidence matrix. In particular, consider the distribution relative to a fixed set of row and column totals of the incidence matrix. Using a result by Katz and Powell in a paper on directed graphs, it is shown how to compute as many factorial moments of the exact distribution as desired and, implicitly, the exact probability of K mutual connections.

10. *Use of bordering in matrix inversion*, by Professor P. S. Dwyer, University of Michigan.

This paper shows how bordering, which is frequently useful in evaluating determinants, may be applied to the problem of matrix inversion. Bordering is especially useful when relations involving elements of the matrix to be inverted are present. The proof is based on the formula for the inverse of a partitioned matrix and the fact that each term of the inverse may be interpreted as the ratio of two determinants. Application is made to the problem of the size of the errors of the inverse resulting from errors in the elements and to the determination of the inverse of a correlation matrix when the factors of the reduced correlation matrix are known.

S. D. CONTE, *Secretary*

THE MARCH MEETING OF THE SOUTHEASTERN SECTION

The annual meeting of the Southeastern Section of the Mathematical Association of America was held March 19–20, 1954 at the University of South Carolina, Columbia, South Carolina. Professor W. V. Parker, Chairman of the Sec-

tion, and Professor W. L. Williams, Vice-Chairman, presided over the general sessions; Professors H. K. Fulmer, J. C. Currie, J. D. Novak, D. F. Barrow and L. A. Dye presided over subsections.

There were about two hundred in attendance including the following one hundred sixteen members of the Association:

P. L. Armstrong, Emil Artin, R. W. Bagley, D. F. Barrow, Helen Barton, R. C. Blackwell, R. G. Blake, A. T. Brauer, J. P. Brewster, G. M. Brown, J. W. Brown, B. F. Bryant, C. C. Buck, E. A. Cameron, N. A. Childress, R. S. Christian, D. H. Clanton, J. B. Coleman, J. A. Cooley, C. L. Cope, F. E. Cothran, R. W. Cowan, R. R. Croxton, J. C. Currie, N. E. Dodson, E. G. Douglas, B. M. Drucker, G. E. Duncan, L. A. Dye, E. D. Eaves, J. C. Eaves, D. O. Ellis, M. H. M. Esser, M. K. Fort, Jr., Tomlinson Fort, H. K. Fulmer, L. L. Garner, Leslie J. Gaylord, D. R. Goodner, M. O. Gonzalez, S. T. Gormsen, B. F. Hadnot, S. W. Hahn, J. C. Harden, Jr., E. A. Hedberg, Marguerite Z. Hedberg, T. F. Hickerson, T. R. Norton, L. P. Hutchison, C. W. Huff, G. B. Huff, J. B. Jackson, F. W. Kokomoor, Stephen Kulik, G. B. Lang, J. W. Lasley, Jr., C. G. Latimer, T. H. Lee, Anne L. Lewis, F. A. Lewis, G. H. Lundberg, R. A. Lytle, E. L. Mackie, C. E. Martin, S. T. Martin, W. N. Mebane, Jr., J. T. Moore, R. H. Moorman, J. C. Morelock, J. A. Nohel, J. D. Novak, G. K. Overholtzer, W. V. Parker, P. B. Patterson, A. H. Payne, Lillian G. Perkins, N. C. Perry, C. G. Phipps, R. L. Plunkett, Ellen F. Rasor, P. A. Reichle, G. E. Reves, J. M. Robertson, H. A. Robinson, F. Virginia Rohde, H. R. Rouse, W. C. Royster, W. A. Rutledge, F. W. Saunders, P. C. Scott, E. B. Shanks, Olivia H. Shanks, D. C. Sheldon, C. Eucebia Shuler, J. L. Sloan, C. B. Smith, F. W. Stallard, E. L. Stanley, L. W. Stark, C. M. Stuart, C. S. Sutton, H. S. Thurston, R. Z. Vause, Jr., J. H. Wahab, D. T. Walker, E. F. Ward, W. W. Weber, W. M. Whyburn, M. C. Wicht, W. L. Williams, N. K. Williamson, R. A. Willoughby, R. L. Wilson, F. J. Witt, J. W. Young, B. K. Youse.

The following officers were elected for the coming year: Chairman, Professor W. L. Williams, University of South Carolina; Vice-Chairman, Professor R. H. Moorman, Tennessee Polytechnic Institute; Secretary-Treasurer, Professor H. A. Robinson, Agnes Scott College.

The following program was presented:

1. *A gap in the treatment of dependent functions*, by Professor C. G. Phipps, University of Florida.

In treating dependent and independent functions a situation unaccounted for occurs where the Jacobian of the functions is only conditionally zero. This condition usually implies that one function has a maximum or a minimum subject to the restraint of fixed values for the others.

2. *Marginal notes*, by Professor J. W. Lasley, Jr., University of North Carolina.

Synthetic division of one polynomial by another, evaluations of a determinant, complex numbers as vectors, rotation of axes, the equations of conics, Newton's correction formula, derivatives of vectors, systems of linear equations, rank of a matrix are topics for which special techniques are developed.

3. *Lissajous figures in the teaching of undergraduate mathematics*, by Professor J. H. Wahab, Georgia Institute of Technology.

A method of drawing Lissajous figures is adapted to the study of parametric equations, composite functions, and the solutions of differential equations with variables separated.

4. *Mathematics in England*, by Professor Tomlinson Fort, University of Georgia.

Professor Fort spent the academic year 1952–1953 in Great Britain. He lectured at various British universities and had an unusual opportunity to see British mathematicians and to observe mathematical organizations and schools. His paper is concerned with his personal observations.

5. *A geometric device to facilitate the use of hyperbolic substitutions*, by Professor J. W. Young, University of Florida.

As a geometric device to facilitate the use of hyperbolic substitutions in integrals involving $\sqrt{u^2+a^2}$, $\sqrt{u^2-a^2}$, $\sqrt{a^2-u^2}$ and a rational function of sh ϕ and ch ϕ, Dr. Young suggested one write $\sqrt{a^2-(iu)^2}$, $i\sqrt{u^2-a^2}$, $\sqrt{a^2+(iu)^2}$, respectively and place results on a right triangle in the complex plane modified by taking the argument pure imaginary. The appropriate substitution follows by the Theorem of Pythagoras and the relations

$$\begin{matrix}\sin\\ \mathrm{sh}\end{matrix}\, i\phi = i \begin{matrix}\mathrm{sh}\\ \sin\end{matrix}\,\phi, \qquad \begin{matrix}\cos\\ \mathrm{ch}\end{matrix}\, i\phi = i \begin{matrix}\mathrm{ch}\\ \cos\end{matrix}\,\phi, \qquad \begin{matrix}\tan\\ \mathrm{th}\end{matrix}\, i\phi = i \begin{matrix}\mathrm{th}\\ \tan\end{matrix}\,\phi.$$

6. *Summer conferences in collegiate mathematics*, by Professor E. A. Cameron, University of North Carolina.

Professor Cameron told of the two 1954 summer conferences and outlined the work of the one to be held at his University.

7. *Some notes on matric computation*, by Professor W. V. Parker, Alabama Polytechnic Institute.

This paper is devoted to a discussion of various techniques in computation with matrices without resorting to the use of determinants. Several well-known results are obtained by use of these techniques.

8. *Exchange of information.*

Representatives from 28 mathematics departments participated in a discussion on certain common administrative problems which had been suggested prior to the meeting. Professor H. K. Fulmer rendered a very helpful report to the section of the findings of this discussion group.

9. *Plücker numbers and the field of values*, by Professor J. C. Currie, Georgia Institute of Technology.

Let A be a square matrix with complex elements. Then the complex numbers $xA\bar{x}'$ with the vector x subject to $x\bar{x}'=1$ fill up a convex region of the complex plane, called the field of values of A. The boundary of this region is an algebraic curve, or, more often, a part of an algebraic curve, some of whose properties are examined with the aid of the Plücker relations.

10. *Criteria for a logarithmic solution of a certain type of linear differential equation of the second order with a regular singular point*, by Professor R. W. Cowan, University of Florida.

Using the method of Frobenius, a compact expression is obtained for a general coefficient of an infinite series solution of the differential equation. The criteria desired arise from a consideration of various values of the difference of the roots of the indicial equation.

11. *On differential equations with variables separated,* by Professors J. A. Nohel and G. K. Overholtzer, Georgia Institute of Technology.

In most popular differential equations texts the problem of finding a solution of a differential equation with variables separated is treated as a mere term-by-term antidifferentiation process. In this teaching note it is pointed out that an existence theorem can be proved by elementary means.

12. *Vibrating nets,* by Professor Tomlinson Fort, University of Georgia.

Professor Fort shows how the study of a vibrating loaded net leads to the study of a boundary-value problem for a partial difference equation of the second order. The solution of these problems in their simpler forms depends upon a study of the characteristic roots of a matrix. In the more complicated cases progress has been made primarily by kinematical methods. In some cases methods similar to those used in the study of analogous problems in the theory of differential equations have proved effective.

13. *An almost regular form,* by Professor P. B. Patterson, University of Florida.

An almost regular form is defined; the form (2, 2, 5, −1, −1, 0) is given as an example; and it is shown that this form does not represent primitively any m^2 where $m \equiv 1 \pmod 4$. The method of approach is essentially that used on indefinite forms by B. W. Jones and E. H. Hadlock in a joint paper presented to the Society by E. H. Hadlock on November 29, 1952.

14. *The Frenet formulas for a ruled surface,* by Professor J. D. Novak, University of South Carolina.

Extending Aoust's notion of *courbure inclinée* V. G. Grove introduced the curvature matrix and the Frenet Formulas for a ruled surface with respect to a curve. In this paper the "first and second curvatures" are used to study ruled surfaces, particularly the line of striction, and to characterize certain classes of ruled surfaces.

15. *On quasi-idempotent matrices,* by Professor G. B. Huff, University of Georgia.

Professor Huff defined a matrix A to be a quasi-idempotent if there exists a matrix polynomial $F(x)$ such that $A^r = F(r)$ for every positive integer r. It was then shown that a given matrix A is a quasi-idempotent if and only if there is an integer k such that $A(E-A)^k=0$ and that the associated $F(x)$ is readily computed for a quasi-idempotent A. The explicit formulas lead to a miniature theory of exponentials and logarithms of certain special matrices.

16. *The numerical quartic, as a synthesis within elementary mathematics,* by Professor F. E. Cothran, Greenwood, South Carolina.

The method of Ferrari indicates parabolic and "hemi-quartic" loci which when "normalized" produce a graphical solution of the resolvent cubic. The devices used were elementary but lead the student to an interest in numerical calculus, the nature of real numbers, combinations of roots, difference and differential calculus, and hyperbolic functions, thus synthesizing several compartments of mathematics.

17. *A criterion for primary crossed products of degree p^2 over fields of formal power series in four variables,* by Professor J. T. Moore, University of Florida.

The paper provides a test for the primary nature of a crossed product algebra of the type indicated, which probably can be extended to apply to a much more general situation.

18. *A note on unstable homeomorphisms*, by Professor B. F. Bryant, Vanderbilt University.

Utz (*Unstable Homeomorphisms*, Proc. Amer. Math. Soc., vol. 1, 1950, pp. 769–774) calls a homeomorphism f of a compact, dense-in-itself, metric space X onto X unstable, with instability constant, δ, provided for each pair of points x, $y \in X$ there exists an integer $n(x, y)$ such that $d(f^n(x), f^n(y)) > \delta$. It was established that if $\omega(x)[\alpha(x)]$ does not contain periodic points then there exists u, $v \in \omega(x)[\alpha(x)]$ whose orbits are negatively [positively] asymptotic, and with the aid of this it was shown that there exist $p, q, r, x \in X$ such that the orbits of p and q are positively asymptotic and the orbits of r and s are negatively asymptotic.

19. *The theory of braids*, by Professor Emil Artin, Princeton University.

The theory of braids gives a good example that illustrates problems of topology and can also be used as an example to the concept of abstract groups. The composition of braids is done in the obvious way by tying one to the other. It can be shown that this leads to a well defined group generated by especially simple braids with several relations between them. The problem of classification amounts to a solution of the word problem in this group. The solution of the word problem can be given in two ways—an algebraic and a geometric way.

20. *An integral transform*, by Professor B. F. Hadnot, Florida State University.

The integral $J(\sigma) \equiv [\sigma^{k+1}/T(k+1)]\int_0^\infty A\lambda^k(\nu)e^{-\sigma\nu}d\nu$ for $\sigma > 0$ is proved to occur as the limit of certain (R, ϕ, κ) sums of $\sum a_n e^{-\sigma\lambda_n}$. The integral $J(\sigma)$ is viewed as an integral transform of $A\lambda^k(\omega)$ for the limit $\sigma \to 0$. One use of this idea has been to simplify certain known results in the theory of Riesz and Abel summation.

21. *Elliptic integrals in terms of Legendre polynomials*, by Professor M. O. Gonzalez, University of Alabama.

For the elliptic integral of the first kind $u = \int_0^\phi (1 - k^2 \sin^2 \phi)^{-1/2} d\phi$ the author obtains the expansion

$$u = 2\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} P_n(\lambda) \tan^{2n+1} \frac{\phi}{2},$$

where $P_n(\lambda)$ is the Legendre polynomial of order n and $\lambda = k'^2 - k^2$.

It is also shown that

$$K = \frac{\pi}{2} P_{-1/2}(\lambda) = 2\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} P_n(\lambda)$$

and that

$$E = \frac{\pi}{4} [P_{-1/2}(\lambda) + P_{1/2}(\lambda)] = 4\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n-1)(2n+1)(2n+3)} P_n(\lambda).$$

Expansions for other elliptic integrals are also given.

22. *Necessary and sufficient conditions for convergence of infinite series*, by Professor E. B. Shanks, Vanderbilt University.

In this paper, the writer proves that most known tests for convergence or divergence of infinite series of positive terms can be deduced simply from the following two theorems.

A necessary and sufficient condition that a series $\sum a_n$ of positive terms converge (diverge) is that there exist real numbers C_n (an unbounded sequence of real numbers D_n) such that

$$C_n - C_{n+1} \geqq a_{v+n} \qquad (0 < D_{n+1} - D_n \leqq a_{v+n})$$

for some fixed $\nu \geqq 0$ and $n = 1, 2, \cdots$. For example, Kummer's test for convergence follows by taking $C_n = a_n D_n / p$, $p > 0$, $\nu = 1$ in the first theorem. Numerous other standard tests are derived similarly in the paper.

The theorems are also of theoretic importance as well as of practical importance, unlike most theorems similar in nature.

23. *Summation of series, interval of differencing one half*, by Professor F. Virginia Rohde, University of Florida.

Given a function $\phi(x)$ whose finite integral is $f(x)$, interval of differencing $h = \frac{1}{2}$. If all differences of $\phi(x)$ are zero after the kth one, an expression for $\sum_{x=1}^{n} \phi(x)$ can be found in terms of $f(x)$, $\phi(x)$, and the k differences of $\phi(x)$ taken between suitable limits.

24. *A mathematics program for high speed computation*, by Professor B. M. Drucker, Georgia Institute of Technology.

Dr. Drucker announced that his institution had received a grant for the purchase of a new electronic digital computer. He outlined the types of courses which will be offered in the fall for the preparation of programs, coding and the operation of an automatic computer.

25. *On definitions of trigonometric functions*, by Professor W. A. Rutledge, Alabama Polytechnic Institute.

The standard text-book definitions of the trigonometric functions as functions of angles are considered and some of the difficulties that arise, both technical and pedagogical, are discussed. Methods of avoiding these difficulties are presented.

26. *The proper approach to trigonometry*, by Professor Tomlinson Fort, University of Georgia.

Professor Fort has developed the subject of trigonometry with the basic idea that the infinite series is a concept with which the student is already informally familiar through his work with infinite decimals, infinite geometric series, infinite binomial series and numerous other limits in his elementary geometry. A review of these topics is proposed, still strictly in an informal manner, to be followed by a discussion of functions and the definition of sin x and cos x by infinite power series. Applications to the solution of triangles appear as one of many uses.

27. *On the least primitive root (mod p)*, by Professor Alfred Brauer, University of North Carolina.

In a paper presented at the meeting of the American Mathematical Society at East Lansing, Michigan, on September 4, 1952, the following theorem was proved.

Let p be a prime of form $4n+1$ and k the number of different prime divisions of $p-1$. We set $2^k = r$. If g is the smallest positive primitive root (mod p), then

$$g < p^{(r-1)/r}. \tag{1}$$

In this paper the following theorem is proved which often gives a better estimate than (1).

Let $s = s(p)$ be the maximal number of consecutive integers which are all not relatively prime to $p-1$. Then

$$g < p^{s/(s+1)}. \tag{2}$$

The bounds (1) and (2) are greater than the well known bound of Vinogradoff, but are obtained by purely elementary methods.

28. *Finitely generated rings*, by Professor Emil Artin, Princeton University.

A ring R is called Noetherian if its ideals satisfy the ascending chain condition. Let R be a Noetherian ring with unit element, S an extension ring with the same unit element such that

$$S = R\omega_1 + R\omega_2 + \cdots + R\omega_n$$

for certain elements ω_v of S. Such a ring S will be called a modul finite extension of R.

If on the other hand one can find a finite number of elements $\xi_1, \xi_2, \cdots, \xi_r$ of S such that every element of S can be expressed as polynomial of $\xi_1, \xi_2, \cdots, \xi_r$ with coefficients in R then we call S a ring finite extension of R. The following theorem is proved:

Let $R \subset T \subset S$ such that S is a ring finite extension of R but a modul finite extension of T, then T is a ring finite extension of R.

As an application one can show: A ring finite extension of a field which is a field, is a finite algebraic extension.

29. *Some inequalities concerning Mills' ratio*, by Professors D. F. Barrow and A. C. Cohen, Jr., University of Georgia, presented by Professor Barrow.

Mills' ratio is obtained by dividing the area under the normal curve to the right of any point by the ordinate of the normal curve at that point. The authors developed a method by which certain inequalities involving Mills' ratio can be more easily proved than by existing methods; and one new inequality was proved which had been only surmised before. This method depends upon the following obvious theorem:

If $F(x)$ is everywhere positive, and if the product $F(x)f(x)$ has a non-negative limit as x grows infinite and a derivative which is everywhere negative, then $f(x)$ is everywhere positive.

30. *A normalizing transformation of the fiducial distribution of point-biserial correlation*, by Professor N. C. Perry, Alabama Polytechnic Institute.

Lev has shown that r_{pbi} (point biserial correlation) has a distribution function related to non-central Student's distribution. Johnson and Welch have pointed out a normal curve approximation to the non-central 't' frequency curve. The present paper combines these two results, obtaining a useful normal approximation to a transformation of r_{pbi}.

31. *Moisture stresses in a certain type of plywood*, by Professor C. B. Smith, University of Florida.

Certain types of plywood plates having the directions of the grain of adjacent plies not perpendicular are considered. Moisture stresses produced by a uniform change in moisture content are discussed mathematically, and it is shown that by a suitable choice of the angles formed by the directions of the grain of adjacent plies warping can be avoided.

32. *Abridged series for numerical evaluation*, by Professor B. K. Youse, University of Georgia.

This paper has appeared in this MONTHLY, March, 1954, page 184.

33. *Function space approximation in laminated orthotropic material under plane strain*, by Professor R. G. Blake, University of Florida.

The method of approximation in elasticity based on the concept of function space is used to obtain a first approximation to the solution of certain problems of plane strain in a rectangle consisting of three layers of orthotropic material cemented together.

This material was prepared under a contract with the Office of Naval Research.

34. *A Cremona involutorial transformation without fundamental curves of the first species*, by Professor L. A. Dye, The Citadel.

The transformation involves a homoloidal web of cones with collinear vertices on a fundamental line of contact of the second species. The principal surfaces are images of isolated fundamental points on this line of contact.

35. *On the non-equivalence of two indefinite ternary forms*, by Professor T. R. Horton, University of Florida.

L. E. Dickson's *Studies in the Theory of Numbers* furnished a table of reduced, indefinite, non-zero, ternary quadratic forms of determinant $d \leqq 83$ except for $d=68$, $d=81$. Dickson stated (p. 147) that of four possible reduced forms of $d=68$, the equivalence of two forms A and B was not decided. This paper gives a necessary and sufficient condition that a form be equivalent to a form of a certain type and applies this condition to show A and B not equivalent.

36. *A theorem concerning mapping of a topological group into the circle*, by Professor R. L. Plunkett, Vanderbilt University.

A factor group of the group of all mappings of a compact, connected, commutative topological group, G, satisfying the second axiom of countability, into the circle is shown to be isomorphic with the character group of G. This is shown to imply that every mapping of such a group into the circle is homotopic to an interior mapping.

37. *The mapping of a second system of algebraic curves invariant under a cyclic involution of period eleven*, by Professor S. T. Gormsen, University of Florida.

In this paper the second of a series of systems of algebraic curves, invariant under an I_{11}, is mapped onto a surface F in S_7. It is found that these curves at one of the branch points on F have five tangent lines at this point. Four of these tangent lines belong to a fifth order tangent cone to F, and the fifth tangent line belongs to the remaining tangent plane to F at this point.

38. *Distinction between some geometric properties arising in non-analytic mappings*, by Professor W. C. Royster, Alabama Polytechnic Institute.

Interesting distinctions between convexity and starlikeness arise when mapping by non-analytic functions are considered. In this note the mapping function is applied to certain families of curves.

39. *On infinite series of sets*, by Professor D. O. Ellis, University of Florida.

Continuing a series of studies (Autometrized Boolean algebras, Boolean functions, Boolean metrized spaces, *etc.*) of similarities and contrasts between Boolean rings and the real field, one considers the convergence (in the Kantorovitch sequential topology) of $\sum_{i=1}^{\infty} A_i$ in a set algebra, where addition is taken as symmetric difference. The main result, quite clear after a little thought, is: $\sum_{i=1}^{\infty} A_i$ *converges if and only if* $\lim_{i\to\infty} A_i = 0$. Certain simple consequences of this result are obtained.

40. *An alternative formalism for the basic notions of algebraic topology*, by Professor C. C. Buck, University of Alabama.

The peculiarities of this notation are that the boundary of an n-cell is a set of $(n-1)$-cells, rather than a chain, and that an n-chain of a complex K over a commutative group G is written as a mapping of the n-cells of K into G. This notation exhibits clearly the relationship between the boundary chain and the coboundary of a cell.

41. *On orbital topologies*, by Professor R. W. Bagley, University of Florida.

Two problems (Hausdorff property and continuity of the topology-inducing mapping) suggested by the paper, David Ellis, *Orbital topologies*, Quart. Jour. of Math. (Oxford), vol. 4, 1953, pp. 117–119, are solved. The topological structure is determined in the special case for which the topology-inducing mapping is a permutation.

42. *Open topological disks in the plane*, by Professor M. K. Fort, Jr., University of Georgia.

A subset X of the plane has the "almost fixed point property" if corresponding to each continuous function f of X into X and each positive number ϵ there is a point p in X such that the distance from p to $f(p)$ is less than ϵ. It is shown that if an open topological disk in the plane is bounded and has a locally connected boundary, then the open disk has the almost fixed point property.

43. *Projections of surfaces obtained from an involution of period 13*, by Professor N. A. Childress, University of Florida.

The image of a planar cyclic involution of period thirteen can be represented as a surface of order thirteen in S_8. This paper concerns five successive projections of this image surface from a certain point on the surface being projected each time. Two results of a peculiar nature are explained.

H. A. ROBINSON, *Secretary*

THE APRIL MEETING OF THE IOWA SECTION

The forty-first annual meeting of the Iowa Section was held jointly with the sixty-sixth annual meeting of the Iowa Academy of Science and the twenty-first convention of the Junior Academy of Science at Iowa State College, Ames, Iowa, on April 30 and May 1, 1954. The Chairman, Professor J. O. Chellevold, Professor H. T. Muhly, acting for the Vice-President, and the Secretary, Professor Fred Robertson, presided in turn.

An amendment to the by-laws was approved by a unanimous vote. The amendment reads as follows:

That Section 2 of Article II of the By-Laws of the Iowa Section of the Mathematical Association of America be replaced by:

Section 2. The Executive Committee shall consist of the officers of the Section and the Sectional Governor.

The registered attendance was sixty-eight including the following forty-four members of the Association:

E. W. Anderson, Fred Brandner, R. H. Bruck, I. H. Brune, E. L. Canfield, J. O. Chellevold, Marian Daniells, A. W. Davis, W. M. Davis, C. B. Germain, B. E. Gillam, H. E. Goheen, Cornelius Gouwens, F. S. Harper, J. J. L. Hinrichsen, D. L. Holl, R. S. Jacobsen, N. W. Johnson, Jr., G. E. Kaldenberg, A. A. Karwath, Dora Kearney, Don Kirkham, O. C. Kreider, R. J. Lambert, C. E. Langenhop, B. F. Laposky, C. H. Lindahl, W. D. Lindstrom, F. W. Lott, C. G. Maple, J. V. McKelvey, H. T. Muhly, E. N. Oberg, Fred Robertson, Hazel M. Rothlisberger, J. A. Schumaker, Augusta L. Schurrer, E. R. Smith, R. D. Stalley, H. P. Thielman, Henry Van Engen, G. P. Weeg, D. V. V. Wend, Roscoe Woods.

Officers elected for the year are: Chairman, Professor H. T. Muhly, State University of Iowa; Vice-Chairman, Professor F. A. Brandner, Iowa State College; Secretary-Treasurer, Professor Fred Robertson, Iowa State College.

The following papers were presented:

1. *Recent advances in the foundations of euclidean plane geometry*, by Professor R. H. Bruck, University of Wisconsin.

Unfortunately for the wide audience whose interest in geometry was awakened in high school or university, recent significant advances in the foundations of the subject are hidden behind a screen of abstract algebra. The present paper offers a pictorial account of some of the geometric axioms, a simple explanation of the algebraic problems which these pose and a brief account (with references rather than proofs) of the answers. The main topic is coordinatization of euclidean planes. Touched on are planar ternary rings, Veblen-Wedderburn systems, division rings with the right inverse property, right alternative and alternative division rings.

2. *Curve families and finite asymptotic values*, by Professor D. V. V. Wend, Iowa State College.

The author gives some relationships between the topological structure of a branched regular curve family F and the finite asymptotic values of the analytic functions contoured by F. Let a b-curve be an element of F separated from some other element of F by no element or chain of F. If F contains a b-curve, then any analytic function contoured by F has a finite asymptotic value. If the trees of F are isolated and F contains no b-curve, then there exists an analytic function contoured by F which has no finite asymptotic value. A precise characterization of the level curve families of harmonic polynomials is given.

3. *Peculiar derivative functions*, by Mr. N. B. Smith, Iowa State College.

Let $f(x)$ be a function defined on an interval x in E_1, and let p be a point property of $f(x)$. The function $f(x)$ is said to be peculiar with respect to the property p if there exists a partition of X into two subsets X_1 and X_2 each everywhere dense in X and such that $f(x)$ has the property p at no point of X. Peculiar derivative functions are discussed in the cases where the property p is continuity, neighborliness of W. W. Bledsoe, *Proc. Amer. Math. Soc.*, vol. 3, 1952, p. 114, and cliquishness of H. P. Thielman, this MONTHLY, vol. 60, 1953, p. 156. In particular it is shown that there exist no derivative functions peculiar on X with respect to neighborliness.

4. *Mathematical theory for the utilization of tagged atoms in determining plant nutrient transformation rates in soils*, by Professors Don Kirkham and W. V. Bartholomew, Iowa State College.

Differential equations are derived and integrated, for determining mineralization rates and immobilization (demineralization) rates, in a tagged atom system of mineral and non-mineral plant nutrients undergoing simultaneous interchange in soil. The results are

$$m = \frac{b - x}{st} L, \qquad i = \frac{x(1 - 1/s)}{t} L, \qquad L = \log \frac{b - sx_0}{b - sx},$$

$$s = \left(\frac{y_0 - y}{a} + \frac{x - x_0}{b} \right) \left(\frac{xy_0 - x_0 y}{ab} \right)^{-1},$$

where m = mineralization rate; i = immobilization rate; t = time; a = mass of tagged atoms, mineral and non-mineral; b = mass of all atoms, mineral, non-mineral, tagged and untagged; x = mass (at time t) of tagged and untagged mineral atoms; y = mass of tagged mineral atoms; x_0 and y_0, values of x and y at $t=0$. The theory is in good agreement with experiments.

5. *The gradient method for the Minkowski metric in the solution of simultaneous linear algebraic equations*, by Professors H. E. Goheen and J. R. Winkelman, Iowa State College.

The system of simultaneous equations

$$Ax - c = 0,$$

in which A is a non-singular $n\times n$ matrix and c is an $n\times 1$ matrix, can be solved by minimizing the positive definite function

$$f(x) = |\epsilon_1| + |\epsilon_2| + \cdots + |\epsilon_n|,$$

in which ϵ_i is the ith component of the $n\times 1$ matrix $Ax-c$. The authors have devised a method for obtaining this minimum.

6. *A modification of the Hartley-Politz scheme for dealing with not-at-home persons*, by Professor Alan Ross, Iowa State College, introduced by the secretary.

A method for obtaining unbiased estimates of population totals and estimates of their reliability is derived which does not require that callbacks be made on the not-at-homes in sample surveys. The theory of sampling with unequal probabilities of selection is expanded to cover the situation in which one utilizes *a priori* probabilities of obtaining schedules from all persons in the universe when these probabilities are actually determined only for those persons interviewed on first call. Horvitz and Thompson, *Journ. Am. Stat. Assoc.* vol. 47, 1952, p. 260, have considered sampling with arbitrary probabilities from the standpoint of a fixed sample size. The extension made here is to consider the case in which the sample size is fixed, but from the "sample" a variable number of schedules is obtained. Tables are presented which give comparisons of efficiency of the no-callback scheme with the usual survey technique involving repeated callbacks on the not-at-homes.

7. *The randomization analysis of a generalized randomized block design*, by Professor M. B. Wilk, Iowa State College, introduced by the secretary.

The problems of estimation of effects and of tests of significance are considered for an experimental design in which t treatments are allotted at random to r blocks, each containing pt experimental units, under the restriction that each treatment appears p times in every block. The statistical inference is referred to the actual set of experimental units employed; a linear model is used, based on means and deviations over a finite population of (conceptual) yields. No distributional assumptions are made.

The means, variances, and covariances under randomization of the analysis of variance mean squares are given. Unbiased estimates of certain linear combinations of the parameters of the model are given, together with the variances of the estimates. Attention is also given to the estimation of these variances. Tests of significance, based on a randomization test, are presented. The possible approximation to the randomization test of procedures which derive from normal theory is considered.

8. *An extension of some limit theorems in probability theory*, by Professor John Gurland, Iowa State College.

Some well-known limit theorems are proved and extended by using properties of sequences of characteristic functions. Special attention is given to stochastic convergence.

9. *Motivation in mathematics*, by Miss Ethel Cain and Students of Roosevelt High School, Des Moines, introduced by the secretary.

The speaker discussed motivation as a two-way path. The teacher may motivate the student

but the student may also motivate the teacher. Three students in the speaker's class, Mr. Jud Harper, Mr. Bob Helmick and Mr. Dick Rothrock, gave illustrations such as short hand multiplication, a method of obtaining certain combinations, a solution of a cubic equation by a system of bodies immersed in a liquid and controlled by a lever, and a set of geometric figures.

10. *A survey course for seniors*, by Professor J. O. Chellevold, Wartburg College.

In this paper the author described a senior-level course designed to show the relationship among various fields of mathematics and to provide a brief introduction to selected topics in modern mathematics.

11. *A study of student achievement and mortality in college mathematics*, by Professor Fred Robertson, Iowa State College.

The author discussed the achievements and mortality, as of the end of the winter quarter 1954, of the 1004 students who entered the college algebra course in September 1952. Success was defined as normal progress or a 2.0 or better grade point in each subject. An achievement and mortality table was shown for the group.

FRED ROBERTSON, *Secretary*

THE APRIL MEETING OF THE OHIO SECTION

The thirty-eighth annual meeting of the Ohio Section of the Mathematical Association of America was held at the Ohio State University, Columbus, Ohio, on Saturday, April 17, 1954. Professor P. R. Rider, Chairman of the Section, presided at the morning and afternoon sessions.

Eighty-nine persons registered in attendance, including the following seventy-five members of the Association:

J. E. Adney, Jr., C. E. Amos, Grace M. Bareis, I. A. Barnett, H. M. Beatty, Foster Brooks, O. E. Brown, Emalou Brumfield, L. E. Bush, C. D. Calhoon, V. B. Caris, M. N. Chase, W. G. Clark, G. M. Clough, C. C. Crell, H. K. Crowder, R. C. Davis, B. B. Dressler, Wade Ellis, R. W. Emmert, P. L. Evans, H. E. Fettis, D. T. Finkbeiner, M. P. Fobes, H. W. Godderz, Marshall Hall, Jr., H. G. Harp, Frances Harshbarger, C. H. Heinke, R. G. Helsel, R. Y. Iwanchuk, L. A. Jehn, E. D. Jenkins, M. L. Johnson, Margaret E. Jones, John Kaiser, Chosaburo Kato, L. C. Knight, Jr., F. A. Kros, H. W. Kuhn, Nathan Lazar, R. F. Liskovec, L. L. Lowenstein, A. D. Martin, Margaret E. Mauch, S. W. McCuskey, E. J. Mickle, L. H. Miller, Knox Millsaps, Emma J. Olson, E. A. Peters, H. S. Pollard, Tibor Radó, P. V. Reichelderfer, P. R. Rider, R. F. Rinehart, D. L. Robb, G. deB. Robinson, Louis Ross, H. J. Ryser, Charles Saltzer, Samuel Selby, L. L. Shetler, R. L. Shively, Ruth B. Smyth, E. T. Stapleford, R. E. Thomas, Harold Tinnappel, H. S. Toney, W. R. Transue, D. R. Whitney, C. O. Williamson, J. A. Wilson, Alberta Wolfe, A. D. Ziebur.

The following officers were elected for the coming year: Chairman, Professor W. R. Transue, Kenyon College; Secretary-Treasurer, Professor Foster Brooks, Kent State University; third member of the Executive Committee, Professor E. J. Mickle, The Ohio State University; Program Committee: Chairman, Professor Marion Wetzel, Denison University; Professor Wade Ellis, Oberlin College; Professor H. D. Lipsich, University of Cincinnati.

The following papers were presented:

1. *Statistical distributions*, by Professor P. R. Rider, Wright-Patterson Air Force Base (Chairman's address).

Some of the statistical distributions in common use were described, methods of deriving them were discussed, and various applications were mentioned.

2. *Calculation of integrals of the form* $\int_0^\theta \sin^p \Phi \cos^q \Phi \, d\Phi$, by Mr. H. E. Fettis, Wright-Patterson Air Force Base.

Attention is directed toward the evaluation of the integral

$$\int_0^\theta \sin^p \Phi \cos^q \Phi d\Phi$$

where $p > -1$ if $\theta < \pi/2$ and $p > -1$, $q > -1$ if $\theta = \pi/2$. Exact expressions are known to be possible when $\theta = \pi/2$ for arbitrary values of p and q, and for integral values of p and q, with θ arbitrary. The present discussion shows that reduction of elliptic integrals can be effected for $p = \pm\frac{1}{2}$, $p = \pm\frac{1}{3}$, $q = 0$ and $\theta \leqq \pi/2$. Also, a rapidly converging series is given for computation with values of p and q for which closed expressions are not possible. Several practical applications are noted.

3. *On the directional derivative*, by Professor Wade Ellis, Oberlin College.

The existence of the directional derivative df/ds of the function $f = f(x, y)$ at the point (x, y) implies the existence of a plane π tangent to the surface S: $z = f(x, y)$ at the point (x, y, z). The relation between df/ds and the dihedral angle formed by π and the z-plane can be exploited to enhance the student's intuitive grasp of this and related material in courses in the calculus.

4. *n-th roots of integral binary matrices*, by Mr. Harvey Weitkamp, University of Cincinnati, introduced by Professor I. A. Barnett.

If X is a binary matrix, then X^n may be written linearly in terms of X and I by the Cayley Hamilton equation, and the coefficients may be given explicitly in terms of the trace and determinant of X. Necessary and sufficient conditions are given in order that a binary matrix with rational or integral elements shall have an nth root with rational or integral elements. The nth roots may be unique or may be infinitely many.

5. *Linear programming*, by Dr. E. Leonard Arnoff, Operations Research Group, Case Institute of Technology. (By title.)

Linear programming, through the transportation and simplex methods, presents recently developed techniques for solving the problem of optimizing a linear functional, subject to restraining conditions which are in the form of linear inequalities. This paper first presents a brief description of the transportation and simplex methods. It then compares the types of optimization problems to which these methods are applicable with those optimization problems which can be handled by the methods usually taught in the college classroom, namely, those methods which use analytic geometry, differentiation, Lagrange multipliers, and the like.

6. *Functional mathematics for the secondary school teacher*, by Professor Emalou Brumfield, Kent State University.

The proposal is made that present mathematics-education students would be better prepared to teach secondary school mathematics if they could study, concurrently with their content mathematics courses, the importances, the applications, and the significance of these courses with respect to secondary school mathematics. The proposed study could be organized and effected by means of a mathematics-education course without affecting the content offerings of the department and without requiring the sectioning of students according to their professional objectives.

7. *The place of algebra and geometry in the undergraduate curriculum*, by Professor Gilbert deB. Robinson, University of Toronto. (By invitation.)

The place of modern algebra in the undergraduate curriculum was discussed by Professor MacLane in Baltimore last Christmas; the author here discusses integrating the algebra with the appropriate geometry. This could be done by treating *linear algebra and geometry* in a one-term course, say in the junior year; the ideas of linear transformation, lattice of subspaces of a projective space, determinants, matrices, the full linear group, *etc.*, would all be coordinated. *Quadratic ideas* involving the reduction of a quadratic form and the application of Taylor's series to a study of polarity and collineation, the inversion transformation and stereographic projection would fall naturally in the senior year. A one-term course in the first postgraduate year dealing with axiomatics, the significance of Pappus' and Desargues' theorems, the non-euclidean metric, the orthogonal group, and the groups of the regular solids in three dimensions would tie the ideas together. Such a three-term sequence could be extended and elaborated, but it would seem to provide a necessary minimum of geometrical background for a student proposing to go on to graduate work, and could replace existing courses in modern algebra and the foundations of mathematics.

8. *The convergence of sequences determined by a simple non-linear difference equation*, by Professor Samuel Goldberg, Oberlin College.

Let a real number x_0 be prescribed and determine $x_1, x_2, \cdots$ by the difference equation $x_{n+1} = x_n^2 - k$. Assume $k \geqq -1/4$. The values of x_0 for which the sequence (x_n) converges are found. In the special case $k=2$, these values form a denumerably infinite set and are determined from an explicit solution of the difference equation. The mode of convergence of (x_n) to its limiting value is also discussed.

9. *Laplace transform and differential operator relations*, by Dr. L. V. Robinson, Wright-Patterson Air Force Base.

Among other relationships, it is shown that

$$\mathcal{L}[f(t)] = pf(-D_p)\frac{1}{p} = p\int_0^\infty e^{-pt}f(t)dt \qquad \left(D_p = \frac{\partial}{\partial p}\right),$$

and

$$\mathcal{L}[F(t)f(t)] = pF(-D_p)\frac{1}{p}[\mathcal{L}f(t)] = pf(-D_p)\frac{1}{p}\mathcal{L}[F(t)],$$

where $\mathcal{L}[f(t)]$ is the Laplace transform of $f(t)$.

10. *Some applications of the finite-difference analogue of Green's third identity*, by Professor Charles Saltzer, Case Institute of Technology.

The finite-difference analogue of Green's third identity is deduced from the finite-difference analogue of Green's second identity (due to Courant, Friedricks and Lewy) by the use of the fundamental solution of the Laplace difference equation developed by McCrea and Whipple, Stohr and Duffin. This identity is applied to the Laplace, Poisson, and biharmonic difference equations.

11. *How mutation geometry says "No" to the trisection problem*, by Dr. Beckham Martin, Owens-Illinois Glass Company.

Two mutation views were developed: one starting from two prototypes resulting in the mutation trisection expression

$$2b\cdot rd\cdot r = (pb - sd)\cdot r$$

where b and d are the orthonormalized bisectors of the given angle and its supplement. The equation was transmuted to a primordial prototype

$$A \cdot r = 0,$$

whose solution was

$$r = 0/0.$$

A second view started from a single prototype which led to a factorable 4th degree equation with a free choice parameter with which to control its irreducibility. A mutation curve was drawn for this which was shown to trisect the angle.

FOSTER BROOKS, *Secretary*

THE APRIL MEETING OF THE ROCKY MOUNTAIN SECTION

The thirty-seventh annual meeting of the Rocky Mountain Section of the Mathematical Association of America was held at Colorado Agricultural and Mechanical College, Fort Collins, Colorado, on Friday afternoon and evening and Saturday forenoon, April 30 and May 1, 1954. Professor M. L. Madison, Chairman of the Section, presided at all three sessions.

Seventy-five registered for the meeting, including the following forty-nine members of the Association:

C. F. Barr, D. L. Barrick, J. R. Britton, R. G. Buschman, R. K. Butz, F. M. Carpenter, A. G. Clark, Sarvadaman Chowla, G. S. Cook, Rev. F. T. Daly, W. E. Dorgan, H. T. Guard, R. R. Gutzman, C. L. Harbison, Leota C. Hayward, I. L. Hebel, LeRoy Holubar, J. E. Householder, P. F. Hultquist, C. A. Hutchinson, B. W. Jones, A. J. Kempner, Claribel Kendall, R. B. Kriegh, J. S. Leech, M. L. Madison, W. E. Mientka, M. W. Milligan, W. K. Nelson, Greta Neubauer, D. O. Patterson, O. M. Rasmussen, O. H. Rechard, A. W. Recht, L. W. Rutland, Jr., Nathan Schwid, S. R. Smith, W. N. Smith, L. C. Snively, K. H. Stahl, J. McD. Staley, P. O. Steen, E. P. Tovani, E. L. Vanderburgh, W. W. Varner, J. F. Wagner, F. J. Wall, C. R. Wylie, Jr., A. Zirakzadeh.

Officers elected at the meeting for 1954–1955 were: Chairman, Professor Nathan Schwid, University of Wyoming; Vice-Chairman, Professor C. R. Wylie, Jr., University of Utah; Secretary-Treasurer, Professor F. M. Carpenter, Colorado School of Mines.

The following papers were presented:

1. *On expressing the matrix A^t as a polynomial in t*, by Professor R. K. Butz, Colorado Agricultural and Mechanical College.

This paper discusses the notion of $E_0F(X)$ matrices as introduced by G. B. Huff (*Matrices such that A^t is a polynomial in t and principal idempotent elements*, Bull. Amer. Math. Soc., vol. 59, 1953, p. 54). Emphasis is placed on the fact that the proofs of the main theorems require only the more elementary concepts of matrix theory and on methods of finding $F(X)$ given a matrix A with elements in the field of complex numbers. The clarity with which some classical results follow by the use of this notion is pointed out.

2. *An optimum solution of N equations in M unknowns with N greater than M*, by Mr. Leon Rutland, University of Colorado.

A problem in engineering design led to a consideration of the system of equations

$$\sum_{j=1}^{m} a_{ij}x_j = t_i \qquad (i = 1, 2, \cdots, n), (n > m),$$

where the desired optimum solution of the system is that set of x's for which the largest absolute value of any of the deltas in the set of equations

$$\Delta_i + \sum_{j=1}^{m} a_{ij}x_j = t_i, \qquad (i = 1, 2, \cdots, n), (n > m),$$

is as small as possible. Several theorems giving solutions to the problem under various conditions were either proved or stated with the proofs being omitted. A numerical example, carried through on a digital computer, was cited to indicate that even though there are as many as forty equations the method is feasible and the answer can be readily attained.

3. *The method of Frobenius*, by Professor R. H. Cook, South Dakota School of Mines and Technology, introduced by the secretary.

The usual textbook presentation of the method of Frobenius effectively camouflages two important points: (1) that the method involves a Taylor's expansion, and quite often leads to just a Taylor's expansion; (2) the conditions under which the method is applicable. This paper suggests a modified approach which has neither of the above disadvantages, is easily taught, and is sufficiently flexible to be applicable to many non-linear problems.

4. *Approximate solutions to a certain functional equation*, by Professor C. A. Rogers, Colorado A and M. College.

The following is investigated: Given a non-negative $g(x)$, defined for all $x>0$, and which is strictly monotonic increasing and everywhere differentiable, to find a closed-form $f(x)$, reasonably computable, such that $f[f(x)]$ is at least approximately equal to $g(x)$. It was indicated how this approximation could be accomplished for certain g-functions, with examples.

5. *Some infinite series*, by Dr. W. E. Briggs, Professor S. Chowla, Professor (Emeritus) A. J. Kempner, and Research Assistant W. E. Mientka, University of Colorado, presented by Mr. Mientka.

It is proved that

$$\sum_{1}^{\infty} \frac{\sigma_n}{n^2} = 2\zeta(3),$$

where

$$\sigma_n = \sum_{t=1}^{n} \frac{1}{t} \cdot$$

6. *Effect of rotation on the normal mode frequencies of transverse vibration of a cantilever beam*, by Professor R. H. Cook and Mr. L. J. Eatherton, South Dakota School of Mines and Technology, presented by Mr. Eatherton.

The differential equation,

$$EI\frac{\partial^4 y}{\partial x^4} - \frac{\partial^2 y}{\partial x^2}\int_x^L \rho A S\Omega^2 dS + \rho A x\Omega^2 \frac{\partial y}{\partial x} + \rho A \frac{\partial^2 y}{\partial t^2} = 0$$

describes the vibrations, in a vertical plane, of a cantilever beam which rotates about a vertical axis through its clamped end. This equation and appropriate boundary conditions are considered

by the use of Taylor's expansion. The normal mode frequencies are calculated in terms of Ω, the rotational speed. Results show the dependence upon both Ω^2 and Ω^4 for the first and second modes and upon Ω^2 for the third mode. They are in excellent agreement with existing experimental data.

7. *Block designs*, by Professor Burton W. Jones, University of Colorado.

The definition and significance of balanced incomplete block designs are briefly described and methods of exclusion sketched.

8. *The mapping of the circles of S_2 into the points of S_3*, by Professor C. R. Wylie, Jr., University of Utah. (By invitation).

If the coefficients a, b, c, d in the general equation of a circle

$$d(x^2 + y^2) - 2ax - 2by + c = 0$$

are interpreted as homogeneous point coordinates in S_3, point circles are mapped into points on the paraboloid

$$V \equiv a^2 + b^2 - cd = 0,$$

proper real circles are mapped into finite points outside V, improper real circles (lines) are mapped into real points at infinity, and imaginary circles are mapped into finite points within V. Pencils and bundles of circles are represented in S_3 by lines and planes, respectively, and may be classified according to the intersection of their images with V. Two circles which are orthogonal are represented by points each of which lies in the polar of the other with respect to V. Conjugate pencils of circles are represented by lines conjugate with respect to V. Various theorems from college geometry were interpreted in S_3, and the classical constructions for circles satisfying three conditions were considered as problems in descriptive geometry in S_3.

9. *The University of Colorado Engineering Experiment Station analog computer—The UCEESAC*, by Mr. Walter W. Varner, University of Colorado.

A description of the Boeing analog computer recently installed at the University of Colorado was given. Types of problems that can be solved as well as restrictions on their solution were given. Finally a simple pair of simultaneous differential equations were considered and the simplicity of forming the wiring diagram shown.

10. *Fitting empirical equations to fluid meter data*, by Professor S. R. Smith, University of Wyoming.

Empirical equations of the form

$$C = \frac{R}{a + bR} + de^{fR}*$$

were fitted to both flow nozzle and orifice meter data and residuals determined. R, C curves were fitted to data for the fluids stream, oil and water for $0 < R \leqq 3{,}040{,}000$. C is the coefficient of discharge of the meter and R its corresponding Reynolds number, both dimensionless.

11. *On $ff(x) = F(x)$, $F(x)$ given (real), $f(x)$ unknown*, by Professor (Emeritus) A. J. Kempner, Dr. W. E. Briggs, Professor S. Chowla and Mr. W. E. Mientka, University of Colorado, presented by Professor Kempner.

For the real case the geometrical interpretation employed in studying $f(x) - x = 0$ can be extended so as to lead immediately to results such as: there exist totally discontinuous functions $f(x)$

* Equation first used by I. D. Ruggles, former graduate student in mathematics, University of Wyoming.

for which $ff(x)$ is single-valued and continuous; or, the function $y=g(x)$ given by $x-y+\pi/2 =\mu \cos (x+y)$, $|\mu| \leqq 1/2$, is a single-valued inverse iterate of $f(x)=x+\pi$. It also makes plain the plausibility of introducing iterations of $f^{(n)}(x)$ of any rational (or even any real) index n.

12. *Figure it out for yourself*, by Professor A. W. Recht, University of Denver.

Trained mathematicians are at a premium; everyone realizes the great part mathematics plays in a technical civilization. Yet election of mathematics in high schools is waning, despite efforts of government and mathematical groups stressing urgent need of training of the talented. The problem is to reach the 85% who will use mathematics in normal pursuits and also train the talented 15%. The paper suggested revisions of policy, including training of really professional mathematics teachers, emphasis on students doing own work, insistence on 100% accuracy, promotion of "first the problem, then the mathematics," and strengthening of fundamental concepts to avoid mathematical accidents in home and industry.

13. *Textbooks on elementary mathematics*, by Professor J. S. Leech, Colorado College.

Attention is called to the fact that in a large majority of textbooks many terms are defined erroneously or ambiguously. Many theorems are stated and "proved" without any or with incomplete hypotheses. In all of these cases, the author believed that the correct definitions and statements of theorems not only do not increase the difficulty of the subjects, but serve to clarify concepts and contribute greatly to understanding.

14. *Freshman mathematics separation*, by Professor I. L. Hebel, Colorado School of Mines.

This paper dealt with the mathematics department's experience over the last seven years in assigning and classifying freshman mathematics students.

F. M. CARPENTER, *Secretary*

THE APRIL MEETING OF THE SOUTHWESTERN SECTION

The fourteenth annual meeting of the Southwestern Section of the Mathematical Association of America was held at Arizona State College, Tempe, Arizona, on April 16, 1954. Professor M. S. Hendrickson, Chairman of the Section, presided.

Forty persons attended the meeting including the following twenty-seven members of the Association:

O. B. Ader, C. E. Aull, C. E. Buell, J. H. Butchart, D. G. Duncan, J. F. Foster, Jr., R. S. Fouch, F. C. Gentry, R. F. Graesser, R. E. Graves, W. P. Heinzman, M. S. Hendrickson, Carol Karp, Max Kramer, Lincoln LaPaz, R. B. Lyon, W. W. Mitchell, Jr., E. D. Nering, J. L. Olpin, E. J. Purcell, L. C. Snively, A. H. Steinbrenner, Deonisie Trifan, Earl Walden, D. L. Webb, Charles Wexler, Oswald Wyler.

The following were elected officers for the year 1954: Chairman, Professor D. L. Webb, University of Arizona; Vice-Chairman, Professor R. L. Westhafer, New Mexico College of Agriculture and Mechanic Arts; Secretary-Treasurer, Professor W. W. Mitchell, Jr., Phoenix College (four year term); Lecturers, Professor Max Kramer, New Mexico College of Agriculture and Mechanic Arts (one year), Professor D. G. Duncan, University of Arizona, (two years).

An invited address "The Story of Palomar" was given following the banquet by Dr. S. B. Nicholson of the Mount Wilson and Palomar Observatories. Dr. Nicholson also read a paper on "Results from Palomar" during the morning session.

The following papers were presented:

1. *Some problems in electrostatics treated synthetically*, by Professor J. H. Butchart, Arizona State College.

The author showed that synthetic methods, notably inversion, are appropriate in problems concerning electrostatic forces in the plane. He discussed in particular the case of concyclic charges.

2. *Nonlinear differential equations that can be reduced to linear*, by Professor M. S. Hendrickson, University of New Mexico. (By title.)

The most general nonlinear equation of the first order which is reducible by a substitution of the form $u=\phi(x, y)$ to the linear equation $u'=up(x)+q(x)$ is of the form $y'=-(\phi_x/\phi_y)+(1/\phi_y)[p\phi+q]$. In particular, any equation of the form $y'=h(y)[F(x)+G(x)\exp -\int 1/hdy]$ is reducible by the substitution $u=\exp[\int 1/hdy-\int Fdx]$ to the equation $u'=G(x)\exp -\int Fdx$ which can be solved by integration alone.

The most general second order equation reducible by a substitution of the form $F(y')=G(y)$ to a first order equation linear in G with y as independent variable is of the form $y''+p(y)F(y')y'/F'(y')+q(y)y'/F'(y')=0$. In particular, any equation of the form $y''+g(y)(y')^2+h(y)(y')^{2-n}=0$ can be reduced by the substitution $G(y)=(y')^n$.

3. *Set representation theorems in implicative models*, by Mrs. Carol R. Karp, New Mexico College of Agriculture and Mechanic Arts.

Implicative Models were introduced by L. Henkin in his paper *An algebraic characterization of quantifiers*, Fundamenta Mathematicae, 1950. They are the algebraic models for the System of Basic Implication, a calculus of propositions with only implication as a propositional connective, yielding as formal theorems only *wffs* valid in both the classical and intuitionistic calculi of propositions.

The result announced was that every Implicative Model $\Psi=\langle X, 0, \dot{-}\rangle$ can be extended to a Brouwerian Algebra with preservation of infs and exactly those tops which satisfy the following conditions: (i) $z=$top Y in the sense of Henkin, *i.e.*, (1) z is an upper bound of Y; (2) for any z', a in X, $y\dot{-}a\leqq z'$ for all y in Y implies $z\dot{-}a\leqq z'$. (ii) for any z', $a_1, a_2, \cdots, a_n$ in X, n any finite integer, $y\dot{-}a_1\dot{-}a_2\dot{-}\cdots\dot{-}a_n\leqq z'$ for all y in Y implies $z\dot{-}a_1\dot{-}\cdots\dot{-}a_n\leqq z'$.

By a result of Tarski and McKinsey [*Closed elements in closure algebras*, Annals of Mathematics, 1946] it follows that every Implicative Model is isomorphic to a subspace of closed sets of a topological space.

4. *Axioms of congruence for absolute geometry in a bounded domain*, by Professor Oswald Wyler, University of New Mexico.

A system of axioms of congruence is given which is valid in any open convex domain (whether bounded or not) of a euclidean, hyperbolic or elliptic geometry. Together with axioms of incidence and of order given elsewhere by the author (*Duke Math. Jour.* vol. 20, 1953; *Composito Math.*, vol. 11, 1953), and with an axiom of continuity, these axioms permit the construction of the full absolute geometry from the geometry of any open convex domain. No axiom of parallelism is used, but the type of a geometry (euclidean, hyperbolic, or elliptic) is given by properties of its absolute polarity.

5. *A method of approximating moments in thin elastic plates*, by Professor Deonisie Trifan, University of Arizona.

A method of approximating the bending and twisting moments in thin elastic was presented. Based on a theory developed by Prager and Synge who regarded a state of stress in a deformed body as a point in function space, the actual solution which satisfies the equation of equilibrium, compatibility conditions, and boundary conditions is approximated by a series of so-called artificial states which satisfy only some of the above conditions.

6. *The case for intuitionism*, by Professor E. J. Purcell, University of Arizona.

7. *A three dimensional model of a lattice*, by Professor D. L. Webb, University of Arizona.

The display of a symmetric model in three dimensions of a free modular lattice of three generators.

8. *On normal subgroups of even order*, by Professor K. A. Fowler, University of Arizona, introduced by the secretary.

Let G be a finite group of even order n which contains exactly m elements of order 2. The following results are obtained: (1) $h-1\geqq(m^2+m)/n$ where h is the number of conjugate classes each of which is equal to its inverse class; (2) there exists a real character, distinct from the 1-character, of degree less than m/n; (3) G properly contains a normal subgroup of bounded index, the bound depending only on the ratio m/n.

9. *Families of lines*, by Dr. Andrew Sobczyk, Los Alamos Scientific Laboratory.

The family of all lines in the plane is in one-to-one correspondence with the Möbius cylinder, since, as may be seen from the normal form, each line is characterized by two coordinates p, α with $-\infty<p<\infty$, $0\leqq\alpha<\pi$ and the boundary $\alpha=\pi$ of the p, α strip must be identified into the boundary $\alpha=0$ with reversal of the sign of p. The family of all lines in space is in one-to-one correspondence with the fiber-bundle of the tangent planes to a sphere. Many new and interesting results on line families are obtained by study of their graphs in these fiber bundles.

10. *Coalition formation in n-person games*, by Professor E. D. Nering, University of Minnesota and Goodyear Aircraft Company.

The von Neumann and Morganstern theory of n-person games is essentially an extrapolation of an intuitive analysis of the zero-sum 3-person game. Except for this case the theory gives little information about the process of formation of coalitions nor, on the other hand, is the effect of such a process on the outcome of the game analyzed. The author presents an intuitive analysis of the zero-sum 4-person game which is an extension of the analysis of the zero-sum 3-person game and which leads to results different from those given by the von Neumann and Morganstern theory.

11. *Mathematics grades of engineers*, by Professor F. C. Gentry, University of New Mexico.

A survey of mathematics grades of a selected group of engineering graduates compared to their grades in other subjects was discussed. It was found that on the whole grades in mathematics are slightly lower than overall grade averages, that very few of these students failed a first year course, but about 25% failed one course in calculus.

R. L. WESTHAFER, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirty-eighth Annual Meeting, University of Pittsburgh, Pittsburgh, Pennsylvania, December 30, 1954.

Thirty-sixth Summer Meeting, University of Michigan, Ann Arbor, Michigan, August 29–30, 1955.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Pittsburgh, Pennsylvania, May, 1955.

ILLINOIS, Monmouth College, Monmouth, May 13–14, 1955.

INDIANA, Butler University, Indianapolis, May, 1955.

IOWA, St. Ambrose College, Davenport, April 15–16, 1955.

KANSAS

KENTUCKY

LOUISIANA-MISSISSIPPI

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Georgetown University, Washington, D. C., December 4, 1954.

METROPOLITAN NEW YORK, Queens College, Flushing, New York, April 30, 1955.

MICHIGAN, Michigan State College, East Lansing, Spring, 1955.

MINNESOTA, University of Manitoba, Winnipeg, October 16, 1954.

MISSOURI, University of Kansas City, Spring, 1955.

NEBRASKA

NORTHERN CALIFORNIA, University of California, Berkeley, January 15, 1955.

OHIO

OKLAHOMA, Oklahoma City University, October 29, 1954.

PACIFIC NORTHWEST

PHILADELPHIA, Princeton University, Princeton, New Jersey, November 27, 1954.

ROCKY MOUNTAIN, University of Wyoming, Laramie, Spring, 1955.

SOUTHEASTERN, Tennessee Polytechnic Institute, Cookeville, March 11–12, 1955.

SOUTHERN CALIFORNIA, Santa Monica City College, March 12, 1955.

SOUTHWESTERN, University of New Mexico, Albuquerque, Spring, 1955.

TEXAS, Abilene Christian College, Abilene, April, 1955.

UPPER NEW YORK STATE, University of Buffalo, May 14, 1955.

WISCONSIN, Cardinal Stritch College, Milwaukee, May, 1955.

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF
THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 61 — NUMBER 7

PART II

Proceedings of the

Symposium on Special Topics in Applied Mathematics

HELD AT
NORTHWESTERN UNIVERSITY, NOVEMBER 27–28, 1953

Number 3
of the
HERBERT ELLSWORTH SLAUGHT
MEMORIAL PAPERS

AUGUST-SEPTEMBER 1954

ONE DOLLAR A COPY

The AMERICAN MATHEMATICAL MONTHLY

(Founded in 1894 by Benjamin F. Finkel)

Carl B. Allendoerfer, *Editor*

EDITORIAL CORRESPONDENCE should be addressed to the Editor, C. B. Allendoerfer, Department of Mathematics, University of Washington, Seattle 5, Washington.

ADVERTISING CORRESPONDENCE should be addressed to J. F. Randolph, Morey Hall, University of Rochester, Rochester 3, N. Y.

NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the Monthly should be sent to the Secretary-Treasurer, H. M. Gehman, University of Buffalo, Buffalo 14, N. Y.

THIS IS THE OFFICIAL JOURNAL OF THE

MATHEMATICAL ASSOCIATION OF AMERICA, INC.

(Devoted to the Interests of Collegiate Mathematics)

Entered as second class matter at the post office at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

Subscription Price: To Members, \$4 a Year, To Others, \$5 a Year.

Published by the Association at Menasha, Wisconsin, and Seattle, Wash. during the months of January, February, March, April, May, June–July, August–September, October, November, December.

PRINTED IN THE UNITED STATES OF AMERICA

Proceedings of the

SYMPOSIUM ON SPECIAL TOPICS IN APPLIED MATHEMATICS

HELD AT

NORTHWESTERN UNIVERSITY, NOVEMBER 27–28, 1953

Sponsored by the Mathematics Division of the
National Research Council under contract
with the National Science Foundation
and the American Mathematical
Society

The Third

HERBERT ELLSWORTH SLAUGHT
MEMORIAL PAPFR

This publication has been financed jointly by the National Science Foundation and the Arnold Buffum Chace Fund of the Mathematical Association of America

Published as a supplement to the AMERICAN MATHEMATICAL MONTHLY

Volume 61 Number 7

August–September, 1954

TABLE OF CONTENTS

INTRODUCTION AND CRITIQUE

F. J. WEYL, Investigator, NRC Committee on Training and Research in Applied Mathematics

1. Sponsorship and Organization. The papers appearing here under a single cover were presented in the course of a somewhat novel type of symposium on special topics in applied mathematics which was organized by the Mathematics Division of the National Research Council and accommodated by the American Mathematical Society as an integral part of its regular meeting at Northwestern University, Evanston, Illinois, on November 27–28, 1953. The character of this symposium was largely set by the fact that its organization constituted one of the phases of a Survey of Training and Research in Applied Mathematics which the National Research Council has been conducting during the past year under provisions made by the National Science Foundation jointly with the three military Offices of Research in response to a recognized need for a comprehensive stock-taking of this country's resources in the way of mathematics and mathematicians likely to play a significant part in the future advances of our science and technology.

The arrangement of one or several conferences on the various phases of training and research in applied mathematics formed part of the plans for the Survey from the beginning. As subsequent work brought the picture better into focus, it became clear that the modern range and depth of the subject matter of applied mathematics was as yet insufficiently understood by many mathematicians. Today applied mathematics is emphatically no longer just the art of solving by dull methods repulsive differential equations whose coefficients carry physical or engineering names! Rather than merely pointing out the necessity of a systematic effort to dispel this deadly myth, the NRC Committee on Training and Research in Applied Mathematics which was in charge of the Survey decided to take at once an active step in this direction. Hence one of the planned conferences was set aside for this particular purpose and developed into the Symposium on Special Topics in Applied Mathematics here reported.

2. Aim and Impact. The Symposium was designed to present characteristic current research to a large audience of mathematicians, illustrating particularly active sectors of the front along which mathematics interacts today with other scientific disciplines. Its philosophy differed therefore markedly from the Applied Mathematics Symposia convened annually by the Mathematical Society or the younger series held under Office of Ordnance Research sponsorship. No attempt was made, as in the latter, to aim the program at the workers and specialists in some unified area, however broadly defined, and the interchange of information among them. Instead, the emphasis was on diversity of topics, on highlighting some exciting developments of the day and doing it in a manner to induce broad understanding among the mathematically literate. What we wanted was a collection of significant examples, not brought as yet too often or

accessibly to the attention of the mathematicians, which would show just how varied and substantial the contributions of the genuinely secular mathematician (as distinct from the monastic one) can be in this interaction of his modes of thought and analysis with scientific endeavors in other fields.

The Symposium was therefore planned in three 90-minute sessions of three half-hour papers each. The first of these sessions was to be concerned with new mathematical theories and techniques which are currently being fashioned for applied purposes; the second one was to be dedicated to the characteristic mixture of combinatorial analysis and probability theory now being developed for the sake of constructing analytic models and the numerical simulation of control and communication systems; the third one, finally, was to deal with situations where the interplay between physical ideas and mathematical construction is at present the critical element. This structure of the sessions is still discernible in the Symposium as it took place in the end, although an unusually large number of last minute changes beyond the planners' control compromised it to some extent. It proved helpful, however, in assuring the diversity and range which was in fact achieved.

The theory of partial differential equations, and especially of the physically relevant non-linear types, shows no signs of exhaustion as a primary source of important and stimulating problems which fascinate the mathematicians of applied interests. There are the basic questions revolving around the idea of what constitutes a well-formulated problem in this area which J. Leray touches in his paper and which have naturally led to the viewpoint, pioneered by Leray and others, of looking at solutions not individually but as families with topological and algebraic structure, capable of being supplemented by ideal elements, *etc.* There are the powerful function theoretical arguments which are being developed for the construction and analysis of important particular solutions, here represented by the methods with which P. Garabedian and his co-workers have recently attacked hydrodynamic flows possessing free surfaces. There are finally the fascinating problems posed by the nature of the singularities which can arise in the solution of these non-linear systems and the structure of the loci on which they can occur: A. H. Taub discusses such a problem using methods of differential geometry to analyze the configuration of interacting shock fronts in compressible flow. All of these are vital areas of substantial mathematical research today.

Ultimately destined, perhaps, to be of comparable importance in its mutually stimulating interaction with modern analysis is the area of signal and noise problems, illustrated by those presented in the paper of M. Kac. Here again the quest for analytic and measure-theoretical information on certain families of functions, containing individually observed time series as members, leads naturally to problems of substantial interest also to current front-line thinking in pure analysis. Another fertile field for mathematical ingenuity in applied contexts is furnished by what, in effect, are elementary problems but which are posed with

such a massive complexity of detail as to render the solution totally inaccessible to the traditional class-room approach. S. Chandrasekhar's discussion of eigenvalue problems of high order furnishes a good example how such an obstacle can be overcome by an ingenious mathematical trick, perhaps even, as in this case, without knowing in a rigorous way what accounts for the success of the trick.

Probably the least familiar aspect of the "new look" in applied mathematics is the steadily growing interaction between the combinatorial and algebraic mathematics of discrete structures on the one hand, and the use and operation of large-scale digital computers on the other. The use of Boolean algebra in the analysis of switching circuits is of course no longer new, but the idea of carrying forward basic research in Boolean algebras, as reported in the sequel by D. E. Muller, with the hope of anticipating the future needs of switching circuit design rather than being merely concerned with supplying the present ones, and to make use, moreover, of high-speed computing equipment in doing it—this gives evidence of the substantial development of a major new province of applied mathematics. In the same general area lies the work of Project SCAMP at the Institute for Numerical Analysis, whose purpose is that of exploring the use of high-speed automatic computers in the analysis of discrete structures. Some of it has been reported for the first time in any completeness at the Symposium by S. S. Cairns. Since summer after summer a sizable group of our ablest mathematicians participate in this work, it cannot fail in the long run to have a noticeable impact on American mathematical research.

Another important development, once more in the traditional vein of applied mathematics, is illustrated by the investigations, summarized by E. Montroll, concerning the energy distribution over the frequencies at which a molecular crystal lattice is capable of oscillating. It turns out that this analysis depends in an essential way on the relations known to exist between the topological structure of a closed manifold, here a certain kind of phase space, and the analytic properties of functions and integrals defined thereon. These problems are among the most intensively investigated ones in modern analysis, and if they should furnish the appropriate terms for analyzing significant aspects of current theories of molecular physics, we might be faced with their proliferous growth similar to the development of Hilbert space methods under the impact of quantum mechanics.

Everyone who scans this list will miss certain topics which he would consider as particularly interesting or important for the development of modern applied mathematics. Let him be assured that but few, if any, of these omissions were inadvertent. It is perhaps indicative of how much needs doing along these lines, that a first attempt could achieve as much and still leave so much untouched. Probably most glaring is the absence of a report, aimed at mathematicians, on the current problems and difficulties faced by quantum mechanical theories of fields. All endeavors of the Committee to provide such a report were unsuccess-

ful, and in hindsight this may have been all to the good, for the matter is serious enough to warrant a special effort.

3. Experimental Features. In arranging for a novel type of meeting, some innovations as regards organization are inevitable. The Symposium on Special Topics in Applied Mathematics introduced two such departures from standard practice. For one, the sessions of the Symposium were interspersed among the regular sessions of the Mathematical Society meeting, rather than grouped into a separate conference preceding or following the former and thus too easily avoided by the non-specialist. The example which was followed here is that of the American Physical Society whose meetings provide for a sizable number of invited half-hour papers, mixed in among the contributed papers either singly or in groups. Thus, the speakers were permitted some expository breadth in order to bring those of their colleagues not immediately concerned with the subject on hand up to date. The second innovation consisted in inviting representatives of other scientific fields to speak to mathematicians. The ones selected for this purpose naturally have extensive contact with mathematics in their daily scientific lives, yet they do not think of themselves as mathematicians. It is from such persons that a mathematician will generally get his most stimulating insight into the mathematical problems which arise in other scientific disciplines. Both of these experiments proved successful at the Symposium under discussion and their adoption, also in other than applied mathematical contexts, is recommended.

THE PHYSICAL FACTS AND THE DIFFERENTIAL EQUATIONS

JEAN LERAY, College de France

1. Introduction. A differential equation raises not only the problem of proving:

(A) the *existence* and *regularity* of its solutions; their uniqueness and their continuity with respect to the datum or at least their *compactness* for the datum running on a compact set.

Indeed, when those properties cannot be established, then the following alternatives arise:

(B) allowing the solution to have convenient singularities, to obtain *existence- and compactness-theorems without the regularity-theorem* (and indeed without the uniqueness-theorem even when uniqueness should be expected),

(C) to construct counter-examples proving that *the compactness-theorem does not hold.*

Assume that the equation is the conclusion of a physical theory expecting the existence, the regularity (and maybe the uniqueness) of its solutions. If (C) is possible, then the properties of the equation contradict the theory. On the contrary, if (A) holds, then the physical phenomenon is actually ruled by the equation. If not, and if (B) is possible, that is if the expected regularity (and uniqueness) is not obtained, then the physical theory neglected something which appears in the end to be not entirely negligible. The theory does not completely describe the phenomenon, which eludes perhaps any complete description. This often happens when the equation is non-linear, especially in the mechanics of fluids; their equations are based on the assumption that the flow is laminar, but the experiment shows turbulence.

Many papers have been devoted to problem (A); few to (B) and (C). The processes required for the establishment of (A) are often difficult; wherever (B) and (C) have been solved, it was by means of very simple processes, although at times somewhat concealed by technical details. We shall describe some of these here.

2. An equation which rules an evolution and for which (B) holds. Consider the Navier-Stokes system ruling incompressible viscous flows; it is parabolic and non-linear; the velocity is given at the initial time $t=0$ and has to be obtained for any $t>0$.

By successive approximations the flow can be calculated from $t=0$ to $t=T_1$, then from $t=T_1$ to $t=T_2$, *etc.* Let $T=\lim T_n$; the problem is solved and its solution is regular if $T=+\infty$. That occurs in some special cases: two-dimensional flows without boundary, sufficiently small data, *etc.* The proof is based on well known properties of the vortex; it can also be based on the relation expressing the conservation of the energy.

But the *homogeneity* of the Navier-Stokes system is such that the conservation of the energy and the smoothing action of the viscosity seem to be insufficient for maintaining the flow so regular that $T=+\infty$.

However, the experiment shows that the flow exists from $t=0$ to $t=+\infty$. Furthermore the Navier-Stokes system can be slightly modified in such a way that:

(a) the energy relation is preserved;

(b) the smoothing properties of the viscosity are so increased that the flow cannot become irregular and hence exists from $t=0$ to $t=+\infty$.

Let this modified system tend to the Navier-Stokes system; its solution converges weakly to one (or several) solutions of the Navier-Stokes system. This solution is not regular; it is *only square integrable* at any moment of time; it satisfies the Navier-Stokes system in the sense of L. Schwartz's distribution theory. Problem (B) is solved.

In some cases (three-dimensional flow without boundary, two-dimensional flow inside a convex boundary) a detailed study shows that such an irregular solution is regular except for a compact subset of values of t of measure zero.

The uniqueness of the flow and *the continuity* of its kinetic energy at the moments of time where it is not regular *are two open problems*; they are related.

No example of irregular flow has been obtained, but the above-mentioned homogeneity of the Navier-Stokes system suggests a process which probably leads to such an example.

3. An equation ruling an equilibrium and for which (A) holds in some cases, (C) in the other cases. Consider *the Dirichlet problem* for *non-linear, second order, elliptic* differential equations *in two independent variables*. The fixed point theory and the theory of the linear elliptic equations reduce the existence theorem to the compactness theorem, more precisely to the obtaining of *a priori* bounds for the solution and its first and second derivatives. These bounds follow from an extension of the maximum principle to the solutions of the studied equation, to their first and to their second derivatives. Assumptions about the equation and the Dirichlet data have to be made; they mean that the equation has a convenient behavior on the vertical cylinders (especially on the cylinder containing the graph of Dirichlet's data) or, if the equation belongs to Monge-Ampère's type, on the curves (especially on this graph).

The proof that those sufficient assumptions are necessary (when the equation behaves regularly on those cylinders and curves) is quite simple. It consists in the construction of a solution of the equation such that:

(a) this solution has at a point a vertical tangent plane or an infinite curvature;

(b) this solution is the limit of regular solutions defined on the same domain.

This is easily done by applying the Cauchy-Kowalewski theorem to the equation transformed by a change of coordinates or by a contact transformation.

The non-linear Dirichlet problem in three or more independent variables is

open. *A priori* bounds cannot be obtained again by a maximum principle; no existence or compactness theorems are known; problem (A) is not solved and probably problem (B) should be considered. However, the construction of the counter-examples showing that (C) holds could easily be extended.

Bibliography

Navier-Stokes system

Its origin, the linear Navier-Stokes system:

C. Oseen, Hydromechanik, (Leipzig, 1927).

Three-dimensional flow without boundary:

J. Leray, Acta math., vol. 63, 1934, pp. 193–248.

Two-dimensional flow inside a convex boundary:

J. Leray, Journal de math., vol. 13, 1934, pp. 331–418.

Regular flows:

J. Leray, Journal de math., vol. 12, 1933, ch. iv, pp. 64–82.
J. Leray, Comptes rendus Acad., vol. 194, 1932, pp. 1892.
W. Wolibner, Math. Zeits., vol. 37, 1933, pp. 698–726.
E. Hölder, Math. Zeits., vol. 37, 1933, pp. 727–738.
J. Leray et L. Robin, Comptes rendus Acad., vol. 205, 1937, p. 18.
P. Casal, Congrès intern. Mécan., 1952.

The general case:

E. Hopf, Math. Nachrichten, vol. 4, 1950, pp. 213–231.

Non-linear Dirichlet's problem

First studied by:

S. Bernstein, Math. Ann., vol. 69, 1910, pp. 82–136; vol. 95, 1926, pp. 585–594; vol. 96, 1927, pp. 633–647; Annales Ecole norm., vol. 27, 1910, pp. 233–256; vol. 29, 192, pp. 431–485.

His preliminaries about linear equations have been largely simplified by

J. Schauder, Math. Zeits., vol. 37, 1933, pp. 623–634; Studia Math., vol. 5, 1935, pp. 34–42.
L. Bers, L. Nirenberg (to be published)

The uniqueness assumption has been removed by

J. Leray, J. Schauder, Annales Ecole Norm. Sup., vol. 51, 1934.

H. Lewy and J. Schauder found a mistake in S. Bernstein's *a priori* bound of the first derivative of a quasi-linear equation (Math. Ann., vol. 69 p. 129: the expression of K is obviously wrong); furthermore, his conclusions are not invariant with respect to a change of coordinates; his study of the Monge-Ampère equation is incomplete. His results have been corrected and completed by

J. Leray, Journal de Math., vol. 17, 1938, pp. 89–104; vol. 18, 1939, pp. 249–284.

RECENT ADVANCES AT STANFORD IN THE APPLICATION OF CONFORMAL MAPPING TO HYDRODYNAMICS

P. R. GARABEDIAN, EDWARD McLEOD, JR., and MARTIN VITOUSEK,
Stanford University

The present paper forms an expository report of researches carried out recently at Stanford University by a group of mathematicians interested in free boundary problems of hydrodynamics. The discussion centers about a unified technique in geometric function theory.

The steady, irrotational motion of an incompressible, inviscid fluid in the z-plane is governed by an analytic complex potential $\zeta(z)=\phi+i\psi$. For a fluid of unit density, the pressure p is given by Bernoulli's equation

$$\tfrac{1}{2}\,|\,\zeta'(z)\,|^2+gy+p=\text{const.},\tag{1}$$

where g is the gravitational constant. This law, in general, determines an additional boundary condition along free streamlines.

Following [3], we consider the uniform plane flow of a liquid past a small bubble of gas. We assume that the bubble is so small that the forces due to surface tension surpass significantly those due to gravity. Setting $g=0$ in (1) and resolving all forces along the normal to the free streamline, we obtain on the boundary C of the bubble a condition of the form

$$|\,\zeta'(z)\,|^2=T\kappa,\tag{2}$$

where κ is the curvature of the free boundary C and T represents the magnitude of the surface tension.

Denoting differentiation with respect to arc length s along C by a dot, we throw (2) into the form

$$\zeta'(z)^2\dot{z}=T\kappa\bar{\dot{z}},\tag{3}$$

and we obtain along C upon integration

$$q(z)=\int\zeta'(z)^2dz=iT\bar{\dot{z}},\tag{4}$$

where $q(z)$ is an analytic function in the flow region D. Hence $q(z)$ maps D conformally upon a Riemann surface bounded by a circle of radius T, while ζ maps D conformally upon the exterior of a horizontal slit. If $w(z)$ maps D upon the exterior of the unit circle, we can therefore write

$$\zeta=w+\frac{1}{w},\tag{5}$$

$$q=Tw\frac{w^2+A^2}{1+A^2w^2},\qquad A^2>1.\tag{6}$$

But since $dqdz = d\zeta^2$, we find

$$z = \frac{1}{T}\int \frac{\left(1 - \frac{1}{w^2}\right)^2 dw}{\frac{d}{dw}\left[w\frac{w^2 + A^2}{1 + A^2w^2}\right]}, \tag{7}$$

where A is to be determined so that the numerator and the denominator of the integrand vanish simultaneously. We obtain the explicit formula

$$z = w - \frac{2}{3w} - \frac{1}{27w^3} \tag{8}$$

by integrating (7), and, together with (5), this determines completely the shape of the bubble and the corresponding flow.

In [4] a treatment is given of a free surface flow representing a water fountain in which gravity, rather than surface tension, is predominant. Taking $g = 1/2$ and choosing suitable coordinates, we can put (1) in the form

$$\frac{dz}{d\zeta}\frac{d\bar{z}}{d\zeta} + \frac{1}{y} = 0 \tag{9}$$

along the free streamline. Following [2], we denote by $\lambda(\zeta)$ an analytic function in the flow region D which coincides with y on the free surface, and we substitute into (9) to obtain by analytic continuation the ordinary differential equation

$$z'(z' - 2i\lambda') + \frac{1}{\lambda} = 0, \tag{10}$$

where the prime indicates differentiation with respect to ζ. Along a vertical linear streamline, this differential equation in the unknown λ is real, and we can expect to obtain a real solution for real initial data.

The desired flow is required to rise along the y-axis until it reaches a free surface at the origin which spreads smoothly to either side and falls, joining two fixed vertical boundaries which descend to infinity. The continuation of the free streamline is linear and vertical, so that we can expect $\lambda(z)$ to map it onto the real axis. Since $\zeta(z)$ has the same mapping property, we are led to set $\lambda = \zeta$ in (10). Integration then gives

$$z = i\zeta - i\int_0^{\zeta} \sqrt{1 + \frac{1}{\zeta}}\, d\zeta, \tag{11}$$

a formula which does, indeed, represent a gravity flow of the type described.

These techniques lead to further interesting free surface flows involving, respectively, surface tension and gravity. However, such examples do not exhaust the method. A successful treatment of cavitation in axially symmetric

flow yields to the present technique. The essential difficulty for this case stems from the more complicated partial differential equation

$$\frac{\partial^2\psi}{\partial z\partial z^*}+\frac{1}{2(z-z^*)}\frac{\partial\psi}{\partial z}-\frac{1}{2(z-z^*)}\frac{\partial\psi}{\partial z^*}=0,\quad z=x+iy,\quad z^*=x-iy, \tag{12}$$

satisfied in the meridian plane by the stream function ψ governing the flow, but it can be largely overcome by appropriate exploitation of the Riemann function

$$R(z,z^*;t,t^*)=\frac{(z-t^*)^{1/2}(t-z^*)^{1/2}}{t-t^*}F\left[\frac{(z-t)(z^*-t^*)}{(z-t^*)(z^*-t)}\right] \tag{13}$$

associated with this equation in the domain of two independent complex variables z and z^*. Here $F[w]=F(-1/2,\ -1/2,\ 1,\ w)$ is the hypergeometric series. Indeed, the stream function ψ of an axially symmetric free surface flow for which the analytic curve $\bar{z}=g(z)$ appears as the free boundary has an explicit representation

$$\psi(z,\bar{z})=\operatorname{Re}\left\{\frac{1}{2i}\int_{z_0}^{z}(z-g(t))^{1/2}(\bar{z}-t)^{1/2}F\left[\frac{(z-t)(\bar{z}-g(t))}{(z-g(t))(\bar{z}-t)}\right]g'(t)^{1/2}dt\right\}, \tag{14}$$

where z_0 is a fixed point on the curve. Analysis of the conformal mapping properties of the analytic function $g(z)$ analogous to the study of $q(z)$ and $\lambda(z)$ carried through above yields explicit examples based on (14) of axially symmetric cavity flow in the large [1].

References

1. P. R. Garabedian, An Example of Axially Symmetric Flow with a Free Surface, Technical Report No. 5, Stanford University, 1952.
2. H. Lewy, On Steady Free Surface Flow in a Gravity Field, Technical Report No. 4, Stanford University, 1951.
3. Edward B. McLeod, Jr., An Application of the Schiffer Variation to the Free Boundary Problems of Hydrodynamics, Technical Report No. 15, Stanford University, 1953.
4. Martin Vitousek, Dissertation, Stanford University, to appear.

SINGULARITIES ON SHOCKS

A. H. TAUB, University of Illinois

1. Introduction. It is our purpose to discuss the situation which arises when a shock front is such that there are points on it at which the tangent to the shock, the curvature of the shock or some derivative of the curvature is discontinuous. Such shocks will be called shocks with singularities. They occur in many shock interaction problems; for example, the shock *MTRO*, in Figure 1, which illustrates the shock configuration arising in the Mach reflection of a plane shock from a rigid wall [1], may be considered as a shock with a singularity at the point T at which point the tangent to this shock is discontinuous.

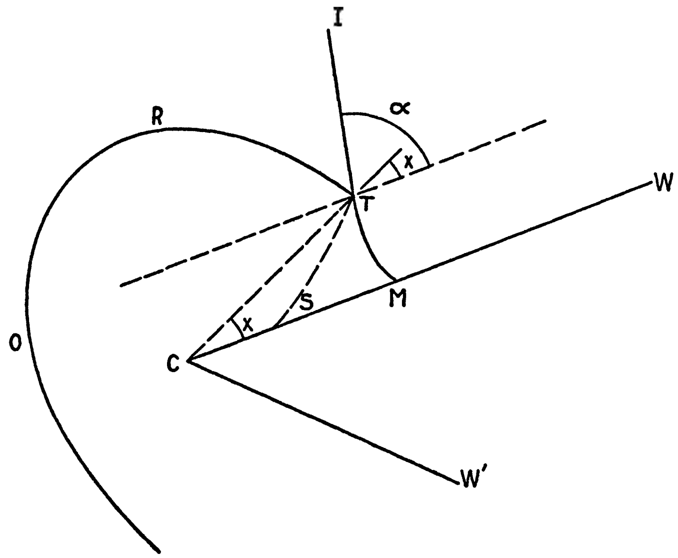


FIG. 1. Incident shock is TI; reflected shock is TRO; Mach shock is TM; slip stream is TS; wall is CW.

We shall restrict the discussion to two dimensional pseudo-stationary and two dimensional stationary flows. It will be sufficient to discuss in detail only one of the cases which will be chosen to be the first one, for the key equations needed in the discussion have been shown, in a previous publication [2], to be identical.

The shock configuration in a two dimensional pseudo-stationary flow at time t may be represented by the equations

$$
x = ta_1(s)
$$
$$
y = ta_2(s), \tag{1.1}
$$

where x and y are fixed Cartesian coordinates in the plane in which the shock

is moving, and s is a parameter along the shock which may be chosen so that

$$\left(\frac{da_1}{ds}\right)^2 + \left(\frac{da_2}{ds}\right)^2 = 1 \tag{1.2}$$

at regular points. At a singular point of the shock some derivative of the functions $a_i(s)$ is discontinuous.

We shall assume that at time t each point of the region behind the shock with a singularity is occupied by a single fluid particle which has crossed the shock at some time t_0 earlier than t.

Let us denote by s_0 the value of s corresponding to a singular point of the shock and refer to the point on the shock as the point s_0. The locus of particles which have crossed the shock at s_0 at times $t_0 < t$ must be a unique locus separating the region behind the shock with a singularity into just two regions in accordance with the assumption made above. However, this locus will be a locus of discontinuities in the flow behind the shock and therefore must be either a slip-stream or a shock. Only the former possibility is allowable as will now be shown.

The position at time t of a particle which has crossed a shock at a point s at time t_0 is given by

$$\begin{aligned} x &= ta_1(s, \tau) \\ y &= ta_2(s, \tau), \end{aligned} \tag{1.3}$$

where

$$\tau = \log \frac{t}{t_0} \tag{1.4}$$

and $a_i(s, \tau)$ are solutions of the equations

$$\frac{da_i}{d\tau} = U_i = u_i - a_i \tag{1.5}$$

where u_i is the particle velocity relative to the fixed Cartesian coordinate system; the $a_i(s, \tau)$ satisfy

$$a_i(s, 0) = a_i(s), \tag{1.6}$$

where the functions on the right hand side of this equation are the functions entering into equations (1.1), the equations specifying the shock.

Equations (1.3) or the functions $a_i(s, \tau)$ for fixed s and variable t_0 give the locus of particles which have crossed the shock at a point s at times $t_0 \leqq t$. We shall call these loci *streak-lines*. Our fundamental assumption implies that for a shock with a singularity at s_0 that the locus given by

$$\lim_{\epsilon \to 0} a_i(s_0 + \epsilon, \tau) = \overset{+}{a_i}(\tau) \tag{1.7}$$

be identical with the locus given by

$$\lim_{\epsilon \to 0} a_i(s_0 - \epsilon, \tau) = a_i^-(\tau) \tag{1.8}$$

where τ is the parameter along this locus, defined by equation (1.4) with t fixed and t_0 variable. The two functions obtained by these two limiting procedures need not be identical in τ but they must represent the same curve. In Figure 1 this curve is represented by TS and called the *slip-stream.*

From equation (1.5) it follows that the tangent vector to the locus of particles that have crossed the shock at s at times $t_0 < t$ has components proportional to U_i. We denote by $U_i^+(a_1, a_2)$ the values of U_i at the point a_i on or above the singular locus described by (1.7) (in Figure 1 this region is STR) and by $U_i^-(a_1, a_2)$ the corresponding quantity at the point a_i on or below the singular locus described by (1.8) (in Figure 1 this region is STM). Since these two loci must be identical we must have

$$\epsilon_{ik} \overset{+}{U_i}(\overset{+}{a_i}(\tau)) \overset{-}{U_k}(\overset{+}{a_i}(\tau)) \equiv 0. \tag{1.9}$$

That is, their tangent vectors at the same points are equal:

$$\frac{\overset{+}{U_i}(\overset{+}{a}(\tau))}{v^+} = \frac{\overset{-}{U_i}(\overset{+}{a}(\tau))}{v^-} \tag{1.10}$$

where

$$(\overset{\pm}{v})^2 = (\overset{\pm}{U_1})^2 + (\overset{\pm}{U_2})^2 \tag{1.11}$$

Equation (1.9) states that there is no flow across the singular locus (1.7) (or (1.8)) and hence this locus must be a slip-stream. It is a consequence of (1.9) and the Rankine-Hugoniot equations given below that

$$\overset{+}{p}(\overset{+}{a_i}(\tau)) \equiv \overset{-}{p}(\overset{+}{a_i}(\tau)) \tag{1.12}$$

where $p^+(a_i)$ is the pressure at a point a_i in the flow on or above the singular locus and $p^-(a_i)$ is the pressure at a point a_i in the flow on or below the singular locus. Equations (1.9) and (1.12) are identities in τ, and $a_i^+(\tau)$ represents a point on the singular locus.

In the subsequent sections of this paper we shall show how equations (1.10) and (1.12) and the derivatives of these equations may be used to determine geometric properties of the shock configuration at a singular point.

Since equations (1.9) and (1.12) are identities in τ, the derivatives of these equations with respect to τ must also hold. It follows from equation (1.9) that

$$\epsilon_{ik} \overset{+}{U}_{i,j} \overset{+}{U_j} \overset{-}{U_k} + \epsilon_{ik} \overset{+}{U_i} \overset{-}{U}_{k,j} \overset{+}{U_j} = 0,$$

where the comma denotes the ordinary derivative with respect to variable a and the argument of the functions U_i^+ and U_i^- is $a_i^+(\tau)$. In view of equation (1.10), this equation may be written as

$$(1.13) \qquad \epsilon_{ik}\frac{\overset{+}{U}_{i,j}\overset{+}{U}_j\overset{+}{U}_k}{(v^+)^3} + \epsilon_{ik}\frac{\overset{-}{U}_i\overset{-}{U}_{k,j}\overset{-}{U}_j}{(v^-)^3} = 0.$$

However, the curvature of a curve whose tangent vector is proportional to U_i is given by

$$(1.14) \qquad K = -\frac{\epsilon_{ik}U_{i,j}U_jU_k}{v^3}\,.$$

Hence it follows from (1.13) that

$$(1.15) \qquad K^+(a^+(\tau)) \equiv K^-(a^+(\tau)),$$

where K^+ is the curvature of the locus described by (1.7) and K^- is the curvature of the locus described by (1.8). Differentiating both sides of equation (1.15) we obtain

$$(1.16) \qquad K^{+\prime} = K^{-\prime}$$

where the prime denotes the derivative with respect to the arc length along the singular locus. Similarly we may show that

$$(1.17) \qquad K^{+(n)} = K^{-(n)}$$

where the superscript n indicates the nth derivative with respect to the arc length of the singular locus.

We may also show that equation (1.12) implies that

$$(1.18) \qquad p^{+(n)} = p^{-(n)}.$$

Equations (1.17) and (1.10) are merely the formal consequences of the requirement that equations (1.7) and (1.8) represent the same locus, for the locus is intrinsically specified in terms of the tangent vector, the curvature, and the derivatives of the curvature.

2. The equations describing the flow. In order to determine the geometric properties of a shock at a singular point we use equations (1.17), (1.10), and (1.18) evaluated at the shock (*i.e.*, at $\tau=0$) and the relations obtained by Thomas [3] relating the curvature and the derivatives of the curvature with respect to arc length of a streak-line in stationary flow with the corresponding quantities of the shock from which the streak-line emanates. The identical relations may be used in pseudo-stationary flows as follows from the results of Taub [2]. Thus we assume analyticity of the flow in regions behind the shock front with a singularity and replace equations (1.9) and (1.12) by sequences of conditions at the singular point of the shock, namely, by equations (1.10), (1.17) and (1.18) evaluated at $\tau=0$. These equations are then replaced by conditions involving the quantities characterizing the shock by using results from consideration involving the differential equations that must be satisfied by the

flow variables and the Rankine-Hugoniot equations that must hold across a shock.

For pseudo-stationary flows the former equations may be written as: the conservation of mass,

$$(2.1)\qquad (\rho U_i)_{,i} + 2\rho = 0,$$

the conservation of momentum,

$$(2.2)\qquad U_j U_{i,j} + U_i + \frac{1}{\rho} p_{,i} = 0,$$

and the conservation of energy,

$$(2.3)\qquad U_i U_j U_{i,j} + v^2 - c^2(U_{i,i} + 2) = 0,$$

where ρ is the density and

$$(2.4)\qquad c^2 = \frac{\gamma p}{\rho}\,.$$

The Rankine-Hugoniot equations may be written as

$$(2.5)\qquad U_t = U_i\lambda_i = U_{1i}\lambda_i = U_{1t}$$

$$(2.6)\qquad y - 1 = \frac{2\gamma}{\gamma + 1}(\sigma^2 - 1)$$

$$(2.7)\qquad \eta = \frac{(\gamma + 1)\sigma^2}{2 + (\gamma - 1)\sigma^2}$$

$$(2.8)\qquad U_n = v_i\epsilon_{ij}\lambda_j = \frac{c_1}{(\gamma + 1)\sigma}((\gamma - 1)\sigma^2 + 2),$$

where λ_i are the components of the tangent vector to the shock,

$$(2.9)\qquad \sigma = \frac{U_{1i}\epsilon_{ij}\lambda_j}{c_1} = \frac{U_{1n}}{c_1}$$

$$(2.10)\qquad y = \frac{p}{p_1}$$

$$(2.11)\qquad \eta = \frac{\rho}{\rho_1}$$

and we have used the subscript 1 to denote quantities evaluated on one side of the shock, a side which we will assume to be uniform. Thus in Figure 1 the subscript 1 may refer to the region *TRI* or to the region to the right of *TM*.

In the work of Thomas referred to above, equations (2.1) to (2.3) and the derivatives of equations (2.5) to (2.8) along the shock when evaluated at the

shock were considered as algebraic equations for the quantities $U_{i,j}$, $p_{,i}$ and $\rho_{,i}$. The solutions of these equations were then used to give explicit expressions for the derivatives of the flow variables behind a shock in terms of properties of the shock. Similar methods were used to obtain relations between higher derivatives of the quantities U_i, p and ρ at a point behind a shock in terms of the derivatives of the equations characterizing the shock.

3. Thomas' results. We shall describe a method for obtaining Thomas' results and the addition equations needed for the carrying out of the procedure outlined above. We consider the functions

$$a_i = a_i(s, \tau) \tag{3.1}$$

which are solutions of equations (1.5) subject to the initial conditions (1.6) as a transformation from the variables a_i to a new set of independent variables s and τ. Then

$$\frac{\partial a_i}{\partial \tau} = U_i \tag{3.2}$$

$$\frac{\partial a_i}{\partial s} = \lambda_i, \tag{3.3}$$

where λ_i is defined by equations (3.3) and is such that

$$\frac{\partial \lambda_i}{\partial \tau} = \frac{\partial U_i}{\partial s}. \tag{3.4}$$

It then follows that

$$\frac{\partial \tau}{\partial a_i} = \frac{\epsilon_{ij}\lambda_j}{U_n} \tag{3.5}$$

and

$$\frac{\partial s}{\partial a_i} = -\frac{\epsilon_{ij}U_j}{U_n}, \tag{3.6}$$

where

$$U_n = \epsilon_{ij}U_i\lambda_j. \tag{3.7}$$

We now have

$$\begin{aligned} U_{i,j} &= \frac{\partial U_i}{\partial \tau}\frac{\partial \tau}{\partial a_j} + \frac{\partial U_i}{\partial s}\frac{\partial s}{\partial a_j} \\ &= \frac{\partial U_i}{\partial \tau}\frac{\epsilon_{jk}\lambda_k}{U_n} - \frac{\partial \lambda_i}{\partial \tau}\frac{\epsilon_{jk}U_k}{U_n}. \end{aligned} \tag{3.8}$$

Hence

$$U_{i,i} = \frac{\partial \log U_n}{\partial \tau} \tag{3.9}$$

and

$$\epsilon_{ij} U_{i,j} = \frac{1}{U_n}\left(2 \frac{\partial U_i}{\partial s} U_i - \frac{\partial U_t}{\partial \tau}\right) \tag{3.10}$$

where

$$U_t = U_i \lambda_i. \tag{3.11}$$

Equations (3.9) enable us to obtain a first integral of the conservation of mass equation. Equation (2.1) may be written as

$$\frac{\partial \rho}{\partial \tau} + \rho U_{i,i} + 2\rho = 0$$

or as

$$\frac{\partial \log \rho}{\partial \tau} + \frac{\partial \log U_n}{\partial \tau} + 2 = 0.$$

Hence we must have

$$\rho U_n = \rho_0 U_{0n} e^{-2\tau}, \tag{3.12}$$

where the arguments of the functions on the left hand side of this equation are s and τ and we have introduced the convention that

$$f_0 = f(s, 0). \tag{3.13}$$

The conservation of momentum equations may be written as

$$\frac{\partial U_i}{\partial \tau} + U_i = -\frac{1}{\rho} p_{,i}. \tag{3.14}$$

Multiplying these equations by U_i and summing we have

$$\frac{\partial U_i}{\partial \tau} U_i + v^2 = -\frac{1}{\rho}\frac{\partial p}{\partial \tau} \tag{3.15}$$

or

$$\frac{1}{2} e^{-2\tau} \frac{\partial}{\partial \tau}(e^{2\tau} v^2) = -\frac{1}{\rho}\frac{\partial p}{\partial \tau}, \tag{3.16}$$

and multiplying by λ_i and summing we have

$$\frac{\partial U_t}{\partial \tau} + U_t = -\frac{1}{\rho}\frac{\partial p}{\partial s} + \frac{1}{2}\frac{\partial v^2}{\partial s}. \tag{3.17}$$

The conservation of energy equation may be written as

$$\frac{\partial}{\partial \tau}(p\rho^{-\gamma}) = 0$$

and admits the first integral

$$p = p_0\left(\frac{\rho}{\rho_0}\right)^{\gamma}. \tag{3.18}$$

Hence in the s, τ coordinate system, in which the curves $s=$constant are the loci of particles which have crossed the shock at earlier times, the streak-lines, and the curves $\tau=$constant are a family of curves which include the shock (for the value $\tau=0$), the conservation equations admit two first integrals and take on the relatively simple form (3.16) and (3.17) in addition. The use of these coordinates in the integration of the equations of pseudo-stationary flows will be discussed elsewhere.

Equation (2.3) may be written as

$$U_i\frac{\partial U_i}{\partial \tau} - \frac{c^2}{U_n}\frac{\partial U_n}{\partial \tau} = 2c^2 - U_iU_i.$$

If we let

$$\lambda^2 = \lambda_i\lambda_i,$$

then

$$\lambda^2 v^2 = \lambda^2 U_iU_i = U_n^2 + U_t^2$$

$$\lambda^2 U_i\frac{\partial U_i}{\partial \tau} + \lambda_i\frac{\partial \lambda_i}{\partial \tau}U_jU_j = U_n\frac{\partial U_n}{\partial \tau} + U_t\frac{\partial U_t}{\partial \tau}.$$

Hence

$$U_i\frac{\partial U_i}{\partial \tau} = \frac{1}{\lambda^2}\left(U_n\frac{\partial U_n}{\partial \tau} + U_t\frac{\partial U_t}{\partial \tau}\right) - \frac{v^2}{\lambda^2}\lambda_i\frac{\partial U_i}{\partial s}$$

and

$$U_n\frac{\partial U_n}{\partial \tau} + U_t\frac{\partial U_t}{\partial \tau} - v^2\lambda_i\frac{\partial U_i}{\partial s} - \frac{c^2\lambda^2}{U_n}\frac{\partial U_n}{\partial \tau} = 2c^2\lambda^2 - v^2\lambda^2.$$

Substituting in this equation from equation (3.17) we obtain

$$\frac{\partial U_n}{\partial \tau} = \frac{U_n}{U_n^2 - c^2\lambda^2}\left(2c^2\lambda^2 - U_n^2 + v^2\lambda_i\frac{\partial U_i}{\partial s} + U_t\left(\frac{1}{\rho}\frac{\partial p}{\partial s} - \frac{1}{2}\frac{\partial v^2}{\partial s}\right)\right)$$

or

$$
\frac{\partial U_n}{\partial \tau} = \frac{U_n}{U_n^2 - c^2\lambda^2}\left[2(c^2\lambda^2 - U_n^2) + v^2\left(\lambda^2 + \lambda_i \frac{\partial U_i}{\partial s}\right) + U_t\left(\frac{1}{\rho}\frac{\partial p}{\partial s} - v\frac{\partial v}{\partial s} - U_t\right)\right]. \tag{3.20}
$$

At a point on the shock, equations (3.17) and (3.20) express the rate of change of the pseudo-velocity vector U along the streak-lines in terms of the derivatives of the flow variables along the shock. That is, the right hand sides of these equations may be computed from the functions a_i occurring in equations (1.1) and from equations obtained by differentiating the Rankine-Hugoniot equations (2.5) to (2.8). We may also compute $\partial p/\partial \tau$ and $\partial \rho/\partial \tau$ behind a shock from the right hand sides of equations (3.17) and (3.20), that is, in terms of the derivatives of the flow variables along a shock.

Thus

$$
\frac{\partial \rho}{\partial \tau} = -2\rho - \frac{\rho}{U_n}\frac{\partial U_n}{\partial \tau}.
$$

Substituting from (3.20) we obtain

$$
\frac{\partial \rho}{\partial \tau} = -\frac{\rho}{U_n^2 - c^2\lambda^2}\left[v^2\left(\lambda^2 + \lambda_i\frac{\partial U_i}{\partial s}\right) + U_t\left(\frac{1}{\rho}\frac{\partial p}{\partial s} - v\frac{\partial v}{\partial s} - U_t\right)\right]. \tag{3.21}
$$

It follows from the equation preceding equation (3.18) that

$$
\frac{\partial p}{\partial \tau} = c^2\frac{\partial \rho}{\partial \tau} = \frac{-\rho c^2}{U_n^2 - c^2\lambda^2}\left[v^2\left(\lambda^2 + \lambda_i\frac{\partial U_i}{\partial s}\right) + U_t\left(\frac{1}{\rho}\frac{\partial p}{\partial s} - v\frac{\partial v}{\partial s} - U_t\right)\right]. \tag{3.22}
$$

The curvature of a streak-line may also be expressed in terms of the flow variables and their derivatives with respect to s. From the definition of curvature we have

$$
\begin{aligned}
-v^3K &= \epsilon_{ik}\frac{\partial U_i}{\partial \tau}U_k \\
&= -\frac{1}{\rho}\epsilon_{ik}p_{,i}U_k
\end{aligned}
$$

as follows from the conservation of momentum equations. However

$$
p_{,i} = \frac{1}{U_n}\left(\frac{\partial p}{\partial \tau}\epsilon_{il}\lambda_l - \frac{\partial p}{\partial s}\epsilon_{il}U_l\right).
$$

Hence

$$v^3K = \frac{1}{\rho U_n}\left(\frac{\partial p}{\partial \tau} U_t - \frac{\partial p}{\partial s} v^2\right), \tag{3.23}$$

where we may substitute from the equation (3.22) for $\partial p/\partial \tau$.

Equations (3.22) and (3.23) when evaluated for $\tau=0$, that is, at the shock, give explicit expressions for $p^{\pm(1)}$ and $K^{\pm(0)}$ needed in the determination of the geometric properties of a shock with a singularity by means of equations (1.17) and (1.18). Explicit expressions for $p^{\pm(n)}$ and $K^{\pm(n)}$ may also be obtained from equations (3.22) and (3.23) respectively as follows: Either of these equations may be differentiated n times with respect to τ. The right hand sides may be reduced to expressions involving only derivatives with respect to s. The derivatives with respect to s may be evaluated by differentiating the Rankine-Hugoniot equations.

We shall demonstrate the method outlined by evaluating equations (3.22) and (3.23) for $\tau=0$. It follows from equations (2.5) to (2.8) that if the flow ahead of the shock is uniform, then (*cf.* [2] eq. (4.2) to (4.6))

$$\begin{aligned} \frac{\partial U_{1i}}{\partial s} &= -\lambda_i \\ \frac{\partial U_{1n}}{\partial s} &= kU_{1t} = c_1\frac{\partial \sigma}{\partial s} \\ \frac{\partial U_{1t}}{\partial s} &= -1 - kU_{1n} = -1 - kc_1\sigma = \frac{\partial U_t}{\partial s} \end{aligned} \tag{3.24}$$

where k is the curvature of the shock and λ_i in these equations represents the tangent vector to the shock. From the definition of s in equations (1.1) it follows that

$$\lambda^2(s, 0) = \lambda_i\lambda_i = 1. \tag{3.25}$$

Further it may be shown that at $\tau=0$

$$\begin{aligned} \frac{\partial p}{\partial s} &= \frac{4kU_tU_n\rho}{\gamma+1} \\ \frac{\partial \rho}{\partial s} &= \frac{4kU_t\rho_1}{(\gamma+1)U_n}(2-(\gamma-1)(\eta-1)) = \rho_1\frac{\partial \eta}{\partial s} \\ \frac{\partial U_n}{\partial s} &= kU_t\left(\frac{2(\gamma-1)}{\gamma+1} - \frac{1}{\eta}\right) \end{aligned} \tag{3.26}$$

and

$$1 + \lambda_i\frac{\partial U_i}{\partial s} = -k(\eta-1)U_n \tag{3.27a}$$

$$U_t + v\frac{\partial v}{\partial s} = -kU_tU_n\left(\eta + \frac{1}{\eta} - \frac{2(\gamma-1)}{\gamma+1}\right). \tag{3.27b}$$

Substituting from these equations into (3.22) and (3.23) we obtain at $\tau = 0$

$$\frac{\partial p}{\partial \tau} = -\frac{\rho c^2 k U_n}{U_n^2 - c^2}\left(\left(\frac{8}{\gamma+1} + \frac{1-\eta}{\eta}\right)U_t^2 - (\eta - 1)U_n^2\right) \tag{3.28}$$

and

$$K = k\frac{U_t}{v^3}\left[\frac{c^2}{U_n^2 - c^2}\left(\frac{8}{\gamma+1} + \frac{1-\eta}{\eta}\right)U_t^2 - (\eta - 1)U_n^2\right] \tag{3.29}$$

respectively.

It has been shown by Thomas [3] for stationary flows and by Taub [2] for pseudo-stationary flows that at $\tau = 0$

$$K^{(n)} = G_n(\eta, U_t/U_n)k^{(n)} + H_n \tag{3.30}$$

where $K^{(n)}$ is the quantity defined in section 1, $k^{(n)}$ is the nth derivative of the curvature of the shock with respect to s, and H_n is a polynomial in k and its derivatives with respect to s of order less than n.

It follows from the arguments given above that at $\tau = 0$

$$p^{(n)} = f_n(\eta, U_t/U_n, k, k^{(1)}, \cdots, k^{(n-1)}). \tag{3.31}$$

That is, behind the shock $p^{(n)}$ is a function of the strength of the shock, its inclination with respect to the flow incident upon it, the curvature of the shock and the first $n-1$ derivatives of this curvature with respect to s.

4. Conditions at singular points of a shock. The formulas of the preceding section enable us to translate the conditions which must obtain at a singular point of a shock, namely, equations (1.10), (1.17) and (1.18), into sequences of algebraic relations between the variables $\eta^{\pm}$, $U_t^{\pm}/U_n^{\pm}$, $k^{\pm}$ and $k^{\pm(i)}$ $(i = 1, 2, \cdots)$.

A shock which has a straight portion and a singular point at which the curvature or some derivative of the curvature jumps to a non-vanishing value cannot have a uniform flow behind the straight portion of the shock at the singular point. This follows from the fact that if the flow were uniform, equations (3.30) would apply for the calculation of $K^{+(n)}$ and hence these quantities would vanish. However, the quantities $K^{-(n)}$ would not vanish and it would be impossible to satisfy equations (1.17). The discussion of a special case of a shock with such a singularity is contained in [2], where a shock with a point at which the curvature jumps from zero to a non-vanishing value is discussed.

In a shock configuration such as shown in Figure 1, a shock with uniform flows incident on it and with a singularity at which the tangent is discontinuous may exist. The sequence of algebraic conditions which must hold may then be paired as follows:

$$\begin{aligned} \overset{-}{v}\,\overset{+}{U_i} &= \overset{+}{v}\,\overset{-}{U_i} \\ \overset{+}{p} &= \overset{-}{p}, \end{aligned} \tag{4.1}$$

the first of these being (1.10) and the second being (1.18) for $n=0$;

$$\begin{aligned} K^{+} &= K^{-} \\ p^{+(1)} &= p^{-(1)} \end{aligned} \tag{4.2}$$

where we have used (1.17) and (1.18); and the remaining equations may be written as

$$\begin{aligned} K^{+(n)} &= K^{-(n)} \\ p^{+(n+1)} &= p^{-(n+1)} \qquad n = 1, 2, \cdots . \end{aligned} \tag{4.3}$$

Equations (4.1) are the fundamental equations of the three-shock theory (*cf.* [1]). They may be solved for η^{+}, U_j^{+}/U_n^{+}, η^{-} and U_j^{-}/U_n^{-} as functions of the strength and angle between the flow incident on and the normal to the shock TI of Figure 1. The solutions are not unique and only very special ones exist for some values of these two parameters characterizing the shock TI.

Equations (4.1) and (4.2) have been solved by Clutterham and Taub [4]. Because the function H_0 of (3.30) vanishes and because $p^{(1)}$ depends linearly on k (*cf.* (3.28)) these equations are independent of $k^{\pm}$ and imply a relation between the two variables characterizing the shock TI. This in turn determines possible world lines of the point T in Figure 1. However, if either k^{+} or k^{-} is given, then the remaining curvatures, k^{-} or k^{+} and K^{+} and K^{-}, are determined.

The sequence of equations (4.3) may be used to determine $k^{\pm(n)}$ and hence $K^{\pm(n)}$ as functions of k^{+} or k^{-}.

The fact that k^{-} cannot be determined from the equations (4.1), (4.2) and (4.3) is to be expected; for in a shock configuration such as in Figure 1, the shock TM must satisfy an additional condition, namely, it must be orthogonal to the wall CW. Such a requirement serves to determine k^{-} (*cf.* [4]).

Thus a shock configuration such as that occurring in Figure 1 may be determined to an arbitrary degree of accuracy by solving equations (4.1), (4.2) and (4.3) with n in the latter equations chosen sufficiently great and satisfying the boundary condition on CW. However, the solution will not be unique for a given shock TI.

References

1. W. Bleakney and A. H. Taub, Interaction of shock waves, Rev. Mod. Phys., vol. 21, 1948, pp. 584–605.

2. A. H. Taub, Curved shocks in pseudo-stationary flows, Ann. Math., vol. 58, no. 3, 1953, pp. 501–527.

3. T. Y. Thomas, The distribution of singular shock directions, J. Math. and Phys., vol. 28, no. 3, 1949, pp. 153–172.

4. D. R. Clutterham and A. H. Taub, Numerical results on the shock configuration in Mach reflection, to appear in the Proceedings of the Sixth Symposium of Applied Mathematics.

SIGNAL AND NOISE PROBLEMS

M. KAC, Cornell University

In this brief report I shall limit myself to a cursory discussion of several problems chosen on grounds of personal predilection and with the view of showing how "pure" and "applied" mathematics can intermingle with profit to both.

1. The problem of detection. This is a purely statistical problem which can be formulated as follows: Let $x(t)$ be a stationary random process (noise) and $s(t)$ a signal. An observer receives a record $y(t)$ which is either $x(t)$ or $x(t)+s(t)$ and is to decide whether the signal is present or not.

In the simplest case, when $s(t)$ is periodic with period θ and a finite number of observations is made at times $t_1, t_1+\theta, \cdots, t_1+(n-1)\theta$ and if furthermore θ is long compared with the correlation time of the noise (*i.e.*, $x(t_1+j\theta)$, $j=1, 2, \cdots, n-1$, are independent), the problem can be solved by the use of the Neyman-Pearson theory. A detailed account can be found in [1]. More recently various authors [2] considered the sequential approach to the problem. In its general form the problem clearly belongs to decision theory and it may be hoped that here the theories of the late A. Wald will play an important part.

2. Spectra and correlation. The relation between the power spectrum $A(\omega)$ and the correlation function $\rho(t)$,

$$\rho(t) = \int_{-\infty}^{\infty} A(\omega) \cos \omega t \, d\omega, \tag{2.1}$$

is a standard tool in the theory of noise. It is often desirable to consider processes for which

$$\rho(t) = 0 \qquad |t| > A. \tag{2.2}$$

The relations (2.1) and (2.2) combined with the fact that $A(\omega) \geqq 0$ induce strong restrictions on $\rho(t)$. One can show, for instance, [3] that if n is an integer

$$\left| \rho\left(\frac{A}{n}\right) \right| \leqq \cos \frac{\pi}{n+1} \rho(0), \tag{2.3}$$

the constant being the best possible.

3. Integral equations. In the theory of radio receivers with square-law detectors one is led to the problem of finding the distribution function of an expression

$$\int_0^{\infty} K(\tau) \{ x^2(t-\tau) + y^2(t-\tau) \} dt, \tag{3.1}$$

where $x(t)$ and $y(t)$ are independent, stationary Gaussian processes with the same correlation function $\rho(t)$.

The characteristic function of the distribution function of (3.1) is given by (see [4])

$$D^{-1}(i\xi),$$

where $D(\lambda)$ is the Fredholm determinant of the integral equation

$$\lambda \int_0^\infty \rho(s - t)K(t)\phi(t)dt = \phi(s). \tag{3.2}$$

Integral equations of the form (3.1) appear in other branches of pure and applied mathematics and it is at least amusing to contemplate solving them by building a corresponding receiver and determining the distribution function experimentally.

4. Zeros of random functions. This is an extremely interesting and difficult problem where much further work needs to be done. If $x(t)$ is a stationary Gaussian process with power spectrum $A(\omega)$, the average number of zeros per unit time is given by Rice's formula [5]:

$$2\left\{\frac{\int_{-\infty}^\infty \omega^2 A(\omega)d\omega}{\int_{-\infty}^\infty A(\omega)d\omega}\right\}^{1/2}. \tag{4.1}$$

The fact that by counting zeros one can obtain information about the spectrum is in itself of great practical interest. It has been, for instance, applied to turbulence by H. W. Liepmann and his group at the California Institute of Technology. A more detailed study of the distribution of zeros of random functions for even the simplest processes encounters great analytical difficulties. A closely related problem is the following:
Let

$$f(t) = \sum_1^n a_k \cos 2\pi(\lambda_k t + \phi_k) \tag{4.2}$$

and assume that the frequencies λ_k are rationally independent. Let $N(T, a)$ be the number of roots of

$$f(t) = a$$

in $0 \leqq t \leqq T$.

It can then be demonstrated [6] that

$$\lim_{T\to\infty} \frac{N(T, a)}{T} = \frac{1}{2\pi^2}\int_{-\infty}^\infty \int \frac{\cos a\xi}{\eta^2}\left\{\prod_{k=1}^n J_0(a_k\eta) - \prod_{k=1}^n J_0(a_k\sqrt{\xi^2 + \lambda_k\eta^2})\right\} d\xi d\eta. \tag{4.3}$$

This is the counterpart of (4.1) and can be interpreted by saying that the average distance between consecutive a-values of $f(t)$ is given by the inverse of the expression on the righthand side of (4.3). If one asks for the average of the *square* of the distance between consecutive a-values one runs into difficulties which, at least at present, appear insurmountable. This is already true for the simplest case

$$f(t) = a_1 \cos \lambda_1 t + a_2 \cos \lambda_2 t$$

(except for special values of a_1, a_2 and a). An interesting application of (4.3) to the theory of unimolecular reaction rates was made recently by N. B. Slater [7]. In conclusion let me mention another related problem, this time with no practical implications.

Consider a polynomial of degree n

$$\sum_{k=0}^{n} X_k t^k \tag{4.4}$$

whose coefficients are independent, normally distributed, random variables each having mean 0 and variance 1. It is then easy to show [8] that the average number of real roots of (4.3) is asymptotically

$$\frac{2}{\pi} \log n. \tag{4.5}$$

Moreover, a tedious but rather elementary calculation shows that the standard deviation about the mean is of lower order and consequently it is very rarely that a random algebraic equation of high degree will have a number of real roots which is significantly different from (4.5).

These conclusions remain valid for a much wider class of independent random variables [9] but proofs become enormously more tedious. For the simplest case

$$\text{Prob. } \{X_k = 1\} = \text{Prob. } \{X_k = -1\} = \tfrac{1}{2}$$

the proof that (4.5) is still asymptotically the average number of real roots is lacking!

The present exposition was naturally permeated with probabilistic considerations. But perhaps it is not too idle and inappropriate to contemplate here the possibility of a statistical approach to various questions in pure mathematics. As an example let me consider the following question: how good is the classical Descartes' rule of signs?

As applied to an individual algebraic equation the question is largely meaningless. Interpreted statistically it can be properly formulated and answered. The average number of changes of sign in (4.4) is $n/2$ (this is to be compared with $\pi^{-1} \log n$ which is the average number of positive real roots of (4.4)).

However, one can do better. If one considers the polynomial

$$\left(\sum_0^n t^k\right)\left(\sum_0^n X_k t^k\right),$$

the number of real roots is the same as for (4.4); the average number of changes of sign can now be shown to be of the order $C\sqrt{n}$. This is about the best one can do, and yet $C\sqrt{n}$ is still so far from the correct order $\pi^{-1}\log n$ that we must conclude that Descartes' rule of sign is extremely unlikely to give a good estimate for equations of high degree.

Bibliography

1. J. L. Lawson and G. E. Uhlenbeck, Threshold Signals, McGraw-Hill, 1950.

2. See *e.g.*, W. C. Fox, Signal detectability, Eng. Rev. Inst., Univ. of Michigan. Here references to previous work, especially that of D. Middleton, are given.

3. R. P. Boas, Jr. and M. Kac, Inequalities for Fourier transforms of positive functions. Duke Math. J., vol. 12, 1945, pp. 189–206. (See also the errata *ibid.* 15, 1948, pp. 105–109.)

4. M. Kac and A. J. F. Siegert, On the theory of noise in radio receivers with square law rectifiers. J. Appl. Phys., vol. 18, 1947, pp. 383–397.

5. S. O. Rice, Mathematical analysis of random noise. Bell. Tech. J., vol. 23, 1944, pp. 282–332.

6. M. Kac, On the distribution of values of trigonometric polynomials with linearly independent frequencies. Am. J. Math., vol. 65, 1943, pp. 609–615. See also a recent paper by N. B. Slater: Some formulas of P. Stein and others concerning trigonometrical sums. Proc. Cam. Phil. Soc., vol. 50, 1950, pp. 33–39.

7. N. B. Slater, Aspects of a theory of unimolecular reaction rates. Proc. Roy. Soc. A194, 1948, pp. 112–131.

8. M. Kac, On the average number of real roots of random algebraic equations. Bull. Amer. Math. Soc., vol. 49, 1943, pp. 314–320.

9. M. Kac, On the average number of real roots of random algebraic equations II, Proc. London Math. Soc., vol. 50, 1949, pp. 390–408.

BOOLEAN ALGEBRAS IN ELECTRIC CIRCUIT DESIGN

D. E. MULLER, University of Illinois

A brief description of how Boolean algebras may be used in the design of computer circuits is followed by an account of studies in the field of Boolean algebras which were carried out with the aid of the ILLIAC computer at the University of Illinois. The computer was used to investigate relationships between functions in Boolean algebras which uncovered symmetries leading to the development of certain theorems. These theorems turned out on later expansion to have a number of applications to computer design. Hence, the work under discussion is quite distinct from the customary use of the computer in carrying out massive algebraic manipulations with the objective of settling some specific design problems.

One of the main motives for the introduction of algebraic techniques into the design of switching circuits is the problem of devising the simplest possible circuit toward a specified end under a given criterion of simplicity. In principle this can, of course, always be achieved by brute force since it can be reduced to the trial of a finite, although generally very large number of combinations. A mathematically satisfying solution, however, has not yet been found and will certainly require a deeper insight into the basic structure of Boolean functions than available at present.

The investigations are concerned with this latter type of functions and therefore apply only to circuits of the so-called static type in which the outputs are expressible as Boolean functions of the inputs. This excludes, in particular, digital circuits with feed-back loops. Any Boolean function of p independent variables can be expanded into a canonical representation consisting of a linear expression in 2^p standard functions with coefficients which are either zero or one. The set of coefficients serves to completely define the function being represented, and conversely any ordered set of 2^p zeros and ones represents a Boolean function. Hence every such function can be regarded as corresponding to a vertex of the 2^p-dimensional unit cube, and the geometric properties of the latter should be reflected in algebraic properties of the corresponding totality of Boolean functions. A particular orthogonal coordinate system, with its origin at the center of the cube, is introduced by passing axes through the vertices corresponding to the independent variables themselves and through the points corresponding to all functions which can be obtained from the latter, using only the operation "exclusive or."

Coordinates of points with respect to these axes for the case $p=4$ were studied by the computer. For each point the number of coordinates of absolute magnitudes 0, 1, 2, $\cdots$, 8 were obtained, and all points for which these nine numbers agreed were put into the same equivalence class. Such equivalence classes represent sets of functions bearing similar relationships to the axes and therefore, presumably, similar algebraic form. There were 8 such classes of

which 3 attracted particular attention, because the number of functions contained in them was 2048, an exact power of two. Jointly they contained all points corresponding to functions, each one of which can be expressed as a sum of products of no more than two of the independent variables, and it turned out that the squared distance between any pair of such points is never less than 4. This suggested giving a special name to the set of all points corresponding to functions which are the sum of products of no more than $p-q$ of the independent variables: this is a *net of order* q. It can then be proved that the distance between any pair of points in a net of order q is at least 2^q.

An unexpected application of this concept was found in the field of error detection which at the same time suggested a number of problems. In this case one wishes to find the most numerous set of vertices of an n-dimensional cube, any pair of which is separated by at least the distance d. The question whether for $n=2^p$, $d=2^q$, the net of order q furnishes a solution has not been completely answered.

The simpler question of whether additional vertices can be found such that the distance property remains valid for the augmented set has been answered affirmatively for large enough p except when $q=0, 1, p-1$, and p. For these latter values of q, the net cannot be augmented for any value of p. The addition of further points is known to be possible when $p=7$, $q=4$. For $p=5$, $q=3$, a computer study has shown that no augmentation of the net is possible. The case $p=6$ remains open.

COMPUTATIONAL ATTACKS ON DISCRETE PROBLEMS

S. S. CAIRNS, University of Illinois

This is a summary of some recent work done at the University of California at Los Angeles under the so-called SCAMP project, of which I was chairman during the summers of 1952 and 1953. This project represents the first organized effort to attack discrete problems with the aid of large-scale computing equipment. It is a pioneer effort and only in the early stages of its expected development. The objectives of the computational procedures are (1) to obtain numerical information useful in the development of general theories, (2) to verify hypotheses which are beyond the reach of hand computations, and (3) to accumulate data on the basis of which to make conjectures as foundations for further research.

Thus far much, and perhaps too much, attention has been devoted to the existence problem for finite projective planes of order m. This can be interpreted as the problem of finding the incidence matrix A of such a plane; that is, a matrix A of order $n=m^2+m+1$, whose elements are all zeros or ones, there being exactly $m+1$ ones in each row and exactly one column in which both members of any given pair of rows have a one. The current state of this problem is the following: Finite projective planes are known to exist whenever m is the power of a prime and are known not to exist provided that $m\equiv 1$ or 2 mod 4 and that the square-free part of m has a prime factor congruent to 3 mod 4. In the particular case $m=6$ this had previously been shown by a direct attack, relying partly on numerical methods.* The question remains therefore unsettled for a sequence of values of m beginning with

$$m = 10, 12, 15, 20 \cdots.$$

The smallest of these is sufficient to present a formidable challenge, yet not so terrifying as to discourage an attack which combines mathematical ingenuity with the capabilities of high-speed automatic computing equipment.

The three basic methods of approach which have been tried are characteristically available in dealing with any problem of this kind, an ever-increasing number of which are presenting themselves in such connections as the theory of switching circuits and communication networks, the scheduling of production and transportation, as well as other assignment and allocation problems. Identifying them as the combinatorial, the algebraic, and the analytic approach, we shall illustrate each of them by summarizing how far it has carried towards solving our problem.

The combinatorial approach. The aim will be that of laying down principles or carrying through constructions which permit on the basis of succinct criteria

* G. Tarry, "Le Problème des 36 Officiers" Comptes Rendus, vol. 1, 1900, pp. 122–123 and vol. 2, 1901, pp. 170–203.

(or a relatively small number of trials) the elimination of large classes of candidates from the frequently astronomical number of possible ones. In the present instance, a matter of proper labelling makes it possible to reduce the problematical portion of A to a right lower corner which consists of an $m \times m$ array of mth order permutation matrices $c_{\rho\sigma}$ ($\rho, \sigma = 1, 2, \cdots, m$), subject to certain conditions resulting from the definition of a finite projective plane. It is possible, moreover, to arrange it so that this $m \times m$ array of matrices will be bordered on top and on the left by unit matrices. There now exists a suitable way of mapping mth order permutation matrices on column vectors. On assembling the column vectors thus corresponding to $c_{\rho 1}, c_{\rho 2}, \cdots, c_{\rho m}$ there results for each $\rho = 2, 3, \cdots, m$, a *latin square.* A set of $(m-1)$ latin squares arising in this manner from the incidence matrix of a finite projective plane consists of mutually orthogonal ones; that is, no ordered pair of elements occurs in identical positions more than once in any ordered pair of squares. *This reduces our problem to the existence problem for a set of* $(m-1)$ *mutually orthogonal latin squares of order* m, thus immersing it into the main stream of a large and important family of combinatorial problems which depend for their solution on the construction of permutation matrices with particular properties (*e.g.*, that of maximizing a suitable function) and which are encountered in the design of experiment as well as in assignment problems and their variants.

Tarry (*loc. cit.*) showed that for $m = 6$ there is not even a pair of such latin squares; Euler had already conjectured the same for any $m \equiv 2 \bmod 4$, but this is still an open question. Even the existence problem for a pair of mutually orthogonal ones of order 10 has resisted all efforts to date: programming the method of exhaustive search, which Tarry carried out by hand in the case $m = 6$, on a modern electronic computer shows it to be beyond the scope of current machines in spite of all improvements thus far devised; and the systematic construction of latin squares most prone to possess orthogonal companions has not as yet produced one which actually does.

The algebraic approach. An incidence matrix A can be characterized as a solution with non-negative integral elements of

$$AA^T = mI_n + U_n, \tag{1}$$

where I_n is the unit matrix and U_n has all its elements equal to 1. Relaxing the requirement that the elements of A be integral, one can find solutions of (1) with rational elements, satisfying at least some of the conditions which define an incidence matrix. For example, this definition can be satisfied as far as the first $(m+1)$ rows are concerned. As yet, however, there is little one can do to exploit these rational solutions in order to come nearer to the ultimate goal. Experiments were conducted, using the machine, in order to get information on the structure of a matrix G which would transform a rational solution S into an incidence matrix. If

$$T = SG$$

where S solves (1) and is normal, then T is also a normal solution of (1) provided G is orthogonal and its row and column sums are either all $+1$ or all -1. If T is integral it is an incidence matrix. Matrices G were therefore studied in cases where S and T are known, in the hope of finding ways of characterizing a manageably small class of possible G's applying also in unknown cases. So far this, too, has proved fruitless.

The analytic approach. This mode of attack consists in imbedding the discrete set of alternatives into a space of continuous variables. What is generally wanted is a procedure which, starting from an arbitrary point, lays down a "search path" on the now continuous manifold of candidates, leading to the desired object. For a computational approach there is required, in addition, a decision procedure when to break off such a search and from where to start anew, in case the original choice of a starting point has turned out to be a very bad one. In the case of our incidence matrix for the finite projective plane of order m, the conditions which define it characterize certain subspaces in the n^2-dimensional space of all matrices of order $n=m^2+m+1$; and the task becomes that of determining whether or not the intersection of these subspaces is empty. Hence the type of search procedures required is of the sort that, if it has failed to yield a common element at a certain stage, then it follows that there can be no such element. Computationally it has turned out more convenient to consider the existence of

$$Y = \frac{-1}{\sqrt{m}}\left(A - (m + \sqrt{m} + 1)^{-1} U_n\right)$$

in place of A, where Y is characterized as follows, if A is an incidence matrix:

(a) $$\sum_i Y_{ij} = \sum_j Y_{ij} = 1,$$

(b) $$-\frac{1}{\sqrt{m}(m + \sqrt{m} + 1)} \leqq Y_{ij} \leqq \frac{\sqrt{m} + 1}{m + \sqrt{m} + 1},$$

(c) Y is orthogonal.

As yet it has not proved possible to determine whether or not these three subspaces have a non-empty intersection for $m=10$.

ON CHARACTERISTIC VALUE PROBLEMS IN HIGH ORDER DIFFERENTIAL EQUATIONS WHICH ARISE IN STUDIES ON HYDRODYNAMIC AND HYDROMAGNETIC STABILITY

S. CHANDRASEKHAR, University of Chicago

1. Introduction. Recent studies in hydrodynamic and hydromagnetic stability have disclosed the existence of a class of characteristic value problems in differential equations of high order—orders as high as twenty-four have been encountered—which appear to have a genuine mathematical interest. A partial list of these problems will be found in the Appendix. These problems have been solved, singly, as they have arisen in the physical connections. But one feels that there must be a general theory which embraces them all though such a theory is lacking at the present time. It is the object of this paper to bring these problems and the methods which have been developed for their solution to the attention of the mathematicians in the hope of stimulating their interest.

In the solution of the problems listed in the Appendix, methods of two different kinds have been found useful. In the first of these methods a variational procedure is developed which in the manner of its application is different from the usual ones. The second of these methods is based on expansion in orthogonal functions but again applied in an unusual manner. We shall illustrate the principles underlying these methods by considering one of the simpler problems listed in the Appendix.

2. A typical problem. G. I. Taylor's [1] investigation in 1923 on the stability of viscous flow between two concentric rotating cylinders provided the first example of a case of hydrodynamic instability for which the criterion was theoretically predicted and experimentally verified. The mathematical problem underlying this classic investigation in hydrodynamic stability is the following:

With certain simplifications suitable to the circumstances under which the experiments were performed, the problem requires the solution of the sixth order equation

$$(D^2 - a^2)^3 v = - a^2 T(1 + \alpha z)v, \tag{1}$$

with the boundary conditions

$$v = (D^2 - a^2)v = D(D^2 - a^2)v = 0 \tag{2}$$

for $z=0$ and 1, where T is the characteristic value parameter, $D=d/dz$ and a and α are assigned (real) constants. In the physical problem α is negative and one is interested in the range $0 \geqq \alpha \geqq -3.0$.

Equation (2) provides six boundary conditions (three at $z=0$ and three at $z=1$) and the requirement that a non-trivial solution of equation (1) satisfy these conditions will lead to a determinate sequence of possible values for T (for given a^2 and α). Among these possible values of T (for given α and varying a^2) there will be a smallest (positive) value; and in the physical problem particu-

lar interest is attached to the minimum of these smallest positive values of T as a function of a^2 (for fixed α).

In discussing the characteristic value problem presented by equations (1) and (2), we shall distinguish the cases $\alpha=0$ and $\alpha\neq 0$. The problem has basically different characters in the two cases: in the former case ($\alpha=0$) it is "self-adjoint" in some sense (yet to be defined!) while in the latter case it is not. Because of this difference the solution of the problem, while it can be made to depend on an extremal principle in the case $\alpha=0$, it cannot be so accomplished in the case $\alpha\neq 0$. As far as the general solution of equations (1) and (2) is concerned, a separate detailed discussion of the case $\alpha=0$ would hardly seem justified; but it does happen that the case $\alpha=0$ is the simplest proto-type of a wide class of problems to which similar methods of solution can be applied. For this reason we shall treat the case $\alpha=0$ at some length.

3. The case $\alpha=0$: example of the variational procedure. When $\alpha=0$, the solution of the characteristic value problem can, in principle, be achieved very simply. For, in this case, the solution of the equation

$$(D^2 - a^2)^3 v = - a^2 T v \tag{3}$$

must clearly be of the form

$$v_i = \sum_{i=1}^{6} A_i e^{q_i z}, \tag{4}$$

where the q_i's, occurring in pairs, are the roots of the characteristic equation

$$(q^2 - a^2)^3 = - a^2 T, \tag{5}$$

and the A_i's are constants of integration. The requirement that the solution represented by equation (4) satisfy the boundary conditions (2) will lead to a system of six linear homogeneous equations; and the determinant of this system must vanish if we are not to have the trivial solution $A_i=0$, $i=1, \cdots, 6$. And the condition that the determinant vanish will provide an equation for determining T. There is, of course, no difficulty of principle in carrying out this procedure. Indeed, it has been carried out by Pellew and Southwell [2] who find for example that

$$T = 1707.8 \qquad \text{for } a = 3.12. \tag{6}$$

(This is the value of a at which T, as a function of a, attains its minimum.)

For the case on hand the carrying out of the direct method of solution described in the preceding paragraph is particularly simple as the roots of equation (5) can be explicitly written down in terms of the known cube roots of -1; thus

$$q^2 = a^2 + \omega\sqrt[3]{(a^2T)}, \qquad (\omega^3 = -1). \tag{7}$$

But this simplification does not arise in other problems which have to be considered; also, the characteristic equations which have to be solved are of higher order. It would therefore be useful if we can devise a method of solution which will avoid the necessity of solving complicated algebraic equations of high order by laborious methods of trial and error. This would be specially important if, as it often happens, we are required to establish the dependence of the characteristic values on two or more parameters rather than on a single parameter, such as a, in the present problem ($\alpha=0$). And it does appear that in many of the cases which arise, convenient variational methods can be devised which we shall now illustrate by considering the problem presented by equations (2) and (3).

Letting

$$G = (D^2 - a^2)v \quad \text{and} \quad F = (D^2 - a^2)G = (D^2 - a^2)^2 v, \tag{8}$$

we can rewrite equation (3) in the form

$$(D^2 - a^2)F = - a^2 T v, \tag{9}$$

while the boundary conditions (2) require that

$$v = G = DG = 0 \qquad \text{for } z = 0 \text{ and } 1. \tag{10}$$

Let T_i denote a particular characteristic value and let the various functions derived from the solution v_i belonging to T_i be distinguished by a subscript i. Multiplying the equation governing v_i by G_j belonging to a different characteristic value T_j and integrating the resulting equation over the range of z, we obtain

$$\int_0^1 G_j(D^2 - a^2)F_i dz = - a^2 T_i \int_0^1 v_i(D^2 - a^2)v_j dz. \tag{11}$$

The integral occurring on the left-hand side of this equation can be reduced by two successive integration by parts; thus,

$$\int_0^1 G_j(D^2 - a^2)F_i dz = [G_j DF_i - (DG_j)F_i]_0^1 + \int_0^1 F_i(D^2 - a^2)G_j dz. \tag{12}$$

The integrated parts vanish in virtue of the boundary conditions (*cf.* equations (10)) and we are left with

$$\int_0^1 G_j(D^2 - a^2)F_i dz = \int_0^1 F_i F_j dz. \tag{13}$$

Similarly, after an integration by parts the right-hand side of equation (11) becomes

$$a^2 T_i \int_0^1 [(Dv_i)(Dv_j) + a^2 v_i v_j] dz. \tag{14}$$

Thus,

$$\int_0^1 F_iF_jdz = a^2T_i\int_0^1 [(Dv_i)(Dv_j) + a^2v_iv_j]dz. \tag{15}$$

Noticing that interchanging i and j in this equation replaces T_i by T_j but leaves the equation otherwise unaffected, we conclude that

$$\int_0^1 F_iF_jdz = 0 \qquad i \neq j; \tag{16}$$

and, further, that when $i=j$ the corresponding characteristic value, T, can be expressed as the ratio of two positive definite integrals in the form:

$$T = \frac{\int_0^1 F^2dz}{a^2\int_0^1 [(Dv)^2 + a^2v^2]dz}. \tag{17}$$

These facts clearly imply the existence of an extremal principle which can be made the basis of a variational method of solving the underlying characteristic value problem. For comparison with the method we shall describe in §4 for the case $\alpha \neq 0$, we shall formulate the extremal principle in the context of a slightly transformed equation.

Operating on equation (3) by (D^2-a^2) we get

$$(D^2 - a^2)^3W = -a^2TW \tag{18}$$

as the equation governing

$$W = (D^2 - a^2)v. \tag{19}$$

According to equations (2) and (3) the corresponding boundary conditions on W are:

$$W = DW = 0 \quad F \equiv (D^2 - a^2)G \equiv (D^2 - a^2)^2W = 0 \qquad \text{for } z = 0 \text{ and } 1. \tag{20}$$

Again rewriting equation (18) in the form

$$(D^2 - a^2)F = -a^2TW, \tag{21}$$

and multiplying the equation governing W_i (belonging to T_i) by F_j (belonging to T_j) we obtain

$$\int_0^1 F_j(D^2 - a^2)F_idz = -a^2T_i\int_0^1 W_i(D^2 - a^2)G_jdz. \tag{22}$$

Making use of the boundary conditions (20) we can reduce equation (22) by one or more integrations by parts to the form

$$\int_0^1 [(DF_i)(DF_j) + a^2F_iF_j]dz = a^2T_i \int_0^1 G_iG_jdz. \tag{23}$$

From this equation it follows that

$$\int_0^1 G_iG_jdz = 0, \qquad i \neq j; \tag{24}$$

and that when $i=j$ we can express T as:

$$T = \frac{\int_0^1 [(DF)^2 + a^2F^2]dz}{a^2 \int_0^1 G^2dz}. \tag{25}$$

This last formula, like (17), expresses T as the ratio of two positive definite integrals.

Consider now the effect on T (evaluated according to equation (25)) of an arbitrary variation, δW, in W compatible only with the boundary conditions on W. We find in a straightforward manner that

$$\delta T = -\frac{2}{a^2 \int_0^1 G^2dz} \int_0^1 \delta F\{(D^2 - a^2)F + a^2TW\}dz, \tag{26}$$

where it should be noted that in the reductions leading to equation (26) the relations implied in the definitions of F and G (*cf.* equation (20)) have been used; in particular

$$\delta F = (D^2 - a^2)^2\delta W. \tag{27}$$

Hence, to the first order, $\delta T \equiv 0$ for all small arbitrary variations in W which satisfy the boundary conditions, provided

$$(D^2 - a^2)F = -a^2TW, \tag{28}$$

i.e., if the differential equation governing W is satisfied. It is evident that the converse of this proposition is also true. Further, it follows from (25) that the true solution of the problem (belonging to the lowest characteristic value T) leads to the minimum value (in the sense of the calculus of variations) for T when evaluated according to (25).

While the minimal principle formulated as above is valid for all small arbitrary variations δW compatible only with the boundary conditions on W, it is true also for all small arbitrary variations δF, the variation, δW, in W being determined in terms of δF by the equation

$$(D^2 - a^2)^2\delta W = \delta F, \tag{29}$$

and the boundary conditions

$$\delta W = D\delta W = 0 \qquad \text{for } z = 0 \text{ and } 1. \tag{30}$$

The validity of this minimal principle in this more restricted form actually provides a more effective basis for a variational procedure for determining T. For on this latter basis the procedure would be the following:

Assume for F an expression involving one or more parameters, A_k, and which vanishes at $z=0$ and 1. With the chosen form of F solve the equation

$$(D^2 - a^2)^2 W = F \tag{31}$$

for W and arrange that the solution satisfies the boundary conditions $W=DW=0$ for $z=0$ and 1; since equation (31) is of the fourth order there will be just enough constants of integration to do this. With W determined in this fashion evaluate T according to (25) and minimize it with respect to the parameters A_k. In this way we shall obtain the "best" value of T for the chosen form of F.

In practice it is found that even with the simplest trial functions, the variational procedure in the foregoing form gives surprisingly high accuracy in the deduced values of T. Thus, with the trial function, $F=\sin \pi z$, with no variational parameter, the method gives $T=1715.1$ for $a=3.12$; while the function, $F=\sin \pi z+A \sin 3\pi z$, with one variational parameter leads to $T=1707.9$ for the same value of a; these values obtained in the "first" and the "second" approximations should be compared with the value $T=1707.8$ obtained from an "exact" solution of the problem. The origin of this high precision in the deduced values of T must clearly be traced to the fact that in satisfying four of the six boundary conditions of the problem (namely $W=DW=0$ for $z=0$ and 1) we have *exactly* satisfied the second of the pair of differential equations,

$$(D^2 - a^2)F = -a^2 TW \quad \text{and} \quad (D^2 - a^2)^2 W = F, \tag{32}$$

which governs the problem.

[It may be noticed here that had we used equation (17) (instead of (25)) as the basis of the variational method, we should have had to assume a form for DG such that not only does it vanish at $z=0$ and 1, but also G, obtained after integration, vanishes at the same points; further, W will have to be obtained as the solution of $(D^2-a^2)W=G$ which vanishes at $z=0$ and 1. Apart from these differences in detail the application of the variational method based on (17) proceeds along essentially the same lines. But the method based on (25) is preferable since it avoids the restrictions implied by the requirement that both G and DG vanish at $z=0$ and 1.]

4. The case $\alpha \neq 0$: example of a method based on expansion in orthogonal functions when no extremal principle exists. Returning to equation (1) and the general case $\alpha \neq 0$, we can readily verify that propositions similar to those embodied in equations (15), (16) and (17), for the case $\alpha=0$, cannot be established now. Thus, if equation (1) and the boundary conditions (2) together represent

a system which might be described as "self-adjoint" when $\alpha=0$, it cannot be so described when $\alpha\neq 0$. It is important to note that this conclusion in no way depends on the sign of α; in particular it is independent of the circumstance that when $\alpha<-1$ the operator whose characteristic values we are seeking becomes singular on account of $(1+\alpha z)$ having a zero in the range $0\leqq z\leqq 1$.

Since equations (1) and (2), in general, do not allow the formulation of an extremal principle on which a variational procedure might be based and since the general solution of equation (1) cannot also be written down in any convenient form, it would appear that the only remaining course is to expand the unknown functions in Fourier series. In Taylor's original investigation this was the method which was adopted: Taylor expanded v in a sine series of the form

$$v = \sum_{n=1}^{\infty} V_n \sin n\pi z \tag{33}$$

and obtained T as the characteristic root of an infinite matrix. But the process of determining the root was not a very convergent one; and he succeeded (with considerable effort) in determining the characteristic root for only one value of $\alpha<-1$. However, we may expect that a more rapidly convergent process will be obtained if in using the method of expansion in Fourier series, we incorporate in the method the same basic idea which led to the high precision of the variational method for the case $\alpha=0$. We shall now indicate how this can be accomplished (see Chandrasekhar [3]).

First we transform equation (1) by rewriting it in the form

$$\frac{1}{1+\alpha z}(D^2-a^2)^3 v = -a^2 T v, \tag{34}$$

and operating on it by (D^2-a^2); in this way, we obtain the differential equation

$$(D^2-a^2)\left\{\frac{1}{1+\alpha z}(D^2-a^2)^2 W\right\} = -a^2 T W, \tag{35}$$

for

$$W = (D^2-a^2)v. \tag{36}$$

The corresponding boundary conditions on W are the same as those given in (20).

Equation (35) is equivalent to the pair of equations

$$(D^2-a^2)^2 W = (1+\alpha z)\psi \tag{37}$$

and

$$(D^2-a^2)\psi = -a^2 T W, \tag{38}$$

while the boundary conditions (20) can be expressed alternatively in the forms

$$W = DW = 0 \quad \text{and} \quad \psi = 0 \qquad \text{for } z = 0 \text{ and } 1. \tag{39}$$

Since ψ has to vanish at $z=0$ and 1, we can expand it in a sine series of the form

$$\psi = \sum_{n=1}^{\infty} C_n \sin n\pi z. \tag{40}$$

Having chosen ψ in this manner, we next *solve* the equation

$$(D^2 - a^2)^2 W = (1 + \alpha z) \sum_{n=1}^{\infty} C_n \sin n\pi z, \tag{41}$$

obtained by inserting (40) in (37), and arrange that the solution satisfies the four remaining conditions on W; since equation (41) is of the fourth order there will be just enough constants of integration to do this. With W determined in this fashion and ψ given by (40), equation (38) will lead to an infinite determinant which must be zero if all the C_n's are not to vanish. In this way we shall obtain a characteristic equation for determining T.

When the details of the method described in the preceding paragraphs are carried out, one finds that the process of solving the infinite order characteristic equation for T, by setting the determinant formed by the first n rows and columns equal to zero and letting n take increasingly larger values, converges very rapidly indeed. Thus for $\alpha = -2.5$ and $a = 5.00$, 5.05 and 5.10 the values of T obtained in the third and the fourth approximations (the "order" of the approximation being the order of the determinant which is set equal to zero in the determination of T) are:

$$a = \begin{cases} 5.00 \\ 5.05 \\ 5.10 \end{cases} \quad T\,(\text{3rd app.}) = \begin{cases} 4.607 \times 10^4 \\ 4.600 \times 10^4 \\ 4.604 \times 10^4 \end{cases} \quad T\,(\text{4th app.}) = \begin{cases} 4.626 \times 10^4 \\ 4.619 \times 10^4. \\ 4.623 \times 10^4 \end{cases} \tag{42}$$

It is seen that the values of T given in the third and the fourth approximations differ by only four parts in a thousand. The origin of this rapid convergence clearly lies in the splitting of the original equation of order six into a pair of order two and four respectively and satisfying the equation of order four exactly. This basic idea underlying the method is capable of extension and application to a wide class of problems.

Appendix

We shall list here some of the more important characteristic value problems in high order differential equations which have occurred in recent studies on hydrodynamic and hydromagnetic stability. The particular physical connections in which they arise are indicated. The references are to papers in which the solutions of the problems will be found.

I. *The inhibition of convection by a magnetic field* [4, 5].

1) To solve

$$(D^2 - a^2)[(D^2 - a^2)^2 - QD^2]W = -a^2RW,$$

together with the boundary conditions

$$W = [(D^2 - a^2)^2 - QD^2]W = 0 \quad \text{on} \quad z = \pm \tfrac{1}{2},$$

and

$$\begin{array}{llll} \textit{either} & DW = 0 & \text{on} & z = \pm \tfrac{1}{2}, \\ \textit{or} & D^2W = 0 & \text{on} & z = \pm \tfrac{1}{2}, \\ \textit{or} & DW = 0 & \text{on} & z = +\tfrac{1}{2} \quad \text{and} \quad D^2W = 0 \quad \text{on} \quad z = -\tfrac{1}{2}, \end{array}$$

where a and Q are assigned positive constants and R is the characteristic value parameter.

The physical problem requires the minimum (with respect to a) of the lowest characteristic value R for various assigned values of Q.

2) To solve

$$(D^2 - a^2)[(D^2 - a^2)^2 - Q(D + i\bar{\omega})^2]W = -a^2RW,$$

together with the boundary conditions

$$|W| = |(D^2 - a^2)^2W - Q(D + i\bar{\omega})^2W| = 0 \quad \text{on} \quad z = \pm \tfrac{1}{2},$$

and

$$\begin{array}{llll} \textit{either} & |DW| = 0 & \text{on} & z = \pm \tfrac{1}{2}, \\ \textit{or} & |D^2W| = 0 & \text{on} & z = \pm \tfrac{1}{2}, \\ \textit{or} & |DW| = 0 & \text{on} & z = +\tfrac{1}{2} \quad \text{and} \quad |D^2W| = 0 \quad \text{on} \quad z = -\tfrac{1}{2}, \end{array}$$

where a, $\bar{\omega}$ and Q are assigned constants and R is the characteristic value parameter.

The physical problem requires the minimum (with respect to a and $\bar{\omega}$) of the lowest characteristic value R for various assigned values of Q. Since W is complex, we have here a genuine problem in an equation of order twelve.

II. *The instability of a layer of fluid heated below and subject to Coriolis acceleration* [6].

3) To solve

$$(D^2 - a^2)Z = -\left(\frac{2\Omega}{\nu} d\right)DW$$

and

$$(D^2 - a^2)\left[(D^2 - a^2)^2W - \left(\frac{2\Omega}{\nu} d^3\right)DZ\right]$$
$$= [(D^2 - a^2)^3 + TD^2]W = -a^2RW,$$

together with the boundary conditions

$$W = (D^2 - a^2)^2 W - \left(\frac{2\Omega}{\nu} d^3\right) DZ = 0 \quad \text{on} \quad z = \pm \tfrac{1}{2},$$

and

either $DW = Z = 0$ on $z = \pm \tfrac{1}{2}$,

or $D^2W = DZ = 0$ on $z = \pm \tfrac{1}{2}$,

or $DW = Z = 0$ on $z = +\tfrac{1}{2}$ and $D^2W = DZ = 0$ on $z = -\tfrac{1}{2}$,

where a, ν, d and Ω are assigned constants, $T = 4\Omega^2 d^4/\nu^2$ and R is the characteristic value parameter.

The physical problem requires the minimum (with respect to a) of the lowest characteristic value R for various assigned values of T.

3a) To solve

$$(D^2 - a^2 - i\sigma)Z = -\left(\frac{2\Omega}{\nu} d\right) DW,$$

$$(D^2 - a^2)(D^2 - a^2 - i\sigma) W - \left(\frac{2\Omega}{\nu} d^3\right) DZ = F$$

and

$$(D^2 - a^2 - i\bar{\omega}\sigma)F = - Ra^2 W,$$

together with the boundary conditions

$$|F| = |W| = 0 \quad \text{on} \quad z = \pm \tfrac{1}{2},$$

and

either $DW = Z = 0$ on $z = \pm \tfrac{1}{2}$,

or $D^2W = DZ = 0$ on $z = \pm \tfrac{1}{2}$,

or $DW = Z = 0$ on $z = +\tfrac{1}{2}$ and $D^2W = DZ = 0$ on $z = -\tfrac{1}{2}$,

where $\bar{\omega}$ is a further assigned constant and σ (which is real) is a parameter to be determined (for given $T = 4\Omega^2 d^4/\nu^2$ and a^2) by the condition that R is real; and the physical problem requires the minimum (with respect to a^2) of these real values of R for various assigned values of T and $\bar{\omega}$.

III. *The instability of a layer of fluid heated below and subject simultaneously to a magnetic field and Coriolis acceleration* [7].

4) To solve

$$(D^2 - a^2)\left[\{(D^2 - a^2)^2 - QD^2\}^2 + TD^2(D^2 - a^2)\right]W = - Ra^2\left[(D^2 - a^2)^2 - QD^2\right]W$$

together with (several different) five pairs of boundary conditions on W at

$z = \pm\frac{1}{2}$, where a, Q and T are assigned constants and R is the characteristic value parameter.

IV. *The stability of viscous flow between rotating cylinders* [**1, 3, 8, 9**].

5) To solve

$$(D^2 - a^2)^3 v = -a^2 T(1 + \alpha z)v,$$

together with the boundary conditions

$$v = D^2 v = D(D^2 - a^2)v = 0 \quad \text{on} \quad z = 0 \quad \text{and} \quad 1,$$

where a and α (<0) are assigned constants and T is the characteristic value parameter. The physical problem requires the minimum (with respect to a) of the lowest (positive) characteristic value T for various assigned values of $\alpha(<0)$.

6) To solve

$$(DD^* - \lambda^2)^3 v = \frac{4A\lambda^2}{\nu^2}\left(A + \frac{B}{r^2}\right)v,$$

where $D = d/dr$ and $D^* = D + 1/r$, together with the boundary conditions

$$v = (DD^* - \lambda^2)v = D^*(DD^* - \lambda^2)v = 0 \qquad \text{for } r = R_1 \text{ and } R_2;$$

λ, ν and B are assigned constants and A is the characteristic value parameter.

V. *The stability of viscous flow between rotating cylinders in the presence of a magnetic field* [**10**].

7) To solve

$$[(D^2 - a^2)^2 + Qa^2]^2\psi = -Ta^2(D^2 - a^2)\psi,$$

together with the boundary conditions

$$D\psi = (D^2 - a^2)\psi = [(D^2 - a^2)^2 + Qa^2]\psi = D[(D^2 - a^2)^2 + Qa^2]\psi = 0 \text{ on } z = \pm\tfrac{1}{2},$$

where a and Q are assigned constants and T is the characteristic value parameter. The physical problem requires the minimum (with respect to a) of the lowest characteristic value T for various assigned values of Q.

VI. *The stability of viscous flow between rotating cylinders in the presence of a radial temperature gradient* [**11**].

8) To solve

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{n^2}{r^2}\right)^3 W = -n^2 S_n W,$$

together with the boundary conditions

$$W = \frac{dW}{dr} = \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{n^2}{r^2}\right)^2 W = 0 \qquad \text{for } r = 1 \text{ and } \eta(<1),$$

where n is an integer and S_n is the characteristic value parameter.

9) To solve

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{n^2}{r}\right)^3 W = -n^2 S_n \frac{W}{r^2},$$

together with the same boundary conditions as in (8) above.

VII. *The thermal instability of fluid spheres and spherical shells* [**12, 13, 14**].

10) To solve

$$\left(\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}-\frac{l(l+1)}{r^2}\right)^3 W = -l(l+1)C_l W,$$

together with the boundary conditions

(a) $$W = \left(\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}-\frac{l(l+1)}{r^2}\right)^2 W = 0$$

and

$$\textit{either}\quad \frac{dW}{dr}\quad \textit{or}\quad \frac{d^2W}{dr^2}=0 \qquad \text{for } r=1,$$

and

$$W = O(r^l) \qquad \text{as } r \to 0;$$

(b) $$W = \frac{d^2W}{dr^2} = \left[\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}-\frac{l(l+1)}{r^2}\right]^2 W = 0 \qquad \text{for } r=1 \text{ and } \eta(<1),$$

where l is an integer and C_l is the characteristic value parameter.

VIII. *The stability of superposed fluids* (*plane problem*) [**15, 16, 17, 18, 19**].

11) To solve

$$D\left\{\left[\rho+\frac{\mu}{\sigma}(-k^2+D^2)\right]Dw+\frac{1}{\sigma}(D\mu)(k^2+D^2)w\right\}$$
$$= k^2\left\{\left[-\frac{g}{\sigma^2}D\rho+\rho+\frac{\mu}{\sigma}(-k^2+D^2)\right]w+\frac{2}{\sigma}(D\mu)(Dw)\right\},$$

together with the boundary conditions

$$w = 0 \quad \text{for} \quad z = 0 \quad \text{and} \quad l$$

and

either	$Dw = 0$ for $z = 0$ and l,	
or	$D^2w = 0$ for $z = 0$ and l,	
or	$Dw = 0$ for $z = 0$ and $D^2w = 0$ for $z = l$,	
or	$D^2w = 0$ for $z = 0$ and $Dw = 0$ for $z = l$,	

where $\rho = \rho(z)$ and $\mu = \mu(z)$ are given functions of z, k is an assigned (real) constant and σ is the characteristic value parameter. (Note that σ can be complex).

IX. *The stability of superposed fluids* (*spherical problem*) [20].

12) To solve

$$\frac{d}{dr}\left\{\rho\,\frac{d}{dr}(rW) + \frac{\mu}{\sigma}\,\frac{d}{dr}(rF)\right\} + \frac{1}{\sigma}\,\frac{d}{dr}\left\{r\,\frac{d\mu}{dr}\left[\frac{d^2W}{dr^2} + \frac{(l+2)(l-1)}{r^2}\,W\right]\right\}$$
$$- \frac{l(l+1)}{r}\left(\rho W + \frac{\mu}{\sigma}\,F\right) = -\,\frac{l(l+1)}{\sigma^2}\left\{\gamma\,\frac{d\rho}{dr}\,W - 2\sigma\frac{d\mu}{dr}\,\frac{d}{dr}\left(\frac{W}{r}\right)\right\},$$

where

$$F = \left[\frac{d^2}{dr^2} + \frac{2}{r}\,\frac{d}{dr} - \frac{l(l+1)}{r^2}\right]W,$$

together with the boundary conditions

$$W = 0 \quad \text{and} \quad \textit{either}\ \frac{dW}{dr}\ \textit{or}\ \frac{d^2W}{dr^2} = 0 \quad \text{on} \quad r = R,$$

and

$$W = O(r^l) \qquad \text{as } r \to 0.$$

In the foregoing ρ, μ and γ are given functions of r, l is an integer and σ (which may be complex) is the characteristic value parameter.

The case when

$$\mu = \mu_1 = \text{constant}, \qquad \rho = \rho_1 = \text{constant for } 0 \leqq r < r_i$$

and

$$\mu = \mu_2 = \text{constant}, \qquad \rho = \rho_2 = \text{constant for } r_i < r \leqq R$$

is of particular importance. In this case the equation reduces to the form

$$\frac{d^2}{dr^2}\left\{r\left(W + \frac{\mu}{\rho\sigma}\,F\right)\right\} - \frac{l(l+1)}{r}\left(W + \frac{\mu}{\rho\sigma}\,F\right) = 0$$

in each of the two regions of constant ρ and μ. And solutions are sought which satisfy the boundary conditions

$$W = 0 \quad \text{and} \quad \frac{d^2W}{dr^2} = 0 \qquad \text{for } r = R$$

$$W = O(r^l) \qquad \text{for } r \to 0$$

and

$$W, \quad \frac{dW}{dr} \quad \text{and} \quad \mu\left\{r^2\frac{d^2W}{dr^2} + (l+2)(l-1)W\right\}$$

are continuous on $r = r_i$.

References

1. G. I. Taylor, Stability of a viscous liquid contained between two rotating cylinders, Philos. Trans. Roy. Soc. London Ser., A, vol. 223, 1923, pp. 289–343.

2. A. Pellew and R. V. Southwell, On maintained convective motion in a fluid heated from below, Proc. Roy. Soc. London Ser. A, vol. 176, 1940, pp. 312–343.

3. S. Chandrasekhar, The stability of viscous flow between rotating cylinders, Mathematika (in press).

4. S. Chandrasekhar, On the inhibition of convection by a magnetic field, Philos. Mag. Ser. 7, vol. 43, 1952, pp. 501–532.

5. S. Chandrasekhar, On the inhibition of convection by a magnetic field. II, Philos. Mag. (in press).

6. S. Chandrasekhar, The instability of a layer of fluid heated below and subject to Coriolis forces, Proc. Roy. Soc. London Ser. A, vol. 217, 1953, pp. 306–327; also a further paper in press.

7. S. Chandrasekhar, The instability of a layer of fluid heated below and subject to the simultaneous action of a magnetic field and rotation, Proc. Roy. Soc. London Ser. A, (in press).

8. D. Meksyn, Stability of viscous flow between rotating cylinders. I, II, III, Proc. Roy. Soc. London Ser. A, vol. 187, 1946, pp. 115–128, 480–491, 492–504.

9. S. Chandrasekhar, (in press).

10. S. Chandrasekhar, The stability of viscous flow between rotating cylinders in the presence of a magnetic field, Proc. Roy. Soc. London Ser. A, vol. 216, 1953, pp. 293–309.

11. S. Chandrasekhar, The stability of viscous flow between rotating cylinders in the presence of a radial temperature gradient, Journal of Rational Mechanics and Analysis, vol. 3, 1954, pp. 181–207.

12. H. Jeffreys and M. E. M. Bland, The instability of a fluid sphere heated within, Mon. Not. R. Astr. Soc. Geophys. Sup. vol. 6, 1951, pp. 148–158.

13. S. Chandrasekhar, The thermal instability of a fluid sphere heated within, Philos. Mag. Ser. 7, vol. 43, 1952, pp. 1317–1329.

14. S. Chandrasekhar, The onset of convection by thermal instability in spherical shells, Philos. Mag. Ser. 7, vol. 44, 1953, pp. 233–241, 1129–1130.

15. Lord Rayleigh, Investigation of the character of the equilibrium of an incompressible heavy fluid of variable density, Scientific Papers, vol. 2, Cambridge University Press, Cambridge, 1900, pp. 200–207.

16. G. I. Taylor, The instability of liquid surfaces when accelerated in a direction perpendicular to their planes. I, Proc. Roy. Soc. London Ser. A, vol. 201, 1950, pp. 192–196; also D. J. Lewis, The instability of liquid surfaces when accelerated in a direction perpendicular to their planes. II, Proc. Roy. Soc. London Ser. A, vol. 202, 1950, pp. 81–96.

17. S. Chandrasekhar, (in press).

18. R. Hide, (in press).

19. Tsung-Ying Teng Fan, (in press).

20. S. Chandrasekhar, The character of the equilibrium of an incompressible fluid sphere of variable density and viscosity subject to radial acceleration, Quart. Jour. of Mechanics and Applied Mathematics, (in press).

FREQUENCY SPECTRUM OF VIBRATIONS OF A CRYSTAL LATTICE

ELLIOTT W. MONTROLL, University of Maryland,* and Office of Naval Research

1. Introduction. The heat capacity of a system of N interacting particles under an external constraint x is defined as $C_x=(\partial Q/\partial T)_x$, the ratio of the added heat to the temperature rise of the system. Experimentally it is easiest to measure the heat capacity of solids at constant pressure, C_p, and of gases at constant volume (in a closed container), C_v. The quantities C_p and C_v are related through the thermodynamic equation $C_p-C_v=9\alpha^2VT/\kappa$, α being the coefficient of thermal expansion, V the volume, and κ the compressibility. Since statistical mechanical formulae lead naturally to C_v, one must include the correction due to lattice expansion in applying theoretical results to the interpretation of experimental C_p measurements in solids.

The modern theory of the heat capacity of solids was originated by Einstein [1] in 1907 through the application of the then new Planck quantum theory. The appropriate basic formulae are those for the internal energy and heat capacity of N independent harmonic oscillators of frequency ν:

$$E = 3Nh\nu\left[\frac{1}{2} - \frac{1}{1-\exp(h\nu/kT)}\right] \tag{1.1}$$

$$C_v = 3Nk\left(\frac{1}{2}\frac{h\nu}{kT}\right)^2\Big/\sinh^2\left(\frac{1}{2}\frac{h\nu}{kT}\right). \tag{1.2}$$

Einstein assumed that each atom in a solid vibrates about its equilibrium position independently of its neighbors and used equation (1.2) to fit the then available heat capacity data. Equation (1.2) is a universal function of the variable $kT/h\nu$. Hence it was predicted that all experimental C_v curves, when plotted on an appropriate scale, could be superimposed. The frequency ν determined the scale. The anomolously low value of the atomic heat of diamond was explained by stating that its frequency ν was larger than that of most materials.

The general form of experimental C_v curves is given in Figure 1. Experimental deviations from the Einstein formula are to be expected at low temperatures (and indeed have been observed for years) because atoms in a solid do not vibraté independently of each other. Interatomic forces are such that the oscillations of a given atom are coupled with those of its neighbors. Hence, the $3N$ equal frequencies of equation (1.2) should be replaced by the set of frequencies of the normal modes of the coupled oscillating atoms.

* The preparation of this article was partially supported by the Office of Scientific Research of the U. S. Air Force.

It is useful to introduce a distribution function $g(\nu)$ which has the property that $g(\nu)d\nu$ is the number of normal modes between ν and $\nu+d\nu$. Then equations (1.1) and (1.2) become

$$E = \int_0^{\nu_L} h\nu\left[\frac{1}{2} - \frac{1}{1-\exp(h\nu/kT)}\right]g(\nu)d\nu \tag{1.3}$$

and

$$C_v = k\int_0^{\nu_L} g(\nu)\left[\left(\frac{1}{2}\frac{h\nu}{kT}\right)^2\Big/\sinh^2\left(\frac{1}{2}\frac{h\nu}{kT}\right)\right]d\nu. \tag{1.4}$$

Here ν_L is the largest normal mode frequency. Optical as well as thermodynamic properties of crystals can be discussed in terms of $g(\nu)$.

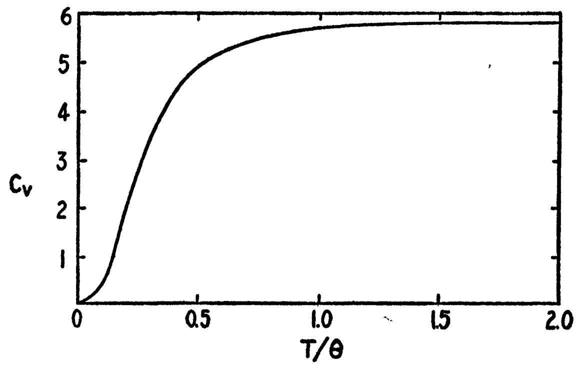


FIG. 1. The variation of the specific heat (in calories per gram mol) of a typical solid as a function of temperature.

The first calculation of the distribution of frequencies of normal modes was made by Debye in 1912. He postulated a solid to be an elastic continuum. Since a continuum has an infinite number of normal modes, Debye cut the frequency spectrum off at a frequency ν_L such that the total number of normal modes was equal to the number of degrees of freedom of the solid. In the Debye [2] theory $g(\nu) = 9N(\nu/\nu_L)^2/\nu_L$ where ν_L, the largest frequency, is a function of the elastic constants of a solid; $g(\nu) = 0$ when $\nu > \nu_L$. Debye's work was one of the great successes of the early quantum theory. The theoretical heat capacities based on his frequency spectrum are in good agreement with experimental results. The simplicity of the Debye theory combined with this fact has given it a long life as the dominant theory of the heat capacity of solids.

The continuum model can be expected to give good results at very low temperatures where only long wave length oscillations are important. However, in the steep part of the C_v curve it is not sufficiently accurate. A discussion of the experimental situation is given in a review by Blackman [3]. Born and von Karman [4] were the first to investigate the dynamics of periodically spaced

atoms. They introduced a model of a solid which is mechanically equivalent to a set of coupled springs and masses.

Our main purpose here is to review the general characteristics of the frequency distribution, $g(\nu)$, of the Born-Karman model. A striking feature of $g(\nu)$ curves is the existence of certain singularities, logarithmic ones in two dimensional lattices and inverse square root singularities in the derivative of $g(\nu)$ in three-dimensional lattices. The existence of peaks was first noticed by Blackman. The analytical nature of these peaks, our main subject of interest here, was first discussed in an investigation [5] of the vibrations in the plane of a two dimensional square lattice with interactions between atoms which were nearest and next nearest neighbors. The logarithmic peaks which occurred in the $g(\nu)$ of this case were also found by Rosenstock and Bowers [6] for the transverse vibrations of atoms in a square lattice and finally by Smollett [7] for the in plane vibrations of the two dimensional lattices with long range Coulomb interactions. Similar results were obtained for the transverse vibrations in the two dimensional hexagonal graphite structure [8, 9].

Van Hove [10] then showed by an application of the critical point theory of M. Morse [11] that the existence of these singularities is a consequence of the periodic lattice structure and not of the detailed character of the atomic force constants. He discussed both two and three dimensional lattices. His theory of the nature of the singularities in $g(\nu)$ for the three dimensional case was verified in specific examples by Newell [12] and Rosenstock [13, 14].

We shall use the abbreviations 2D and 3D in their usual connotation and FS to mean frequency spectrum. RL will denote reciprocal lattice.

Many of the results to be reviewed here are applicable to the general problem of wave propagation in 2D and 3D periodic structures. For example the electron energy level distribution in metals has the same behavior as the $g(\nu)$ of lattice vibrations.

2. Fundamental formulae for analysis of doubly periodic lattices. We shall introduce the discussion of the vibration of periodic lattices by analyzing some 2D examples and then generalizing to 3D systems.

Let us consider a square lattice which contains N rows and N columns of identical particles of mass M. The lattice points are identified by (l,m) (where both l and m range through the integers $1, 2, \cdots, N$). The components of the displacement of the particle (l,m) are designated by $u_{l,m}$ and $v_{l,m}$. Our notation is summarized in Figure 2.

We shall assume that our lattice is formed on a torus so that we can apply the simplifying Born-Karman boundary conditions:

$$u_{l,m} = u_{l+N,m}, \qquad v_{l,m} = v_{l+N,m} \tag{2.1a}$$

$$u_{l,m} = u_{l,m+N}, \qquad v_{l,m} = v_{l,m+N}. \tag{2.1b}$$

Physically one expects thermodynamic functions which are proportional to the number of particles in a system to be independent of surface effects in three

dimensions and of boundary effects in two dimensions. We shall discuss the mathematical theory of this point in section 5. The boundary conditions (2.1) are generally used in solid state problems.

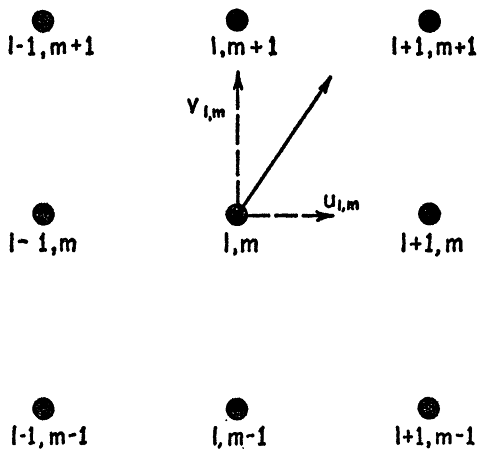


FIG. 2. Designation of lattice points in a two dimensional lattice.

When the particles are displaced from their equilibrium positions the total potential energy of interactions can be expanded in a Taylor series about its equilibrium value:

$$\Phi = \Phi_0 - \sum C_{l,m}u_{l,m} - \sum C'_{l,m}v_{l,m} + \text{terms quadratic, cubic, } etc., \text{ in } u_{l,m} \text{ and } v_{l,m}.$$

Since $C_{l,m}$ and $C'_{l,m}$ represent the forces in the horizontal and vertical directions acting on particle (l, m) while it is at equilibrium, they must vanish, equilibrium being defined as that state in which no forces act on any particles. By making the hypothesis that displacements from equilibrium are small, we are justified in neglecting the cubic and higher order terms in the displacements.

The forces which act in the horizontal and vertical directions respectively on the (l,m)-th particle are

$$-\frac{\partial \Phi}{\partial u_{l,m}} = -\sum_{\lambda,\mu} a_{\lambda,\mu}u_{l+\lambda,m+\mu} - \sum_{\lambda,\mu} b_{\lambda,\mu}v_{l+\lambda,m+\mu} \tag{2.2a}$$

$$-\frac{\partial \Phi}{\partial v_{l,m}} = -\sum_{\lambda,\mu} a'_{\lambda,\mu}u_{l+\lambda,m+\mu} - \sum_{\lambda,\mu} b'_{\lambda,\mu}v_{l+\lambda,m+u} \tag{2.2b}$$

where $-a_{\lambda,\mu}u_{l+\lambda,m+\mu}$ is the contribution of the horizontal force on (l,m) due to a horizontal displacement of a particle λ lattice distances to the right and μ above (l,m), *etc.*

The equations of motion of our system of coupled particles are

$$(2.3a)\qquad M d^2 u_{l,m}/dt^2 = -\partial\Phi/\partial u_{l,m}$$

$$(2.3b)\qquad M d^2 v_{l,m}/dt^2 = -\partial\Phi/\partial v_{l,m}.$$

These equations are linear in the displacements and contain only constant coefficients to the $u_{l,m}$'s and $v_{l,m}$'s. Hence we can seek solutions of the form

$$(2.4a)\qquad u_{l,m} = ue^{2\pi i\nu t}e^{i(\phi_1 l+\phi_2 m)}$$

$$(2.4b)\qquad v_{l,m} = ve^{2\pi i\nu t}e^{i(\phi_1 l+\phi_2 m)}$$

where u, v, ν, ϕ_1, and ϕ_2 are constants which must be chosen so that (2.4) satisfies (2.3), the boundary conditions (2.1), and whatever initial conditions that might be imposed.

The boundary conditions (2.1) are satisfied if ϕ_1 and ϕ_2 are solutions of

$$e^{i\phi_1 N} = 1 \quad \text{and} \quad e^{i\phi_2 N} = 1,$$

that is, if

$$(2.5)\qquad \phi_1 = 2\pi k/N \quad \text{and} \quad \phi_2 = 2\pi j/N$$

with $k, j=1, 2, \cdots, N$ or (if N is even) k, j may run through $-\frac{1}{2}N+1, -\frac{1}{2}N+2, \cdots, 0, 1, 2, \cdots, \frac{1}{2}N$. This gives N^2 possible choices of the pair (ϕ_1, ϕ_2). The frequency ν is related to ϕ_1 and ϕ_2 by noting that if (2.4) is substituted into (2.2) and (2.3) the constants u and v are solutions of the homogeneous linear equations

$$(2.6a)\qquad u\{F_{11}(\phi_1, \phi_2) - 4\pi^2\nu^2 M\} + vF_{12}(\phi_1, \phi_2) = 0$$

$$(2.6b)\qquad uF_{21}(\phi_1, \phi_2) + v\{F_{22}(\phi_1, \phi_2) - 4\pi^2\nu^2 M\} = 0$$

where

$$(2.6c)\qquad F_{11}(\phi_1, \phi_2) = \sum_{\lambda,\mu} a_{\lambda,\mu}e^{i(\phi_1\lambda+\phi_2\mu)}; \qquad F_{12}(\phi_1, \phi_2) = \sum_{\lambda,\mu} b_{\lambda,\mu}e^{i(\phi_1\lambda+\phi_2\mu)}$$

$$(2.6d)\qquad F_{21}(\phi_1, \phi_2) = \sum_{\lambda,\mu} a'_{\lambda,\mu}e^{i(\phi_1\lambda+\phi_2\mu)}; \qquad F_{22}(\phi_1, \phi_2) = \sum_{\lambda,\mu} b'_{\lambda,\mu}e^{i(\phi_1\lambda+\phi_2\mu)}.$$

In order for solutions of (2.6) to exist, the determinant of the coefficients of u and v must vanish:

$$(2.7)'\qquad \begin{vmatrix} F_{11}(\phi_1, \phi_2) - 4\pi^2\nu^2 M & F_{12}(\phi_1, \phi_2) \\ F_{21}(\phi_1, \phi_2) & F_{22}(\phi_1, \phi_2) - 4\pi^2\nu^2 M \end{vmatrix} = 0.$$

This equation relates the frequency ν to the constants (ϕ_1, ϕ_2). There are two values of ν^2 for each (ϕ_1, ϕ_2) pair. Hence there is a total of $2N^2$ normal modes, or solutions (2.4) of our equations of motion (2.3). The most general solution is a linear combination of these $2N^2$ normal modes.

3. Curves of constant frequency in (ϕ_1, ϕ_2)-space and distribution of normal mode frequencies. Equation (2.7) is a quadratic equation in the square of the circular frequency $\omega = 2\pi\nu$:

$$\omega^4 - \omega^2(F_{11} + F_{22}) + (F_{11}F_{22} - F_{12}F_{21}) = 0. \tag{3.1}$$

The solutions of this equation are

$$[\omega(\phi_1, \phi_2)]^2 = \tfrac{1}{2}(F_{11} + F_{22}) \pm \tfrac{1}{2}[(F_{11} - F_{22})^2 + 4F_{12}F_{21}]^{1/2}. \tag{3.2}$$

Since each F_{ij} is a doubly periodic function of (ϕ_1, ϕ_2), ω also has this character and $\omega(\phi_1, \phi_2) = \omega(\phi_1 + 2\pi m, \phi_2 + 2\pi n)$ when m and n are positive or negative integers. We shall refer to all those frequencies which are generated as (ϕ_1, ϕ_2) runs through the N^2 values defined by (2.5) as a branch of the frequency spectrum. Our 2D lattice has two branches. We shall refer to the (ϕ_1, ϕ_2) space over which our doubly periodic function $\omega(\phi_1, \phi_2)$ is defined as a doubly periodic space.

Henceforth all of our discussions will be concerned with lattices which contain a very large number, N^2, of particles. Our various formulae will refer to limit results as $N \to \infty$. In this limit the uniformly distributed points in (ϕ_1, ϕ_2)-space (those defined by 2.5) which correspond to a branch of frequencies of normal modes become dense. The number of normal modes associated with a closed region in (ϕ_1, ϕ_2)-space is proportional to the area of that region. The proportionality constant is $N^2/4\pi^2$ since the area of a single period of our doubly periodic (ϕ_1, ϕ_2)-space is associated with N^2 normal modes.

It is clear that a doubly periodic function defined on a two dimensional space is equivalent to a function defined on a torus. Since the F_{ij}'s are continuous functions of (ϕ_1, ϕ_2), two neighboring points in (ϕ_1, ϕ_2)-space yield frequencies which differ only slightly from each other. Curves of constant frequency can be constructed in (ϕ_1, ϕ_2)-space. They are closed curves on the (ϕ_1, ϕ_2) torus. The curve $\omega = \omega_1 =$ constant separates the torus into two regions, one which corresponds to frequencies $\nu = \omega/2\pi$ which are less than $\nu_1 = \omega_1/2\pi$ and the other to frequencies greater than ν_1 (there may actually be several closed curves of a single frequency of interest; in this case the regions of frequencies $< \nu_1$ may not be connected).

If we let $N_\alpha(\nu)$ be the number of frequencies less than ν in the α-th branch, we have in the limit as $N \to \infty$

$$N_\alpha(\nu) = \frac{N^2}{4\pi^2} \int_R \int d\phi_1 d\phi_2 \tag{3.3}$$

where the integration extends over the set R of all values of (ϕ_1, ϕ_2) for which $\nu_\alpha(\phi_1, \phi_2)$ (the definition of $\nu_\alpha^2(\phi_1, \phi_2)$ is $\omega_\alpha^2(\phi_1, \phi_2)/2\pi$ with ω_α^2 given by (3.2)) is less than ν^2. Since the number of frequencies between ν and $\nu + d\nu$ in the α-th branch is

$$g_\alpha(\nu)d\nu = N_\alpha(\nu + d\nu) - N_\alpha(\nu) = (\partial N_\alpha/\partial \nu)d\nu,$$

we have

$$g_\alpha(\nu) = \frac{\partial N_\alpha}{\partial \nu} = \frac{N^2}{4\pi^2}\frac{\partial}{\partial \nu}\int_R\int d\phi_1 d\phi_2. \tag{3.4}$$

An alternative formula, which has been derived by Bowers and Rosenstock [6], in terms of δ-functions, allows one to express $g_\alpha(\nu)$ as an integral over a complete period of (ϕ_1, ϕ_2)-space. Now we write

$$\begin{aligned} g_\alpha(\nu)d\nu &= N_\alpha(\nu + \tfrac{1}{2}d\nu) - N_\alpha(\nu - \tfrac{1}{2}d\nu) \\ &= \frac{N^2}{4\pi^2}\int_S\int d\phi_1 d\phi_2, \end{aligned}$$

where S is the set of values of (ϕ_1, ϕ_2) for which

$$-\nu d\nu < \nu_\alpha^2(\phi_1, \phi_2) - \nu^2 < \nu d\nu.$$

If $F_a(x)$ is a function with the property

$$F_a(x) = \begin{cases} 1 & \text{if} \quad -a < x < a \\ 0 & \text{if} \quad |x| > a \end{cases}$$

then

$$\int_{-\infty}^{\infty} \frac{1}{2a} F_a(x)dx = 1,$$

so that $(1/2a)F_a(x)$ approaches the delta function $\delta(x)$ as $a \to 0$. Since

$$g_\alpha(\nu)d\nu = \frac{N^2}{4\pi^2}\int_{-\pi}^{\pi}\int F_{\nu+d\nu}[\nu_\alpha^2(\phi_1, \phi_2) - \nu^2]d\phi_1 d\phi_2,$$

we have

$$g_\alpha(\nu) = \frac{2\nu N^2}{4\pi^2}\int_{-\pi}^{\pi}\int \delta[\nu_\alpha^2(\phi_1, \phi_2) - \nu^2]d\phi_1 d\phi_2. \tag{3.5}$$

The Fourier integral representation of the δ-function

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ixy}dy$$

will sometimes be useful in the detailed calculation of (3.5). It leads to

$$g_\alpha(\nu) = \frac{\nu N^2}{(4\pi)^2\pi}\int_{-\infty}^{\infty}\int_{-\pi}^{\pi}\int \exp\left\{iy[\nu_\alpha^2(\phi_1, \phi_2) - \nu^2]\right\}dy d\phi_1 d\phi_2. \tag{3.6}$$

Obvious 3D generalizations of (3.4) and (3.6) exist.

4. The frequency spectrum of some special examples of two-dimensional lattices. We shall now apply the ideas developed in the two previous sections to the calculation of the frequency spectrum of 2D lattices whose vibrations are transverse (normal to the plane of the lattice) and of several special cases with vibrations in the plane.

A considerable insight into the behavior of curves of constant frequency and the FS can be achieved by studying a simple model in which each particle on a 2D square $N\times N$ lattice has only one degree of freedom. This we shall regard as the displacement of a particle in a direction normal to the equilibrium plane of the lattice. We shall postulate each particle to interact only with its nearest neighbors but in an unsymmetrical manner with the force constant for interactions along a row differing from that for interactions along a column.

We let $z_{l,m}$ be the displacement of the (l,m)-th particle from equilibrium, and choose the potential energy of interaction between particles to be

$$\Phi = \tfrac{1}{2}\alpha_1 \sum_{l,m} (z_{l,m} - z_{l,m+1})^2 + \tfrac{1}{2}\alpha_2 \sum_{l,m} (z_{l,m} - z_{l+1,m})^2. \tag{4.1}$$

The equations of motion of the particles on our lattice are

$$M \frac{d^2 z_{l,m}}{dt^2} = -\frac{\partial \Phi}{\partial z_{l,m}} = -2(\alpha_1 + \alpha_2) z_{l,m} + \alpha_1 (z_{l,m+1} + z_{l,m-1}) + \alpha_2 (z_{l+1,m} + z_{l-1,m}).$$

As usual we use the periodic boundary condition

$$z_{l,m} = z_{l+N,m} = z_{l,m+N} = z_{l+N,m+N}$$

and find the frequencies of normal modes to be given by

$$4\pi^2 \nu^2 M = 2(\alpha_1 + \alpha_2) - 2\alpha_1 \cos \phi_1 - 2\alpha_2 \cos \phi_2 \tag{4.2}$$

where

$$\phi_1 = 2\pi j/N, \quad \phi_2 = 2\pi k/N, \quad \text{and} \quad j,\, k = 0, 1, 2, \cdots, N-1. \tag{4.3}$$

If N is even we can choose j and k to run from $-\tfrac{1}{2}N+1$ to $\tfrac{1}{2}N$.

The largest frequency, ν_L, appears at $\phi_1 = \phi_2 = \pi$. Then

$$4\pi^2 M \nu_L^2 = 4(\alpha_1 + \alpha_2). \tag{4.4a}$$

Hence, if we let

$$r = \alpha_1/\alpha_2 \leqq 1 \quad \text{and} \quad f = \nu/\nu_L \tag{4.4b}$$

(4.2) becomes

$$(1 - 2f^2)(1 + r) = r \cos \phi_1 + \cos \phi_2. \tag{4.5}$$

Schematic curves of constant frequency are plotted in Figure 3.

We can apply equation (3.4) to the calculation of $g(\nu)$. A special integral must be evaluated in the three cases

i. $$f^2 \leqq r/(1+r)$$

ii. $$r/(1+r) \leqq f^2 \leqq 1/(1+r)$$

iii. $$1/(1+r) \leqq f^2 \leqq 1.$$

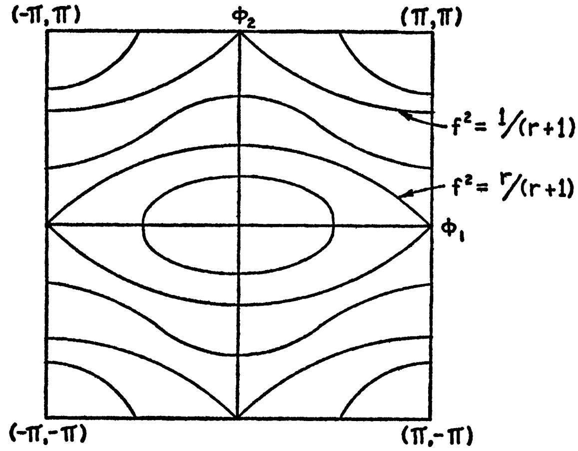


FIG. 3. Schematic curves of constant frequency in the reciprocal lattice of a two dimensional crystal.

The typical curves in one quadrant of each of three regimes are given in Figure 4. We now calculate the integrals of each of the shaded areas in the figures.

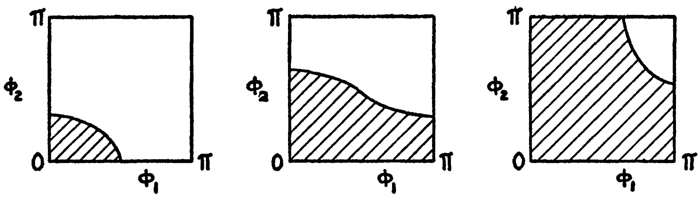


FIG. 4. Typical curves of constant frequency for three ranges of frequency.

The FS is proportional to the derivatives of these areas with respect to frequency (equation 3.4).

Case i. The intercept at $\phi_1=0$ is

$$\phi_2^{(0)} = \cos^{-1} [1 - 2f^2(1+r)]$$

while that at $\phi_2=0$ is

$$\phi_1^{(0)} = \cos^{-1} [1 - 2(1+r)f^2/r]$$

Hence the shaded area in Figure 4a is

$$\int_0^{\phi_1^{(0)}} \cos^{-1}\{(1-2f^2)(1+r)-r\cos\phi_1\}\,d\phi_1$$

so that (remembering that each quadrant yields only one fourth of the frequencies)

$$\nu_L g(\nu) = \frac{N^2}{\pi^2}\frac{\partial}{\partial f}\int_0^{\phi_1^{(0)}} \cos^{-1}\{(1-2f^2)(1+r)-r\cos\phi_1\}\,d\phi_1 = \frac{4N^2(1+r)f}{\pi^2}\times$$

$$\int_{r-2(1+r)f^2}^{r} \frac{dx}{\{(r-x)(r+x)[2(1+r)f^2-r+x][2+r-2f^2(1+r)-x]\}^{1/2}}.$$

In our interval of interest $0 \leqq f^2 \leqq r/(1+r)$, the roots of the polynomial of the denominator of the integrand are ordered with

$$2+r-2f^2(1+r) > r \geqq r-2f^2(1+r) \geqq -r.$$

Hence a direct application of equation 553 in Peirce's Table of Integrals yields

$$(4.6) \qquad \nu_L g(\nu) = \frac{4N^2(1+r)f}{\pi^2\sqrt{r}}K\left((1+r)f\sqrt{\frac{1-f^2}{r}}\right) \quad \text{if} \quad 0 \leqq f^2 \leqq r/(1+r)$$

where $K(k)$ is the complete elliptic integral

$$K(k) = \int_0^1 \frac{dx}{[(1-x^2)(1-k^2x^2)]^{1/2}}.$$

Case ii. Here we compute the shaded area in Figure 4b. This is

$$\int_0^{\pi} \cos^{-1}\{(1-2f^2)(1+r)-r\cos\phi\}\,d\phi_1.$$

Hence, if we let $x = r\cos\phi$, and apply (3.4) we find

$$(4.7) \qquad \nu_L g(\nu) = \frac{4N^2}{\pi^2(1-f^2)^{1/2}}K\left(\frac{r^{1/2}}{(1+r)f\sqrt{1-f^2}}\right)$$

$$\text{if} \quad r/(1+r) \leqq f^2 \leqq 1/(1+r).$$

Case iii. In the range $f^2 > 1/(1+r)$, the curves of constant frequency intersect the line $\phi_2 = \pi$ at

$$\phi_1^{(\pi)} = \cos^{-1}\{[(2+r)-2f^2(1+r)]/r\}.$$

Hence the area of the shaded portion of Figure 4c is

$$\pi\phi_1^{(\pi)} + \int_{\phi_1^{(\pi)}}^{\pi} \cos^{-1} \{(1 - 2f^2)(1 + r) - r \cos \phi_1\} d\phi_1$$

so that

$$\nu_L g(\nu) = \frac{4N^2(1 + r)f}{\pi^2 r^{1/2}} K((1 + r)f\sqrt{(1 - f^2)/r}) \quad \text{if} \quad 1 \geqq f^2 \geqq 1/(1 + r). \tag{4.8}$$

It is to be noted that (4.8) and (4.6) have exactly the same form.

In a symmetrical two-dimensional lattice $r = 1$ and the range in which (4.7) exists vanishes. The elliptic integral $K(k)$ behaves as $\log [4/(1 - k^2)^{1/2}]$ as $k \to 1$. This logarithmic singularity occurs in $g(\nu)$ as ν/ν_L approaches $[r/(1+r)]^{1/2}$ or $[1/(1+r)]^{1/2}$. In fact,

$$\nu_L g(\nu) \sim \begin{cases} -(N^2/\pi^2)(1 + r)^{1/2} \log \{\nu/\nu_L - [r/(1 + r)]^{1/2}\}^2 & \text{as } (\nu/\nu_L)^2 \to r/(1 + r) \quad (4.9\text{a}) \\ -(N^2/\pi^2)r^{-1/2}(1 + r)^{1/2} \log \{\nu/\nu_L - (1 + r)^{-1/2}\}^2 & \text{as } (\nu/\nu_L)^2 \to 1/(1 + r). \quad (4.9\text{b}) \end{cases}$$

As the lattice becomes more and more symmetrical, (*i.e.*, as $r \to 1$) the two logarithmic singularities approach each other and become one in the limit. In other words the singularity is split by reducing the symmetry. We have plotted $g(\nu)$ for several values of r in Figure 5 to exhibit this property.

The result for a symmetrical lattice agrees with that of Bowers and Rosenstock who showed that the logarithmic singularity persists even when next nearest neighbors are taken into account. As long as the nearest neighbor interactions predominate, the singularity moves toward higher frequencies with increasing next nearest neighbor force constants. Curves based on the formulae of Bowers and Rosenstock are given in Figure 6.

An especially important feature of the results given above (as well as of those of Bowers and Rosenstock [6]) is that the logarithmic singularities in the FS occur at frequencies which correspond to saddle points in (ϕ_1, ϕ_2)-space. In Figure 3 the saddle point on the $f^2 = (1+r)^{-1}$ curve occurs at $(0, \pi)$ and that on the $f^2 = r/(1+r)$ curve appears at $(\pi, 0)$. In a symmetrical lattice with $r = 1$, both of these saddle points are on the curve with $f^2 = \frac{1}{2}$.

Since $K(k) \sim \frac{1}{2}\pi$ as $k \to 0$, we have as $\nu \to 0$

$$\nu_L g(\nu) \sim 2N^2(1 + r)\nu/\pi r^{1/2}\nu_L.$$

In the Debye continuum theory of a two dimensional lattice $g(\nu)$ is proportional to ν for all frequencies. It is not surprising that the continuum and discrete models give similar results in the range of wave lengths long compared to lattice spacings.

Equations (4.6)–(4.8) reduce properly to the one dimensional FS as $r \to 0$.

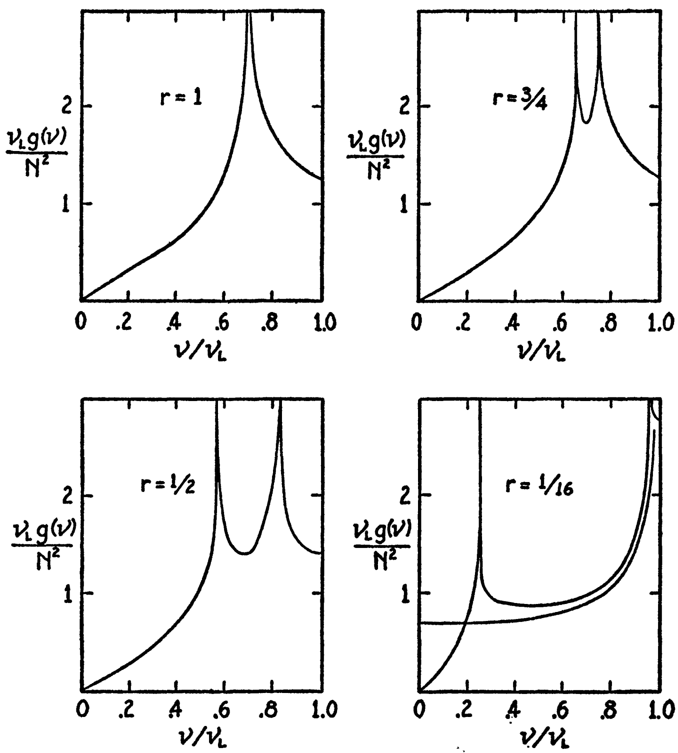


FIG. 5. Variation of frequency spectrum of two dimensional lattice with one degree of freedom per lattice point with increasing anisotropy. The parameter r equals 1 in an isotropic lattice and zero in the absence of coupling in one direction.

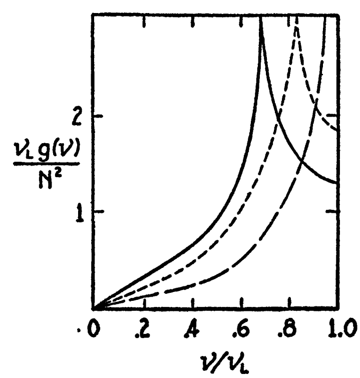


FIG. 6. Variation of frequency spectrum with ratio of next nearest to nearest neighbor force constants in two dimensional lattice with one degree of freedom per lattice point. Solid line to left represents no next nearest neighbor coupling. As this coupling increases the peak moves to the right. The curve whose peak falls at $\nu/\nu_L = 1$ corresponds to equal nearest and next nearest neighbor coupling.

In this limit (4.7) describes the entire spectrum. Since $K(0)=\frac{1}{2}\pi$, (4.7) becomes

$$\nu_L g(\nu) = N\left[\frac{2N}{\pi}(1-f^2)^{-1/2}\right],$$

the one-dimensional $g(\nu)$.

Equations 4.6–4.8 could also have been derived from the formula (3.6) which involves the Fourier transform of the δ-function. In our problem

$$\begin{aligned} g(\nu) = \frac{N^2\nu}{\pi(4\pi)^2}\int_{-\infty}^{\infty}dy\int_{-\pi}^{\pi}\int \exp\{iy[\nu^2 - \nu_L^2\{(1+r) \\ - r\cos\phi_1 - \cos\phi_2\}/2(1+r)]\}d\phi_1 d\phi_2. \end{aligned} \tag{4.10}$$

An application of the formula

$$J_0(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ix\cos\theta}d\theta \tag{4.11}$$

and a simple transformation yields

$$\nu_L g(\nu) = 2N^2(1+r)f\pi^{-1}\int_{-\infty}^{\infty} J_0(x)J_0(rx)e^{i(1+r)(2f^2-1)x}dx. \tag{4.12}$$

With the aid of the two Bessel function formulae

$$\pi J_0(z)J_0(Z) = \int_0^{\pi} J_0(\sqrt{Z^2+z^2-2Zz\cos\theta})d\theta$$

and

$$\int_0^{\infty} J_0(at)\cos bt\,dt = \begin{cases}(a^2-b^2)^{-1} & \text{if } a > b \\ 0 & \text{if } a < b,\end{cases}$$

(4.12) can easily be shown to be equivalent to (4.6)–(4.8).

We shall now show that the saddle points and logarithmic singularities discussed above also exist in 2D lattices of particles which vibrate in the plane of the lattice. It was in this type of system that the existence of the logarithmic singularities was first noticed [5].

The simplest example of a 2D system with vibrations in the plane is one with interactions between nearest neighbors only. One would expect the coupling between horizontal and vertical components of lattice vibrations to be very weak. On that basis let us consider as a model a system with the potential energy function

$$\begin{aligned} \Phi = \tfrac{1}{2}\alpha_1\sum_{l,m}(u_{l,m}-u_{l+1,m})^2 + \tfrac{1}{2}\alpha_2\sum_{l,m}(u_{l,m}-u_{l,m+1})^2 \\ + \tfrac{1}{2}\beta_1\sum_{l,m}(v_{l,m}-v_{l,m+1})^2 + \tfrac{1}{2}\beta_2\sum_{l,m}(v_{l,m}-v_{l+1,m})^2. \end{aligned} \tag{4.13}$$

The constants α_1 and β_1 are force constants associated with displacements parallel to the line connecting two particles while α_2 and β_2 go with displacements perpendicular to the connecting line. If all forces between particles are central, $\alpha_2=\beta_2=0$. Since the non-central contribution to forces in a crystal is usually small, we assume that $\alpha_2\ll\alpha_1$ and $\beta_2\ll\beta_1$. The 3D analogue of this model has been discussed by Newell and Rosenstock [13].

The characteristic determinant (2.7) for this system is diagonal. The reader can easily verify that the frequencies of normal modes are

$$4\pi^2\nu^2M = 2(\alpha_1+\alpha_2) - 2\alpha_1\cos\phi_1 - 2\alpha_2\cos\phi_2$$
$$4\pi^2\nu^2M = 2(\beta_1+\beta_2) - 2\beta_1\cos\phi_2 - 2\beta_2\cos\phi_1$$

where ϕ_1 and ϕ_2 satisfy (2.5).

Since these frequencies are generated by exactly the same equation, (4.2), as the generating equation for normal mode frequencies in a system with transverse vibrations, the FS, $g(\nu)$, must have the form as that plotted in Figure 5. In a symmetrical lattice $(\alpha_1, \alpha_2)=(\beta_1, \beta_2)$, and there are two logarithmic singularities as long as $\alpha_2\neq\alpha_1$ (the case $\alpha_1=\alpha_2$ is quite unrealistic physically). In an unsymmetrical lattice with $(\alpha_1, \alpha_2)\neq(\beta_1, \beta_2)$, $\alpha_2\neq\alpha_1$, $\beta_2\neq\beta_1$, and none of the α's or β's equal to zero, there are four singularities; the removal of a symmetry doubles the number of singularities.

When the forces are central $(\alpha_2=\beta_2=0)$, $g(\nu)$ becomes exactly that of a one-dimensional system. In order to restore the two-dimensional character to a lattice of particles with central forces and vibrations in the lattice plane only, one must take into account interactions between more distant neighbors.

The calculation of $g(\nu)$ becomes rather complicated when interactions between more than nearest neighbors are included. The case of interactions between nearest and next nearest neighbors has been discussed at considerable length by the author [5]. Here we shall merely give a statement of the problem and summarize the results.

We restrict ourselves to symmetrical systems with central forces (with interactions only along lines which connect particles). A typical set of interactions are those along lines through the vectors given in Figure 7. The total potential energy of interaction between all particles is ($u_{l,m}$ and $v_{l,m}$ represent horizontal and vertical displacement of the (l, m)-th particle)

$$\begin{aligned}\Phi &= \tfrac{1}{2}\alpha\sum_{l,m}\left[(u_{l,m}-u_{l+1,m})^2+(v_{l,m}-v_{l,m+1})^2\right]\\ &\quad+\tfrac{1}{2}\gamma\sum_{l,m}\left[(u_{l,m}-u_{l+1,m+1}+v_{l,m}-v_{l+1,m+1})^2\right.\\ &\qquad\left.+(u_{l,m}-u_{l+1,m-1}-v_{l,m}+v_{l+1,m-1})^2\right].\end{aligned}\tag{4.15$'$}$$

In the usual manner the characteristic equation is found to be

$$\begin{vmatrix}(1-\tau)(1-c_1)+\tau(1-c_1c_2)-2f^2 & \tau s_1s_2\\ \tau s_1s_2 & (1-\tau)(1-c_2)+\tau(1-c_1c_2)-2f^2\end{vmatrix}=0\tag{4.16}$$

where

$$c_j = \cos \phi_j, \qquad s_j = \sin \phi_j, \quad \text{and} \quad f = \nu/\nu_L. \tag{4.17}$$

The largest frequency is given by

$$\nu_L = [(4\alpha + 8\beta)/4\pi^2 M]^{1/2} \tag{4.18}$$

and

$$\tau = 8\gamma/4\pi^2 M\nu_L^2 = 1/[1 + (\alpha/2\gamma)]. \tag{4.19}$$

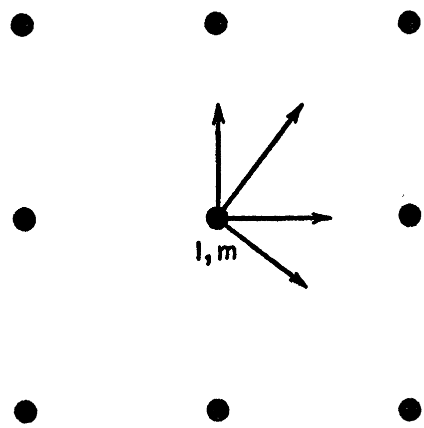


FIG. 7. Typical directions of interaction between nearest and next nearest neighbors in a two dimensional lattice.

The frequencies of normal modes are generated by

$$\begin{aligned} f^2 = (\nu/\nu_L)^2 &= \tfrac{1}{4}[2\tau(1 - c_1c_2) + (1 - \tau)(2 - c_1 - c_2)] \\ &\pm \tfrac{1}{4}[4\tau^2(c_2^2 - 1)(c_1^2 - 1) + (1 - \tau)^2(c_2 - c_1)^2]^{1/2} \end{aligned} \tag{4.20}$$

as the values of ϕ_j run through their usual range (2.5).

An unfortunate feature of (4.20) is that it involves radicals. The integration of (3.4) or (3.6) does not give a closed form result in terms of well known functions. However, when $\tau = 1/3$, the quantity inside the radical is a perfect square, and the two branches of frequencies become:

$$f_+^2 = \tfrac{1}{6}\{2(1 - c_1c_2) + 2 - c_1 - c_2\} \tag{4.21a}$$

$$f_-^2 = \tfrac{1}{6}\{2 - c_1 - c_2). \tag{4.21b}$$

The branch (4.21b) has a form that has already been discussed ($r = 1$ in (4.5)). The "+" branch can be analyzed easily and also leads to a FS that is a complete elliptic integral of the second kind. Both branches have logarithmic

singularities (the "+" branch at $f^2=3/4$ and the "−" branch at $f^2=1/3$) at values of f which correspond to saddle points. The curves of constant frequency (in one quadrant of (ϕ_1, ϕ_2)-space) of the "+" branch are drawn schematically in Figure 8.

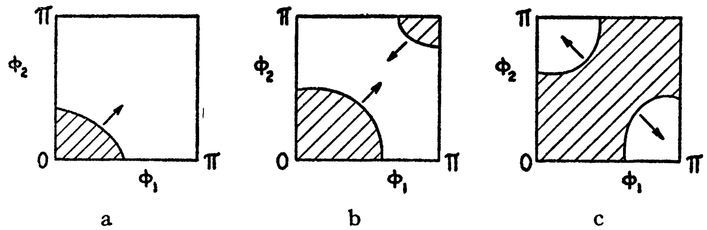


FIG. 8. Curves of constant frequency in the three ranges. The arrows point in the direction of increasing frequency.

Although one cannot obtain simple expressions for $g(\nu)$ for other values of τ, it is not difficult to plot the curves of constant frequency for any fixed τ. For all values of $\tau<1/2$ the "−" branch has a saddle point at $f^2=\tau$ and the "+" branch has one at $f^2=1-\tau$ if $\tau<1/5$ and at $f^2=(1+3\tau)^2/16\tau$ if $1/5<\tau<1/2$. It can be shown that logarithmic singularities exist at each of these saddle points and therefore that there is a singularity for each branch of the spectrum.

Smollett [7] has plotted out the curves of constant frequency for a lattice of alternate positively and negatively charged particles, all of equal mass, to determine the effect of long range coulomb forces on their character. These curves are somewhat more complicated than those with short range forces, but the saddle points and singularities still exist.

The qualitative similarity between the frequency spectra of all the models discussed leads one to ask three questions; (a) do saddle points generally lead to logarithmic singularities; (b) is there a physical reason for the existence of the saddle points; and (c) would the singularities remain if the periodic boundary conditions were replaced by some other ones? We shall now deduce an affirmative answer to all of these questions.

5. Topological basis for logarithmic singularities of the 2D FS.* We first show that saddle points in (ϕ_1, ϕ_2)-space always lead to logarithmic singularities in $g(\nu)$ (see also Smollett). If the saddle point is not on the ϕ_1 or ϕ_2 axis, let us rotate the coordinate system so that this becomes the case. If it is not symmetrical so that

$$\frac{\partial^2 f^2}{\partial \phi_1^2} = -\frac{\partial^2 f^2}{\partial \phi_2^2} = -2a^2 < 0, \tag{5.1}$$

* The topological discussion is too oversimplified for many applications to FS calculations. The reader is referred to the work of van Hove for a more detailed and rigorous analysis. The author is indebted to Professor van Hove for an informative discussion on which part of this section is based.

let the scale of one of the variables be changed so that this becomes true at the saddle point, $(\phi_1^{(s)}, 0)$. Since

$$\frac{\partial f^2}{\partial\phi_1} = \frac{\partial f^2}{\partial\phi_2} = \frac{\partial^2 f^2}{\partial\phi_1\partial\phi_2} = 0$$

at $(\phi_1^{(s)}, 0)$, the Taylor expansion of f^2 about $(\phi_1^{(s)}, 0)$ is

$$f^2 = f_s^2 - a^2\left((\phi_1^{(s)} - \phi_1)^2 - \phi_2^2\right) + \cdots \tag{5.2}$$

where f_s is the value of ν/ν_L on the curve of constant frequency which goes through $(\phi_1^{(s)}, 0)$. The value of the ϕ_1 intercept, $\phi_1^{(0)}$, in the neighborhood of the saddle point is related to f through

$$\phi_1^{(s)} - \phi_1^{(0)} = (f_s^2 - f^2)^{1/2}/a. \tag{5.3}$$

The contribution of the shaded area of Figure 9 to $\nu_L g(\nu)$ in a range

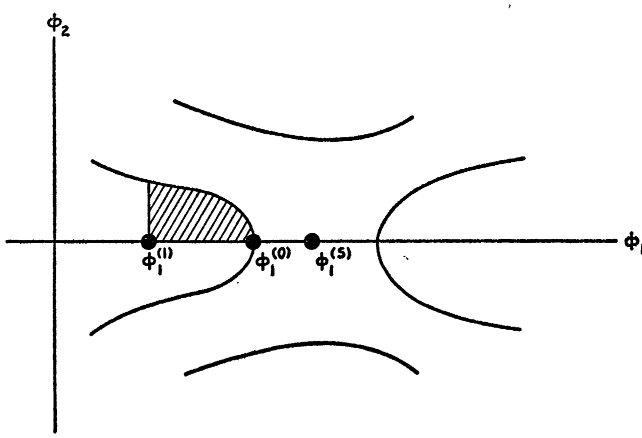


FIG. 9. Curves of constant frequency in the neighborhood of a saddle point.

$\phi_1^{(1)} \leqq \phi_1 \leqq \phi_1^{(0)}$ in which (5.2) is a good approximation to f^2 is

$$\begin{aligned}
&\frac{N^2}{4\pi^2}\frac{\partial}{\partial f}\int_{\phi_1^{(1)}}^{\phi_1^{(0)}} \left[(\phi_1^{(s)} - \phi_1)^2 - (f_s^2 - f^2)/a^2\right]^{1/2} d\phi_1 \\
&= \frac{N^2 f}{4\pi^2 a^2}\int_{\phi_1^{(1)}}^{\phi_1^{(0)}} \frac{d\phi_1}{\left[(\phi_1^{(s)} - \phi_1)^2 - (f_s^2 - f^2)/a^2\right]^{1/2}} \\
&= -\frac{N^2 f}{4\pi^2 a^2}\int_{\phi_1^{(s)} - \phi_1^{(1)}}^{(f_s^2 - f^2)^{1/2}/a} \frac{du}{\left[u^2 - (f_s^2 - f^2)/a^2\right]^{1/2}}
\end{aligned}$$

$$= -\frac{N^2 f}{8\pi^2 a^2}\log(f_s - f) + \text{bounded function of } f. \tag{5.4}$$

As the frequency ν approaches ν_s (*i.e.*, as $f \rightarrow f_s$) this term contributes the logarithmic singularity to the FS as we set out to prove. In a similar manner it is easy to obtain the same result as ν_s is approached from higher frequencies.

Van Hove [10] has recently used the critical point theory of the calculus of variations in the large (developed by M. Morse [11]) to show that our saddle points are a direct consequence of the periodicity of a crystal lattice and the Born-Karman boundary conditions.

In preparation for the application of Van Hove's arguments we present a heuristic discussion of a special case of one of M. Morse's theorems. Let $G(\phi_1, \phi_2)$ be a continuous doubly periodic function of ϕ_1 and ϕ_2 with continuous first and second derivatives, the derivatives vanishing only at isolated points. Then Morse's theorem states that G has at least two saddle points. As was mentioned in section 3, a doubly periodic function is equivalent to a function defined on a torus; hence this theorem is equivalent to the statement that a function $G(\phi_1, \phi_2)$ with the properties postulated above and defined on a torus has at least two saddle points.

Since G has been chosen to be continuous it must have at least one maximum and one minimum point in each of its periods. Let the location of one of these maxima be denoted by an × in each period in Figure 10 and let the location of one set of equivalent minima be represented by heavy dots (points D, E, *etc.*)

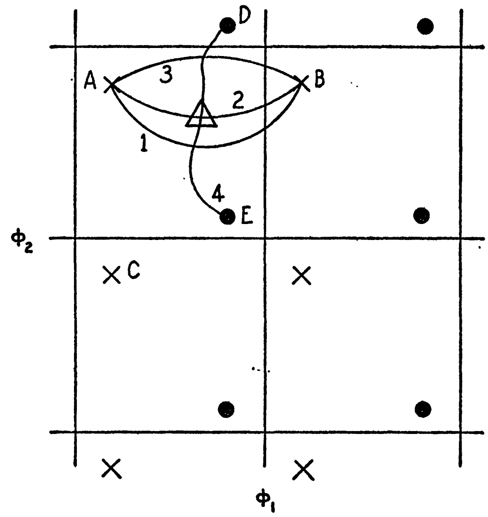
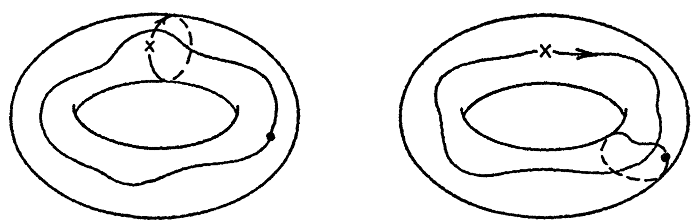


FIG. 10. Curves connecting maxima and minima of functions defined on a doubly periodic space.

If the maxima A and B are connected by a curve such as 1 in Figure 10, there is at least one point on the curve where G has a smaller value than at neighboring points on the curve. G also achieves a smallest value along any other curves, for example on 2 and 3, which connect A and B. The locus of these points forms a continuous curve which passes through E and D. G must achieve

a largest value somewhere along this locus of smallest values. Let this be accomplished at the point represented by the small triangle in Figure 10. This point must be a saddle point because as one passes along 4 from E to D it is a relative maximum, and as one passes from A to B it is a relative minimum.

The same argument could be applied by the examination of paths which connect A and C. Hence the function G has at least two saddle points. It is easy to trace out the same proof when one period of (ϕ_1, ϕ_2)-space is wrapped into a torus. Those paths which start at × and loop the torus, as in Figure 11a, correspond to paths from A to C in Figure 10, while those that go around the



a b

FIG. 11. Closed curves through maxima and minima of a function defined on a torus.

hole (as in Figure 11b) correspond to paths from A to B. No matter how the path which goes around the hole is deformed or displaced it cannot be transformed continuously into the one which goes through the hole. The number of saddle points of a function defined at all points on the surface of a 3D body is at least equal to the maximum number of closed paths which cannot be transformed into each other or into a point by a continuous deformation of the paths. This number is known as the Betti number of the surface.

The fact that there might be two saddle points in our reciprocal (ϕ_1, ϕ_2)-space does not guarantee the existence of two logarithmic singularities in $g(\nu)$. The value of $\nu(\phi_1, \phi_2)$ at the two saddle points may correspond to the same frequency. This is the case when a square lattice is elastically symmetrical. We see in Figure 3 that the two saddle points, one at $(\phi_1, \phi_2) = (0, \pi)$ and the other at $(\pi, 0)$ (as well as those at $(0, -\pi)$ and $(-\pi, 0)$) both correspond to the frequency $\nu = (\frac{1}{2})^{1/2}\nu_L$ when $r = 1$.

On the basis of the general theorem concerning saddle points, Van Hove has pointed out that the logarithmic singularities in the FS of a 2D lattice are a direct consequence of the postulate of the harmonicity of the interatomic forces and of the doubly periodic structure of the lattice. The number of interacting neighbors and the detailed nature of the force constants is not critical as long as the series F_{ij} in (3.6) converge and as long as the lattice is not so weakly coupled that the relaxation of an initial displacement of a given atom propagates a wave only along a single row and column (as is the case of a square lattice with central force interactions which act only between nearest neighbors).

The last of our questions about the singularities of the 2D $g(\nu)$ is whether their existence is sensitive to the choice of periodic boundary conditions. To show that the detailed choice of boundary conditions is unimportant we apply the following argument of Ledermann [15]. It is based on Ledermann's theorem, *If, in a Hermitian matrix the elements of r rows and their corresponding columns are modified in any way whatsoever, provided only that the matrix remains Hermitian, then the number of characteristic values which lie in any interval cannot increase or decrease by more than 2r.*

The particles in an $N\times N$ square lattice with interactions between nearest and next nearest neighbors can be divided into $(N-2)^2$ internal particles and $(4N-4)$ boundary particles. We might identify the internal ones by the numbers $1, 2, \cdots, (N-2)^2$ and the boundary ones by $(N-2)^2+1, \cdots, N^2$. The equations of motion of the internal particles are all of the same form.

The matrix of the secular determinant can then be expressed in the form

$$\left(\begin{array}{c|c} M_{ii} & M_{ib} \\ \hline M_{bi} & M_{bb} \end{array}\right) \begin{array}{l} 2(N-2)^2 \text{ rows.} \\ \hline 2(4N-4) \text{ rows.} \end{array} \tag{5.5}$$

where M_{ii} represents the matrix elements of the interactions between internal particles, M_{ib} and M_{bi} those between internal and boundary particles, and M_{bb} those between boundary particles. The corresponding matrix for Born-Karman boundary conditions would be of the form

$$\begin{pmatrix} M_{ii} & B_{ib} \\ B_{bi} & B_{bb} \end{pmatrix} \tag{5.6}$$

and hence differs from (5.5) in the elements of $2(4N-4)$ rows. Then the maximum number of normal modes which move into or out of a given frequency interval is at most $16(N-1)$ out of the order of $2N^2$. Hence in the limit as $N\rightarrow\infty$ a negligible fraction make the move, and in this limit $g(\nu)/2N^2$ is independent of the boundary conditions.

Even in systems with longer (but fixed) range interactions, the boundary thickness is independent of N so that the number of rows changed in (5.5) in going to periodic boundary conditions is still of $O(N)$.

6. The nature of the singularities in three-dimensional lattices. Van Hove has also discussed the singularities of the 3D FS by topological considerations. The reciprocal lattice of a 3D one component cubic crystal is also 3D. For convenience we shall restrict ourselves to simple cubic lattices. We let $\phi_3=2\pi l/N_3$, with l ranging from 1 to N_3, be the third dimension in the RL. Then each normal mode frequency is a triply periodic function of the ϕ's, $\nu(\phi_1, \phi_2, \phi_3)$. The 3D RL can be visualized as a set of parallel layers of 2D RL's with ϕ_3 constant in each layer. The complete FS of a crystal is the sum of the spectra of each layer in the RL.

The Morse theorem is applicable to each of these layers. The FS associated

with each has at least one or two saddle points, the minimum number depending on symmetry. If the saddle point remains at the same frequency on all layers, the total FS, being a sum of functions each with a logarithmic singularity at the same point, has a logarithmic singularity. This, however, is an exceptional case which occurs only when the force constants satisfy special conditions.

Generally the position of the saddle point varies from layer to layer. A schematic plot of $\nu_c(\phi_3)$, the singular frequency, is made in Figure 12. This frequency, being a continuous periodic function of ϕ_3 has at least one maximum (say $\nu_c^{(2)}$ at $\phi_3^{(2)}$) and one minimum (say $\nu_c^{(1)}$ at $\phi_3^{(1)}$). In the case of a single maxi-

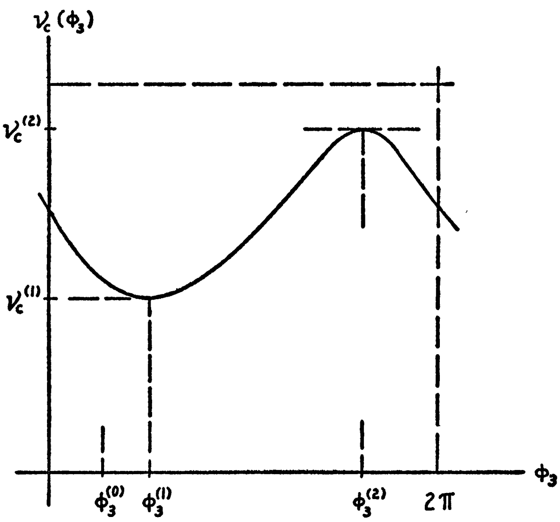


FIG. 12. Variation of singular frequency with layer.

mum and minimum all the logarithmic singularities are located between $\nu_c^{(1)}$ and $\nu_c^{(2)}$. Hence we can expect $g(\nu)$ to have a sharp rise in the neighborhood of $\nu_c^{(1)}$ and a sharp drop in the neighborhood of $\nu_c^{(2)}$. In the limit as $N_3 \to \infty$ these singularities become dense so that if one divides the FS by N_3 as the limit is taken, the final $g(\nu)$ should vary only slowly in the frequency interval $(\nu_c^{(1)}, \nu_c^{(2)})$ (Figs. 13 and 14).

We shall now investigate this behavior analytically for frequencies near $\nu_c^{(1)}$ and $\nu_c^{(2)}$. The singular frequencies on layers of the RL with ϕ_3 close to $\phi_3^{(1)}$ are given by

$$\nu_c(\phi_3) = \nu_c^{(1)} + b^2(\phi_3 - \phi_3^{(1)})^2 + \cdots \tag{6.1$'$}$$

where

$$b^2 = \frac{1}{2} \left.\frac{\partial^2 \nu_c(\phi_3)}{\partial \phi_3^2}\right]_{\phi_3^{(1)}} > 0.$$

The FS of a ϕ_3 layer close to $\phi_3^{(1)}$ and at frequencies near $\nu_c^{(1)}$ is proportional to (see (5.4))

$$-\log|\nu_c(\phi_3)-\nu|. \tag{6.2}$$

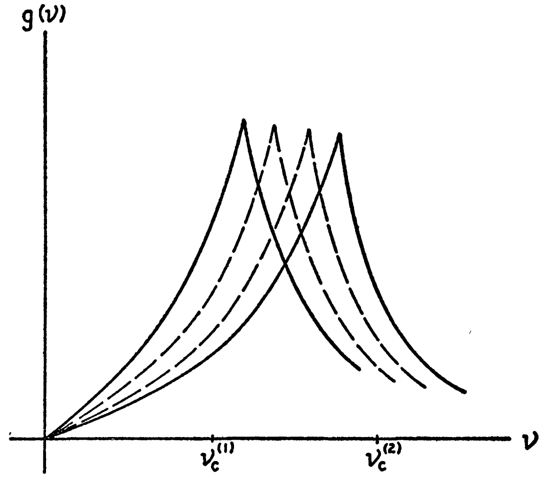


FIG. 13. Frequency spectra associated with various layers in a thin crystal.

We choose $\phi_3^{(0)}$ to be an arbitrary number such that (6.1) is valid for all ϕ_3 in the range $\phi_3^{(0)}<\phi_3<\phi_3^{(1)}$. Then the total contribution of the region of the RL in the neighborhood of the saddle points near $\phi_3^{(1)}$ to the frequency spectrum is proportional to

$$-2\frac{N_3}{2\pi}\int_{\phi_3{}^{(0)}}^{\phi_3{}^{(1)}}\log|\nu_c(\phi_3)-\nu|\,d\phi_3 \tag{6.3}$$

for those frequencies near $\nu_c^{(1)}$. The factor 2 is introduced so that the integration can stop at $\phi_3^{(1)}$ instead of proceeding to a distance $\phi_3^{(1)}-\phi_3^{(0)}$ beyond $\phi_3^{(1)}$.

We shall now show that the FS rises sharply as $\nu\to\nu_c^{(1)}$ (see Fig. 14) from the left. When $\nu<\nu_c^{(1)}$, (6.3) becomes

$$\begin{aligned}I_1&=-\frac{N_3}{\pi}\int_{\phi_3{}^{(0)}}^{\phi_3{}^{(1)}}\log\left[(\nu_c^{(1)}-\nu)+b^2(\phi_3-\phi_3^{(1)})^2\right]d\phi_3\\&=-\frac{N_3}{\pi b}\int_0^c\log(a^2+u^2)\,du\\&=-\frac{N_3}{\pi b}\left\{c\log(a^2+c^2)-2c+2a\tan^{-1}c/a\right\}\end{aligned}$$

where

$$a = \left|\nu_c^{(1)} - \nu\right|^{\frac{1}{2}},\ u = b\,(\varphi_3^{(1)} - \varphi_3),\ \text{and}\ c = (\phi_3^{(1)} - \phi_3^{(0)})b > 0.$$

In the limit as $\nu \to \nu_c^{(1)}$ (since $\tan^{-1} \infty = \frac{1}{2}\pi$)

$$I_1 \sim \frac{N_3}{\pi b} \left\{ -\pi(\nu_c^{(1)} - \nu)^{1/2} + 2c(1 - \log c) \right\}. \tag{6.4}$$

Since the slope of I_1 is proportional to $1/(\nu_c^{(1)} - \nu)^{1/2}$, the FS, $g(\nu)$, has a vertical tangent at $\nu = \nu_c^{(1)}$ (see Fig. 14).

It can now be shown that $g(\nu)$ is continuous at $\nu_c^{(1)}$ but that $g'(\nu)$ is discontinuous. When $\nu > \nu_c^{(1)}$ we define ϕ_3^* so that

$$\nu - \nu_c^{(1)} = b^2(\phi_3^{(1)} - \phi_3^*)^2 = a^2.$$

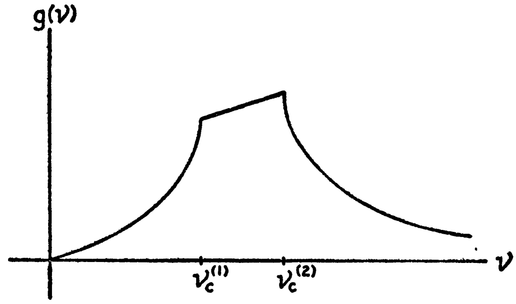


FIG. 14. Typical peak in a 3-D frequency spectrum.

Then (6.3) becomes

$$\begin{aligned} I_2 &= -\frac{N_3}{\pi}\left\{\int_{\phi_3^{(0)}}^{\phi_3^*} \log\,[b^2(\phi_3^{(1)} - \phi_3)^2 - a^2]d\phi_3 \right. \\ &\quad \left. + \int_{\phi_3^*}^{\phi_3^{(1)}} \log\,[a^2 - b^2(\phi_3^{(1)} - \phi_3)^2]d\phi_3\right\} \\ &= \frac{N_3}{\pi b}\left\{\int_c^a \log\,(u^2 - a^2)du + \int_a^0 \log\,(a^2 - u^2)du\right\} \\ &= -\frac{N_3}{\pi b}\left\{c \log\,(c^2 - a^2) + a \log\,[(c + a)/(c - a)] - 2c\right\} \\ &\sim \frac{N_3}{\pi b}\left\{2c(1 - \log c) - (\nu - \nu_c^{(1)})/c + O(\nu - \nu_c^{(1)})^{3/2}\right\}; \end{aligned} \tag{6.5}$$

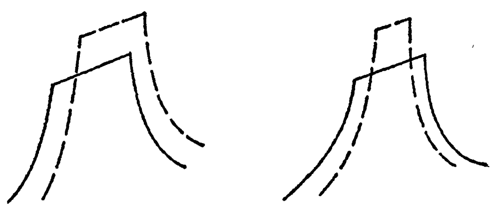
as $\nu \to \nu_c^{(1)}$, (6.4) and (6.5) approach each other. Hence $g(\nu)$ is continuous at $\nu_0^{(1)}$. However, $\partial I_1/\partial\nu \to \infty$ when $\nu \to \nu_c^{(1)}$ from the left while $\partial I_2/\partial\nu \to -N_2/\pi bc$, a

finite number, as $\nu \rightarrow \nu_c^{(1)}$ from the right. Hence $g'(\nu)$ is discontinuous.

One can proceed with a similar analysis to find that $g(\nu)$ has a rapid drop with a vertical tangent at $\nu_c^{(2)}$, the location of the highest frequency saddle point in the RL. This behavior is exhibited in Figure 14.

The 3D RL can also be built up in layers which are normal to either the ϕ_1 or ϕ_2 axes. When the surfaces of constant frequency are not symmetrical functions of ϕ_1, ϕ_2 and ϕ_3 the argument used above can be repeated in each of the new alternative layer constructions to locate other singularities.

Let us consider a case which has symmetrical surfaces of constant frequency and which has simple peaks of the type given in Figure 14. Then let a symmetry be removed by making a small change in the force constants so that contours in planes normal to two of the axes are identical but different from those on planes perpendicular to the third axis. The total FS is the superposition of pairs of peaks with their edges slightly displaced from each other as in Figures



a b

FIG. 15. Displacement of peaks by slight perturbation in symmetry.

a b c d

FIG. 16. Typical peaks in the FS for anisotropic 3-D crystals.

15a or 15b. The total curve, being the sum of these pairs, generally has peaks of the form of those in Figure 16a. Under special conditions only one of the edges might be displaced by the perturbation of the force constants, thus yielding curves such as those in Figure 16b. When larger perturbations are made one of the members of the pair of peaks may be displaced sufficiently to split the original peak into two as in Figure 16c.

Finally when small displacements in force constants are made which remove all symmetries with respect to ϕ_1, ϕ_2, and ϕ_3, three layered peaks such as that in Figure 16d may appear.

We shall now examine some models in which these effects can be studied analytically.

7. Some examples of 3D frequency spectra. As in the case of 2D lattices we shall first discuss a model in which one degree of freedom is associated with each lattice point in a $N \times N \times N$ 3D lattice with interactions between nearest neighbors. If the potential energy of interaction is chosen to be

$$\begin{aligned}\Phi = \tfrac{1}{2}\alpha_1 \sum_{l,m,n} (z_{l,m,n} - z_{l,m,n+1})^2 + \tfrac{1}{2}\alpha_2 \sum_{l,m,n} (z_{l,m,n} - z_{l,m+1,n})^2 \\ + \tfrac{1}{2}\alpha_3 \sum_{l,m,n} (z_{l,m,n} - z_{l+1,m,n})^2\end{aligned} \tag{7.1}$$

and periodic boundary conditions are employed, the frequencies of normal modes are given by

$$4\pi^2\nu^2 M = 2(\alpha_1 + \alpha_2 + \alpha_3) - 2\alpha_1 \cos \phi_1 - 2\alpha_2 \cos \phi_2 - 2\alpha_3 \cos \phi_3 \tag{7.2}$$

where

$$\phi_1 = 2\pi j/N, \qquad \phi_2 = 2\pi k/N, \qquad \phi_3 = 2\pi s/N, \qquad j, k, s = 0, 1, 2, \cdots, N-1.$$

As usual we let $f = \nu/\nu_L$ so that

$$(1 - 2f^2)(\alpha_1 + \alpha_2 + \alpha_3) = \alpha_1 \cos \phi_1 + \alpha_2 \cos \phi_2 + \alpha_3 \cos \phi_3 \tag{7.2a}$$

with

$$4\pi^2 \nu_L^2 M = 4(\alpha_1 + \alpha_2 + \alpha_3). \tag{7.2b}$$

The saddle points on planes normal to the ϕ_3 axis occur at the frequencies

$$f_c^2(\phi_3) = \frac{\alpha_3 + 2\alpha_2 - \alpha_3 \cos \phi_3}{2(\alpha_1 + \alpha_2 + \alpha_3)} \tag{7.3}$$

and

$$f_c^2(\phi_3) = \frac{\alpha_3 + 2\alpha_1 - \alpha_3 \cos \phi_3}{2(\alpha_1 + \alpha_2 + \alpha_3)}. \tag{7.4}$$

The largest value of f_c^2 at a saddle point associated with (7.3) is

$$(f_c^{(1)})^2 = (\alpha_2 + \alpha_3)/(\alpha_1 + \alpha_2 + \alpha_3) \tag{7.5a}$$

while the smallest is

$$(f_c^{(1)})^2 = \alpha_2/(\alpha_1 + \alpha_2 + \alpha_3). \tag{7.5b}$$

The edges of the peak associated with (7.4) are at

$$(f_c^{(2)})^2 = (\alpha_1 + \alpha_3)/(\alpha_1 + \alpha_2 + \alpha_3) \tag{7.6a}$$

on the right and

$$(7.6b) \qquad (f_c^{(2)})^2 = \alpha_1/(\alpha_1 + \alpha_2 + \alpha_3)$$

on the left. A similar analysis of saddle points on planes normal to the ϕ_1 and ϕ_2 axes yield the following four peaks with edges at f^2 equal to

$$(7.7) \qquad \alpha_1/(\alpha_1 + \alpha_2 + \alpha_3) \quad \text{and} \quad (\alpha_1 + \alpha_2)/(\alpha_1 + \alpha_2 + \alpha_3)$$

$$(7.8) \qquad \alpha_3/(\alpha_1 + \alpha_2 + \alpha_3) \quad \text{and} \quad (\alpha_3 + \alpha_2)/(\alpha_1 + \alpha_2 + \alpha_3)$$

$$(7.9) \qquad \alpha_2/(\alpha_1 + \alpha_2 + \alpha_3) \quad \text{and} \quad (\alpha_2 + \alpha_1)/(\alpha_1 + \alpha_2 + \alpha_3)$$

$$(7.10) \qquad \alpha_3/(\alpha_1 + \alpha_2 + \alpha_3) \quad \text{and} \quad (\alpha_3 + \alpha_1)/(\alpha_1 + \alpha_2 + \alpha_3).$$

The qualitative appearance of the FS depends on the particular values assigned to the force constants α_1, α_2, and α_3. We shall consider several interesting special cases.

When $\alpha_1 = \alpha_2 = \alpha_3$ we get a single peak with edges at $f^2 = 1/3$ and $f^2 = 2/3$ as in Figure 17a (and as was first obtained by Bowers and Rosenstock [6]).

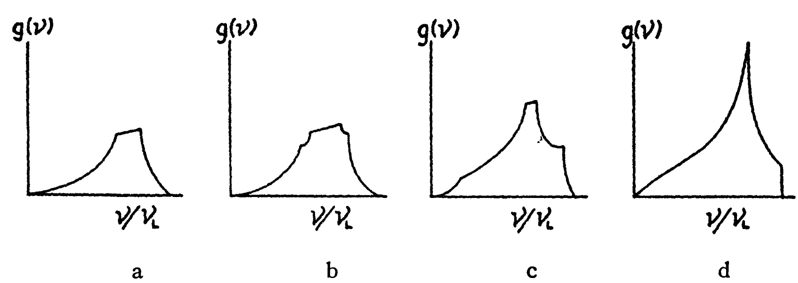


FIG. 17. Variation of frequency spectrum of 3-D lattice with one degree of freedom per lattice point with the reduction of coupling between layers.

If we let $\alpha_3 = \alpha_2 = \alpha$ and $\alpha_1 = \alpha(1+\delta)$ with $|\delta| \ll 1$, the peak edges appear at f^2 equal to $(\frac{1}{3}, \frac{2}{3})$, $(\frac{1}{3}(1+\delta), \frac{1}{3}(2+\delta))$ $(\frac{1}{3}, \frac{1}{3}(2+\delta))$, so that the FS has the form of Figure 17b.

If $\delta = \epsilon - 1$ with ϵ a small positive number the edges are at $(\frac{1}{3}, \frac{2}{3})$; $(\frac{1}{3}\epsilon, \frac{1}{3}(1+\epsilon))$; $(\frac{1}{3}, \frac{1}{3}(1+\epsilon))$. In the limit as $\epsilon \to 0$ the system degenerates into a set of independent layers while the peak of the FS narrows (Fig. 17c) until in the limit it has the 2D logarithmic singularity (17d).

The peak in 17a splits into two when $\delta > 1$, for then $\frac{1}{3}(1+\delta) > \frac{2}{3}$ and the FS has the appearance of Figure 18a.

Let us now consider a model which is more representative of a real crystal; namely, a simple cubic lattice of springs and masses with three degrees of

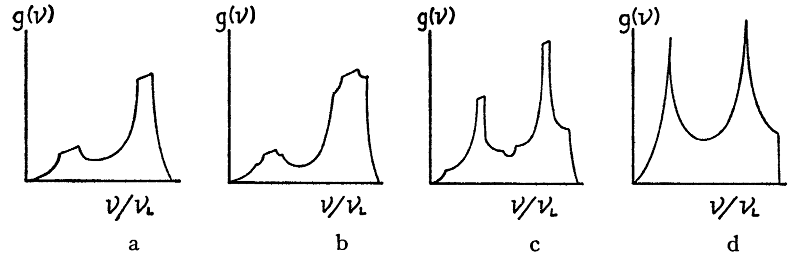


FIG. 18. Variations of frequency spectrum of simple cubic lattice (with interaction between nearest neighbors only) as coupling between layers weakens.

freedom each and with interactions between nearest neighbors only. Then generalizing (4.13) we choose the potential energy function to be

$$\begin{aligned}\Phi = {} & \tfrac{1}{2}\alpha_1 \sum_{l,m,n} (u_{l,m,n} - u_{l+1,m,n})^2 + \tfrac{1}{2}\alpha_2 \sum_{l,m,n} (u_{l,m,n} - u_{l,m+1,n})^2 \\ & + \tfrac{1}{2}\alpha_3 \sum_{l,m,n} (u_{l,m,n} - u_{l,m,n+1})^2 \\ & + \tfrac{1}{2}\beta_1 \sum_{l,m,n} (v_{l,m,n} - v_{l,m+1,n})^2 + \tfrac{1}{2}\beta_2 \sum_{l,m,n} (v_{l,m,n} - v_{l,m,n+1})^2 \\ & + \tfrac{1}{2}\beta_3 \sum_{l,m,n} (v_{l,m,n} - v_{l+1,m,n})^2 \\ & + \tfrac{1}{2}\gamma_1 \sum_{l,m,n} (w_{l,m,n} - w_{l,m,n+1})^2 + \tfrac{1}{2}\gamma_2 \sum_{l,m,n} (w_{l,m,n} - w_{l+1,m,n})^2 \\ & + \tfrac{1}{2}\gamma_3 \sum_{l,m,n} (w_{l,m,n} - w_{l,m+1,n})^2.\end{aligned}$$

The u, v, and w's represent components of displacements in the x, y, and z directions while α_1, β_1, and γ_1 are force constants associated with displacements in the direction of the lines which connect particles. If forces are central all other α, β, and γ's vanish. The introduction of periodic boundary conditions yields the following normal mode frequencies:

$$(7.11)\quad 4\pi^2\nu^2 M = 2(\alpha_1 + \alpha_2 + \alpha_3) - 2\alpha_1 \cos\phi_1 - 2\alpha_2 \cos\phi_2 - 2\alpha_3 \cos\phi_3$$

$$(7.12)\quad 4\pi^2\nu^2 M = 2(\beta_1 + \beta_2 + \beta_3) - 2\beta_3 \cos\phi_1 - 2\beta_1 \cos\phi_2 - 2\beta_2 \cos\phi_3$$

$$(7.13)\quad 4\pi^2\nu^2 M = 2(\gamma_1 + \gamma_2 + \gamma_3) - 2\gamma_2 \cos\phi_1 - 2\gamma_3 \cos\phi_2 - 2\gamma_1 \cos\phi_3;$$

the ϕ's have the usual range. These equations are of the same form as (7.2). Hence the discussion given above is immediately applicable.

First we consider the symmetrical simple cubic lattice with $\alpha_1 = \beta_2 = \gamma_3 = \alpha(1+\delta)$ and $\alpha_2 = \alpha_3 = \beta_1 = \beta_3 = \gamma_1 = \gamma_2 = \alpha$. Since the noncentral character of the forces acting on atoms in a solid is small, we assume $\delta \gg 0$. When $\delta > 1$ two peaks exist as is shown in Figure 18a. Detailed calculations have been made on this

model by Newell and Rosenstock [13]. This type of FS is typical for single cubic lattices with interactions between more distant neighbors. Newell [12] has shown this through several exact numerical calculations.

It is interesting to see how the FS in Figure 18a degenerates into that of a two dimensional lattice as the interactions between layers are made weaker. We let $\alpha_1=\gamma_3=\alpha(1+\delta)$, $\beta_2=\epsilon\alpha(1+\delta)$, $\beta_1=\alpha_2=\beta_3=\gamma_2=\alpha$ and $\gamma_1=\alpha_3=\epsilon\alpha$ and let $\epsilon\rightarrow 0$. Then the peaks become progressively narrower until in the limit they become logarithmic.

If one wishes to find the FS of a real crystal, several courses are available. He should first locate the various singularities and the low frequency behavior by analytical methods. Then either by numerical integrations based on the 3D analogues of (3.4) or (3.6) or by actual calculation and sorting of normal mode frequencies on a high speed computing machine, a FS curve can be sketched which is consistent with the location of singularities. Another scheme is to choose an empirical function which is a linear combination of one with properly located singularities and a polynomial (or the product of these two types of functions) and find the values of the constants which lead to independently calculated moments of the FS. These moments are easily obtained from the characteristic equations which define the normal mode frequencies [16].

References

1. A. Einstein, Ann. Phys., vol. 22, 1907, p. 180.
2. P. Debye, Ann. Phys., vol. 39, 1912, p. 789.
3. M. Blackman, Reports on Progress in Physics, vol. 8, 1942, p. 11.
4. M. Born and von Karman, Phys. Zeit., vol. 13, 1912, p. 297.
5. E. Montroll, J. Chem. Phys., vol. 15, 1947, p. 575.
6. W. Bowers and H. Rosenstock, J. Chem. Phys., vol. 18, 1950, p. 1056.
7. M. Smollett, Proc. of Phys. Soc. (London) A65, 1952, p. 109.
8. J. Holsen and W. Nierenberg, Phys. Rev., vol. 89, 1953, p. 662.
9. H. B. Rosenstock, J. Chem. Pys., vol. 21, 1953, p. 2064.
10. L. van Hove, Phys. Rev., vol. 89, 1953, p. 1189.
11. M. Morse, Trans. Am. Math. Soc., vol. 27, 1925, p. 345; Calculus of Variations in the Large, Colloquium Lectures, Am. Math. Soc., 1934.
12. G. Newell, J. Chem. Phys., vol. 21, 1953, p. 1877.
13. H. B. Rosenstock and G. Newell, J. Chem. Phys., vol. 21, 1953, p. 1607.
14. H. B. Rosenstock and H. M. Rosenstock, J. Chem. Phys., vol. 21, 1953, p. 1608.
15. W. Ledermann, Proc. Roy. Soc., A 182, 1944, p. 362.
16. E. Montroll, J. Chem. Phys., vol. 10, 1942, p. 219; vol. 11, 1943, p. 481.

THE CARUS MONOGRAPHS

•

The Carus Monographs are expository presentations of the best thought and keenest researches in pure and applied mathematics, set forth in a manner comprehensible not only to teachers and students specializing in mathematics, but also to scientific workers in other fields. They are intended especially for the wide circle of thoughtful people who, having a moderate acquaintance with elementary mathematics wish to extend their knowledge without prolonged and critical study of the mathematical journals and treatises.

•

The complete list of Carus Monographs is:

No. 1. Calculus of Variations, by G. A. Bliss. 1925.

No. 2. Analytic Functions of a Complex Variable, by D. R. Curtiss. 1926.

No. 3. Mathematical Statistics, by H. L. Rietz 1927.

No. 4. Projective Geometry, by J. W. Young. 1930.

No. 5. History of Mathematics in America before 1900, by D. E. Smith and Jekuthiel Ginsburg. 1934.

No. 6. Fourier Series and Orthogonal Polynomials, by Dunham Jackson. 1941.

No. 7. Vectors and Matrices, by C. C. MacDuffee. 1943.

No. 8. Rings and Ideals, by N. H. McCoy. 1948.

No. 9. The Theory of Algebraic Numbers, by Harry Pollard. 1950.

No. 10. The Arithmetic Theory of Quadratic Forms, by B. W. Jones. 1950.

The price of each monograph is $1.75 per copy to members of the Mathematical Association, one copy to each member, when ordered directly through the office of the Secretary of the Mathematical Association of America, University of Buffalo, Buffalo 14, New York.

Additional copies and copies for non-members of Monographs 1-8 are priced at $3.00 each, and must be purchased from the Open Court Publishing Co., LaSalle, Illinois.

In the case of Monographs 9 and 10, additional copies and copies for nonmembers must be purchased at $3.00 from John Wiley and Sons, 440 Fourth Ave., New York 16, N.Y.

THE MATHEMATICAL ASSOCIATION OF AMERICA

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

THE AMERICAN
MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 61 NUMBER 8

CONTENTS

OCTOBER 1954

The AMERICAN MATHEMATICAL MONTHLY

(Founded in 1894 by Benjamin F. Finkel)

EDITORIAL CORRESPONDENCE should be addressed to the Editor, C. B. Allendoerfer, Department of Mathematics, University of Washington, Seattle 5, Washington.

ADVERTISING CORRESPONDENCE should be addressed to J. F. Randolph, Morey Hall, University of Rochester, Rochester 3, N. Y.

NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the Monthly should be sent to the Secretary-Treasurer, H. M. Gehman, University of Buffalo, Buffalo 14, N. Y.

THIS IS THE OFFICIAL JOURNAL OF THE

MATHEMATICAL ASSOCIATION OF AMERICA, INC.

(Devoted to the Interests of Collegiate Mathematics)

Entered as second class matter at the post office at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926

Subscription Price: To Members, $4 a Year, To Others, $5 a Year.

Published by the Association at Menasha, Wisconsin, and Seattle, Wash. during the months of January, February, March, April, May, June–July, August–September, October, November, December.

PRINTED IN THE UNITED STATES OF AMERICA

EXTENDED ANALYTIC GEOMETRY AS APPLIED TO SIMULTANEOUS EQUATIONS

R. S. UNDERWOOD, Texas Technological College

1. Preview. We begin with a question which goes to the heart of the matter discussed in this paper. Is it true that analytic geometry, with a suitable coordinate system and the aid of algebra, can readily find solutions of some simultaneous equations which are not solvable within reasonable time limits by known purely algebraic methods?

The answer is, or seems to be, "Yes." In some cases, to be sure, the algebra as guided by geometry still remains involved and tedious; but often drastic simplification is possible. To suggest some accomplishments before dealing with details, we shall lead up to one interesting result by considering first the pair

$$(1)\qquad 4x + 3y + 4z + 3u = 25, \qquad \frac{x^2}{7} + \frac{y^2}{11} + \frac{z^2}{18} + \frac{u^2}{14} = 1.$$

Throughout this paper a *solution* of a set of simultaneous equations will be understood to be a common *real* solution, and the set is *consistent* or not according as a solution exists. With this understanding equations (1) are consistent but just barely so, for the *only* solution is (28/25, 33/25, 72/25, 42/25). It will be apparent that an attack based upon the elimination of one letter would lead promptly to an algebraic mess. The example also illustrates the point that it is not always easy to decide whether two equations in more than two unknowns are consistent, quite aside from the matter of producing in that case a real solution.

While the explanation of the solution of (1) must come later, the point of the method need not wait. It will be shown that in one of the many possible coordinate systems the loci of (1) are respectively a line and a "filled circle" to which the line is tangent. From the point of tangency we get at once the sole solution above. And we could have used n unknowns instead of four as in (1), adding for good measure in the new equations $n-2$ parameters in terms of which the lone answer would appear. The point is that one who attacks such problems with algebraic devices alone is handicapping himself unnecessarily, since in effect he is working in the dark, however cleverly.

For maximum usefulness, of course, the geometric methods should be applicable to more general situations in which, for example, a line or curve is not accidentally tangent to another locus. The following theorem shows one of such applications. For simplicity only four unknowns are used, but the generalization to n variables is both obvious and valid. The proof will be deferred to Article **11**.

THEOREM 1. *The simultaneous equations*

$$Ax + By + Cz + Du = E, \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} + \frac{u^2}{d^2} = 1, \tag{2}$$

are consistent or not according as the line $X+Y=E$ does or does not touch the ellipse $X^2/P^2+Y^2/Q^2=1$, where $P^2=a^2A^2+b^2B^2$ and $Q^2=c^2C^2+d^2D^2$. If the line touches the ellipse at (X, Y) the corresponding solution is $(a^2AX/P^2, b^2BX/P^2, c^2CY/Q^2, d^2DY/Q^2)$, and this will be the only solution if (X, Y) is a point of tangency.

An interesting result is that the first sentence of the theorem is still true if $P^2=a^2A^2$ and $Q^2=b^2B^2+c^2C^2+d^2D^2$, or if the four terms are distributed in any manner between P^2 and Q^2. The coordinates in the second sentence must be adjusted in an obvious manner in each case. If $P^2=a^2A^2+b^2B^2+c^2C^2+d^2D^2$ the ellipse is replaced by the vertical lines $X^2/P^2=1$, and we get the simple algebraic result:

THEOREM 2. *If $E=\pm\sqrt{a^2A^2+b^2B^2+c^2C^2+d^2D^2}$, equations* (2) *of Theorem* 1 *have the sole solution $(a^2A/E, b^2B/E, c^2C/E, d^2D/E)$.*

This gives immediately the previously stated result for equations (1)' though without the possible $n-2$ parameters mentioned below the equations. The obvious generalization to n variables is valid, as for Theorem 1. If we delete the word "sole" Theorem 2 may be verified by direct substitution, and may also be reached by an n-space approach. Theorem 1, though slower to apply, gives in addition two specific solutions when the line crosses the ellipse.

To illustrate another type of result, it will be shown that the loci of the two equations

$$\begin{aligned} \frac{(x-1)^2}{1} + \frac{(y-2)^2}{2} + \frac{(z+1)^2}{4} + \frac{(u-3)^2}{5} &= 1, \\ \frac{(x+1)^2}{3} + \frac{(y+1)^2}{2} + \frac{(z-1)^2}{5} + \frac{(u-1)^2}{6} &= 1 \end{aligned} \tag{3}$$

do not overlap when the axes are chosen properly, and hence the equations are inconsistent. We can reach this conclusion by inspection of (3) by use of an algebraic test for intersection of the "conics" which may be derived from the geometric method.

So much for the general direction and purpose of this article. Since in the development of the basic idea improvements and simplifications have naturally been made, it has seemed advisable to repeat, with appropriate changes, enough introductory material to make this paper self-contained. Additional aspects of the field are dealt with in earlier papers listed at the end of this one.

2. The three essential features. The coordinate system which is here called

"extended analytic geometry" provides a generalization of Descartes' system, both plane and solid, which widens the field to n variables instead of just two or three. Its basic features are (1) a set of n axes on a plane which radiate from a common origin and are designated by small letters, (2) a superimposed pair of orthogonal axes denoted by the capitals X and Y, and (3) a third-dimensional Z-axis through the origin and perpendicular to the n-axes plane.

When the first two axes coincide with the X and Y axes, as is often desirable, the first member of the plane sequence, starting with $n=2$, is that of plane analytic geometry, while the corresponding sequence of solid systems begins with solid analytic geometry.

3. Features of special systems. While the X and Y axes are inviolate, the small-letter axes may be shifted at will, or even on occasion practically dispensed with. Since it turns out that adaptation of the arrangement of axes to the particular problem is a simple and useful device, it is advisable to discuss more than one system. On the other hand, for easier reading it will be understood that, except where otherwise indicated, *System* A *is used throughout this paper.*

System A. Odd-numbered axes coincide with the X-axis, and even-numbered ones with the Y-axis. The coincidence is in direction as well as position, so that positive directions are to the right or upward. (Fig. 1.)

When not more than six axes are involved, they will be designated in order as x, y, z, u, v, and w, so that x, z, v are horizontal coordinates. For the case of n axes the letters in order are x_1, y_1, x_2, $\cdots$, where x_i (or y_i) stands for the i-th horizontal (or vertical) axis, together with its corresponding coordinate. It follows that the fundamental equations of the n-axes plane are

$$X = \sum_1^m x_i; \qquad Y = \sum_1^k y_i; \qquad m + k = n; \qquad k = m \text{ or } k = m - 1. \tag{4}$$

The last condition ($k=m$ or $m-1$) is not essential to the theory and is merely incidental to the scheme of numbering axes in System A. It is worth noting that $n-1$ horizontal axes and one vertical one, for example, may be used to advantage in a particular problem, with only obvious and minor changes in the theorems that follow.

System B. Here the successive positive directions of the axes are "right, up, left, down, right, up, . . . " and so on. (Fig. 5.) The basic equations are

$$X = x_1 - x_2 + \cdots \pm x_m; \qquad Y = y_1 - y_2 + \cdots \pm y_k. \tag{5}$$

Some consequences of special orientations are noted herewith.

(a) (Applies to A, B, and other variations.) The lattice points of plane (and solid) analytic geometry are preserved. This is useful in dealing with Diophantine equations, for example.

(b) (A and others, but not B.) When positive directions are to the right or upward, only points in the first quadrant can have all-positive coordinates.

(c) (All two-dimensional systems. Special systems of one dimension can be

useful, as will be shown.) The n coordinates of a given point have $n-2$ parameters, or, from another point of view, $n-2$ degrees of freedom.

4. Generalized coordinates. From (4) or (5) or the corresponding equations of other systems, we may easily find the *generalized coordinates* (called G.C.) of a fixed point (X, Y). For example, assuming a 5-axes plane, (4) becomes

$$X = x + z + v; \qquad Y = y + u. \tag{6}$$

Therefore the G.C. of (X, Y) are

$$(x, y, z, Y - y, X - x - z), \tag{7}$$

where x, y, and z are arbitrary. This illustrates (c) above when $n=5$. Note that any three, or, in the general case, any $n-2$, of the coordinates could have been the arbitrary ones. Indeed, we shall have use for the special coordinates

$$(X, Y, 0, 0, \cdots) \tag{8}$$

of the point (X, Y), in which the last $n-2$ coordinates are given the value zero. But it will be seen that 2-axes coordinates, such as (2, 3), for example, are preferable to the G.C., or even to $(2, 3, 0, 0, \cdots)$ for describing a particular point on the n-axes plane.

5. Two types of loci. The *locus* of an equation, or the totality of points having coordinates which satisfy the equation, is either *normal* (usually an area) or *degenerate* (usually a curve). But a simplification of previous results [**3**, Articles 5–6] is effected by use of the *revised* definition below.

(9) DEFINITION: *The locus of*

$$f(x_1, y_1, x_2, \cdots) = 0 \tag{10}$$

is degenerate if and only if (10) *can be written in the form*

$$F(X, Y) = 0. \tag{11}$$

For example, in the 3-axes case the locus of

$$x^2 + y^2 + z^2 + 2xz = (x + z)^2 + y^2 = X^2 + Y^2 = 0 \tag{12}$$

is degenerate, while the locus of

$$x^2 + y^2 + z^2 = 0 \tag{13}$$

is normal, though each locus consists of a single point (0, 0). Again, if zero is replaced by -1 in each of (12) and (13), the loci are imaginary but still degenerate and normal respectively.

6. Properties of normal and degenerate loci. Since it is important to distinguish between the separate roles of the two types of loci in dealing with simultaneous equations, we shall state the appropriate theorems.

THEOREM 3. *The locus of the equation* $f(x_1, y_1, x_2, \cdots)=0$ *is degenerate if and only if*

$$\frac{\partial f}{\partial x_i} \equiv \frac{\partial f}{\partial x_j} \quad \text{and} \quad \frac{\partial f}{\partial y_i} \equiv \frac{\partial f}{\partial y_j} \tag{14}$$

for every pair x_i, x_j *and* y_i, y_j *involved.*

For if the locus is degenerate it follows from definition (9) that

$$f = F(X, Y) = F(x_1 + x_2 + \cdots + x_m, y_1 + y_2 + \cdots + y_k). \tag{15}$$

Then

$$\frac{\partial f}{\partial x_i} = \frac{\partial F}{\partial X}\frac{\partial X}{\partial x_i} = \frac{\partial F}{\partial X},$$

which is independent of i. Similarly, so is $\partial f/\partial y_i$. Hence (14) holds. Again, since (15) is a solution of (14) f must be expressible in the form (15) when (14) holds, so that the locus is degenerate by the definition.

By way of an example, the locus of

$$f(x, y, z) \equiv 2x^2 + 3xy - y^2 + 2z^2 + 4xz + 3yz - 1 = 0 \tag{16}$$

is degenerate, since $\partial f/\partial x = 4x+3y+4z = \partial f/\partial z$.

Note that symmetry of the equation with respect to each pair of like coordinates (both horizontal or both vertical) is a necessary but not sufficient condition for degeneracy of the locus. This is illustrated in equations (12) and (13).

THEOREM 4. *All sets of coordinates of a point on a degenerate locus satisfy the corresponding equation.*

For by (9) the equation may be written in the form $F(X, Y)=0$, and hence the G.C. of (X, Y) must satisfy it.

THEOREM 5. *The equation of a degenerate locus is obtained in terms of X and Y merely by replacing the first two variables by X and Y and the remaining ones by zero.*

For since all sets of coordinates of a point on the locus satisfy the equation, the special set $(X, Y, 0, 0, \cdots)$ must do so.

To illustrate, we go back to (16), whose locus was shown by test to be degenerate. The substitution of Theorem 5 yields

$$f(x, y, z) = F(X, Y) = 2X^2 + 3XY - Y^2 - 1 = 0. \tag{17}$$

Theorems 3 and 5 provide the simple technique for "spotting" equations in n variables which have degenerate loci in System A, and then changing them to equations in two variables with their respective loci unchanged.

THEOREM 6. *At least one but not all of the sets of coordinates of a point on a normal locus satisfy the corresponding equation.*

Proof. Without the phrase "but not all," the theorem merely restates the definition of a locus. But if all of the coordinates of (X, Y) satisfy the equation, then the special set $(X, Y, 0, 0, \cdots)$ must do so, and the locus is degenerate rather than normal.

In illustration of Theorem 6, the locus of

$$2x + y + z = 1 \tag{18}$$

is the whole 3-axes plane. But selecting the point (2, 1) at random, we find, upon using the G.C. $(x, 1, 2-x)$, that $(-2, 1, 4)$ are the only coordinates of the point which satisfy (18).

It may be well to point out here that the curves of plane analytic geometry $(n=2)$ are all degenerate according to definition (9), so that Theorem 6 is inapplicable. Theorems 3, 4, and 5 do apply, though trivially.

THEOREM 7. *For each point at which a curve locus touches or crosses an area locus, there is at least one common real solution of the corresponding equations.*

This is a direct corollary of Theorems 4 and 6.

It follows that only the special coordinates of the points described which satisfy the equation of the normal locus need be sought, since these coordinates will surely satisfy the other equation.

THEOREM 8. *If two normal loci have points in common, the corresponding equations may still be inconsistent.*

For example, no two distinct members of the family of equations

$$x + y - z = a \tag{19}$$

have solutions in common, though their common locus is the whole 3-axes plane. But note that if we change to System B, in which $X = x - z$ and $Y = y$, the locus of (19) becomes a family of parallel lines, illustrating the next theorem.

THEOREM 9. *If the loci of two equations have no point in common, the equations have no common real solution.*

For otherwise this solution will locate one and only one point, contradicting the hypothesis.

Obviously the theorem is valid for *any* system of axes which locates one point per set of coordinates (the one system, of course, being used for both equations). We may even, on occasion, [cf. (2)] discard the n axes and let X and Y be single-valued functions of the n coordinates. This wide range of validity of the theorem gives it scope in testing equations suspected of inconsistency. The decision is not always easy—witness (1). But often it is not hard to show that two given equations are consistent. Since in this case the common locus

points cannot be removed by a change of axes, it is often best to proceed at once to the examination of overlapping areas by the methods to be discussed.

7. A general method for normal loci. Since normal loci are usually areas on the n-axes plane, the technique for picturing them must disclose the bounding curves. One generally applicable method is described here, but often there are shortcuts or canonical forms (Article 8) which make it unnecessary.

If U is a function of n variables, measured in the Z-axis direction (*i.e.*, perpendicular to the n-axes plane), then *over the point* (X, Y) U is a function of $n-2$ variables plus the constants X and Y. The finite absolute maximum or minimum of this latter function, when it exists, may often be found by the method of calculus. It will be the Z-value of the surface $Z=F(X, Y)$ which forms the top or bottom of the normally solid U-function locus. Paper [3] dealt primarily with such solids and surfaces, but here we are concerned with loci on the n-axes plane. So, to get the equation of the bounding curves, we merely set $Z=0$.

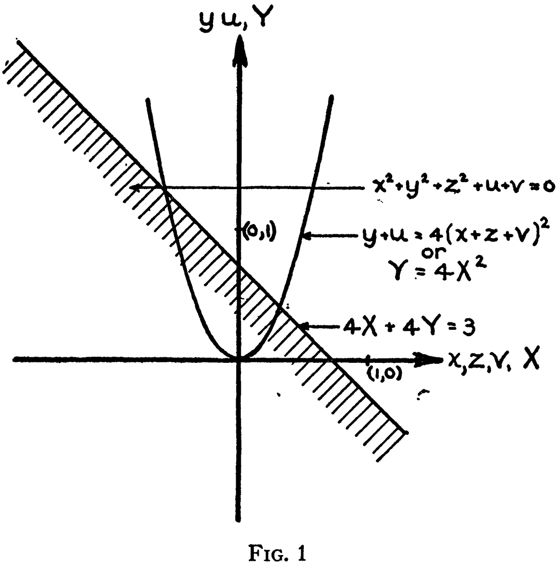


FIG. 1

To illustrate, we shall find the locus on the 5-axes plane of the equation

$$f(x, y, z, u, v) = x^2 + y^2 + z^2 + u + v = 0. \tag{20}$$

By (7) we have

$$\begin{aligned} U &= f(x, y, z, Y - y, X - x - z) \\ &= x^2 + y^2 + z^2 + (Y - y) + (X - x - z). \end{aligned} \tag{21}$$

The necessary conditions for an extremal of U, namely $U_x = U_y = U_z = 0$, yield $x=y=z=\frac{1}{2}$, at which U takes the special value represented by Z in the equation,

$$Z = X + Y - 3/4. \tag{22}$$

The Z of the plane (22) is seen to be the *minimum* value of U over the point (X, Y) either by inspection of (21) or by noting that $U_{xx} > 0$, $U_{yy} > 0$, and $U_{zz} > 0$.

In the final step, with $Z = 0$, we get the line

$$4X + 4Y = 3 \tag{23}$$

plus all the points on one side of it, as the locus of (20). Since the lower-bound plane (22) lies below the origin, the locus of (20) is the half-plane shown in Figure 1.

Some further play with the locus of Figure 1 may be enlightening. Upon substituting $X = Y = 0$ in (21), completing the squares, and equating the result to zero, we get

$$(x - \tfrac{1}{2})^2 + (y - \tfrac{1}{2})^2 + (z - \tfrac{1}{2})^2 = \tfrac{3}{4}, \tag{24}$$

so that obviously the origin has sets of real coordinates which satisfy (20). But using $X = 1$, $Y = 0$, for instance, to get a point on the right of the bounding line, we find that the right side of (24) becomes negative, showing that (1, 0) is not on the locus. Finally, with $Y = (3 - 4X)/4$ in (21), the right side of (24) becomes zero, and the G.C. of a point (X, Y) on the bounding line are the unique coordinates $(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, (1-4X)/4, X-1)$. Hence if we find the equation of a parabola tangent to this line on the right, say

$$16(X - Y)^2 - 88X - 8Y + 61 = 0, \tag{25}$$

for which the point of tangency is $(1, -\tfrac{1}{4})$, we can say without further calculation that the *one and only* common real solution of (20) and

$$16(x - y + z - u + v)^2 - 88(x + z + v) - 8(y + u) + 61 = 0 \tag{26}$$

is the set of coordinates $(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{3}{4}, 0)$. (Try it without the geometry.)

Again, for an application in number theory, let us find common *integral* solutions, if they exist, of (20) and

$$y + u = 4(x + z + v)^2. \tag{27}$$

The locus of (27) is the parabola $Y = 4X^2$. Since this crosses the half-plane, (20) and (27) have infinitely many real solutions by Theorem 7. But common integral solutions must be coordinates of the common lattice point (0, 0); in other words, they must be integers $(x, y, z, -y, -x-z)$ such that the values of x, y, and z satisfy (24). The general method would be to use the lattice points in the locus of (24) on the 3-axes plane, but here it is unnecessary. We simply note that (24) is satisfied in integers when x, y, and z are assigned the values 0 and 1 independently. Thus there are exactly eight integral solutions of (20) and (27), ranging from the obvious (0, 0, 0, 0, 0) to (1, 1, 1, −1, −2).

8. Special methods and canonical forms. When the G.C. of (X, Y) are substituted in an equation we can sometimes see directly that real coordinates of any given point (X, Y) may be found, in which case the locus is the whole plane. For example, given

$$\begin{aligned} 0 &= x^2 + y^2 + z^2 - u^2 - v^2 \\ &= x^2 + y^2 + z^2 - (Y - y)^2 - (X - x - z)^2 \\ &= 2xX + 2yY + 2zX - 2xz - X^2 - Y^2, \end{aligned}$$

we may let $x = z = 0$ when $Y \neq 0$, solving for y, or let $y = z = 0$ when $X \neq 0$, solving for x, or let $x = y = z = 0$ when $X = Y = 0$.

Again, to illustrate another device, consider

$$x^2 + y^2 + z^2 + xy + xz + yz - 1 = 0. \tag{28}$$

Using the G.C. $(x, Y, X - x)$ of (X, Y) in (28) we find that its locus is the ellipse

$$3X^2 + 4XY + 4Y^2 = 4 \tag{29}$$

plus the points inside, since the discriminant of the quadratic in x is negative for points outside.

Probably the most far-reaching of the "inspection" methods for finding loci is the use of canonical forms applying to the general case of n variables. These are illustrated in Article 9.

9. Generalizations of the conic sections. The following theorem deals with the plane aspect (with System A instead of B) of Theorem 6 in [**3**].* It is a generalization to the case of n axes of an ellipse whose center is at the origin. Since the 2-axes formulas for translation of axes carry over easily to the general case, the theorem will enable us to draw readily the loci of all quadratic equations in which each of the n variables appears as a square with a positive coefficient, and no cross-product terms are present, as soon as we have completed the squares and put the equations in the familiar standard form pattern. (An illustration appears near the end of Article 10.)

THEOREM 10. *The locus of the equation*

$$\frac{x_1^2}{a_1} + \frac{y_1^2}{b_1} + \frac{x_2^2}{a_2} + \cdots = 1, \tag{30}$$

where all denominators are positive and the left side of (30) *terminates with the n-th term* $(n \geq 2)$, *is the ellipse*

$$\frac{X^2}{A} + \frac{Y^2}{B} = 1, \qquad A = \sum_1^m a_i, \qquad B = \sum_1^k b_i, \qquad m + k = n, \tag{31}$$

---

* In the proof of the theorem in [**3**], a_i should have appeared in the numerator of the expression for x_i.

plus the points inside when $n>2$. Furthermore, the only coordinates of the point (X, Y) on the bounding ellipse (31) *which satisfy* (30) *are $x_i=a_iX/A$ and $y_i=b_iY/B$.*

Proof. Starting with the equation

$$U = -1 + \sum_{i}^{m} \frac{x_i^2}{a_i} + \sum_{1}^{k} \frac{y_i^2}{b_i}, \tag{32}$$

we get, when the values of x_m and y_k from (4) are used,

$$\begin{aligned} U = &-1 + \sum_{i}^{m-1} \frac{x_i^2}{a_i} + \sum_{1}^{k-1} \frac{y_i^2}{b_i} + \frac{1}{a_m}(X - x_1 - \cdots - x_{m-1})^2 \\ &+ \frac{1}{b_k}(Y - y_1 - \cdots - y_{k-1})^2. \end{aligned} \tag{33}$$

Hence

$$\frac{1}{2} U_{x_i} = \frac{x_i}{a_i} - \frac{1}{a_m}(X - x_1 - \cdots - x_{m-1}) \qquad i = 1, 2, \cdots, m-1, \tag{34}$$

with a similar expression for $\frac{1}{2}U_{y_i}$. For a minimum U (obviously it has no maximum) it is necessary that the $m-1$ equations obtained by setting $U_{x_i}=0$ be solved simultaneously. This is accomplished when $x_i=a_iX/A$. Similarly, $y_i=b_iY/B$ for a minimum U. Substituting in (32) and replacing U by Z we get

$$\begin{aligned} Z &= -1 + \frac{X^2}{A^2}(a_1 + a_2 + \cdots + a_m) + \frac{Y^2}{B^2}(b_1 + b_2 + \cdots + b_k) \\ &= -1 + \frac{X^2}{A} + \frac{Y^2}{B}. \end{aligned} \tag{35}$$

Thus when $Z=0$, (31) is obtained. Inspection of (33) shows that U has a minimum value over each point (X, Y), and this must be the Z of the surface (35) obtained from the necessary conditions for an extremal.

It should be noted that nothing in the proof requires that the number of horizontal and vertical axes differ by zero or one, as they do in System A. For example, consider the locus of

$$\frac{x^2}{1} + \frac{y^2}{2} + \frac{z^2}{3} + \frac{u^2}{4} = 1. \tag{36}$$

In System A it is the filled ellipse $X^2/4+Y^2/6=1$; if $X=x$, $Y=y+z+u$, the corresponding ellipse is $X^2/1+Y^2/9=1$, where 9 is obtained as the sum of 2, 3, and 4; if $X=x+y+z$, $Y=u$, the ellipse is $X^2/6+Y^2/4=1$; and if $X=x+y+z+u$, $Y=0$, the locus is the line segment on the X-axis with endpoints

$X^2/10=1$, or $X=\pm\sqrt{10}$. Again, the locus of

$$\frac{(x-1)^2}{4}+\frac{(y-2)^2}{5}+\frac{(z-3)^2}{6}+\frac{(u+4)^2}{7}=1, \tag{37}$$

for example, is a filled ellipse with center at $(1, 2, 3, -4)$, wherever that point may be. In System A it is the point $(4, -2)$, so that the locus of (37) is then the filled ellipse

$$\frac{(X-4)^2}{10}+\frac{(Y+2)^2}{12}=1. \tag{38}$$

A similar argument, which will be omitted, leads to

THEOREM 11. *The locus of the equation*

$$\sum_1^k y_i=\sum_1^m \frac{x_i^2}{a_i}, \qquad a_i>0, \tag{39}$$

is the parabola

$$Y=\frac{X^2}{A}, \qquad A=\sum_1^m a_i \tag{40}$$

plus the points inside when $m>1$. *The coordinates of the point* (X, Y) *on* (40) *which satisfy* (39) *are:* $x_i=a_iX/A$, *with* y_i *subject only to the condition that* $\sum_1^k y_i=Y$.

The generalizations of the hyperbola are not so simple. When we allow some of the denominators in (30) to be negative and the curve (31) exists, it is sometimes but not always a boundary of the locus. In any case it is a useful test curve, and the locus is found easily by testing points on opposite sides of the curve to see whether the locus extends across it. For example, the locus of

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}-\frac{u^2}{d^2}=1, \qquad a^2>c^2,\ b^2>d^2, \tag{41}$$

is an ellipse plus the points *outside*, the ellipse being

$$\frac{X^2}{a^2-c^2}+\frac{Y^2}{b^2-d^2}=1. \tag{42}$$

To illustrate other possibilities, the locus of

$$2x+3y+4z=\frac{(u-1)^2}{4}+\frac{(v-2)^2}{5} \tag{43}$$

is "by inspection" the filled parabola

$$Y = \frac{(X-3)^2}{9}, \quad \text{if} \tag{44}$$

$$X = u + v; \qquad Y = 2x + 3y + 4z. \tag{45}$$

However, a somewhat complicated transformation like (45) should in general be used for a set of simultaneous equations only when it makes one locus degenerate—always a desirable goal.

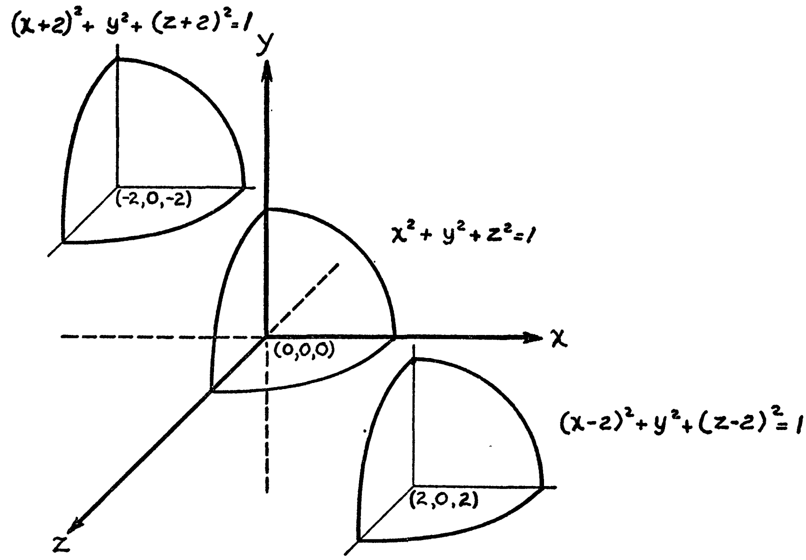


FIG. 2. Three dimensions.

We are now ready to consider equations (1) again, using System A. By Theorems 3 and 5 the first locus is that of the line $4X+3Y=25$, while by Theorem 10 the second locus is the filled circle $X^2/(7+18)+Y^2/(11+14)=1$, or $X^2+Y^2=25$. The line is tangent to the circle at (4, 3), and again by Theorem 10 the only coordinates satisfying both equations are (28/25, 33/25, 72/25, 42/25). This, as we have seen, could also be obtained from Theorem 2. But now more appears. For if the denominators are replaced by a, b, $25-a$ and $25-b$ respectively (all positive), the filled circle locus remains unchanged, and the unique solution in terms of the parameters becomes $(4a/25,\ 3b/25,\ (100-4a)/25,\ (75-3b)/25)$. Evidently n unknowns instead of four could have been used. For another variation of this theme we can say that the equations

$$5x - 9y + 5z - 9u = -7, \qquad \frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{25-a} + \frac{u^2}{25-b} = 1, \tag{46}$$

with a and b arbitrary except that the denominators are positive, have infinitely many real solutions since the line crosses the filled circle, but no integral solutions since the chance for this is lost at the one common lattice point $(4, 3)$.

10. "Parallax" in loci. Consideration of the loci of

$$x^2 + y^2 + z^2 = 1, \tag{47}$$

$$(x - 2)^2 + y^2 + (z - 2)^2 = 1, \quad \text{and} \tag{48}$$

$$(x + 2)^2 + y^2 + (z + 2)^2 = 1 \tag{49}$$

in solid analytic geometry will suggest why a change of axes sometimes serves a useful purpose. In Figure 2 the spherical surfaces of the three-dimensional

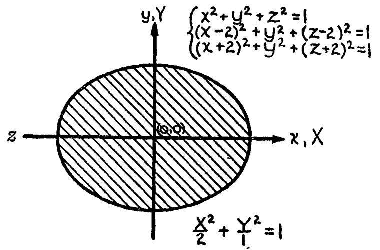


FIG. 3. System B.

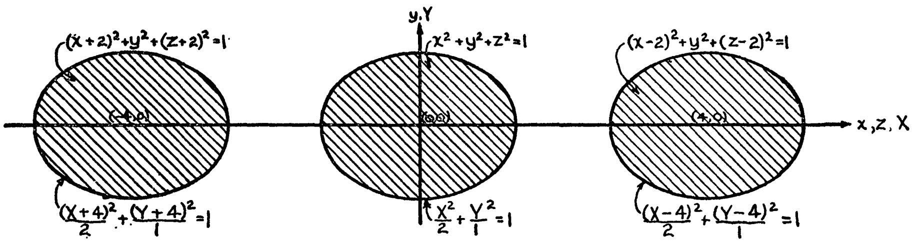


FIG. 4. System A.

system are shown. Now suppose the observer to be located "at infinity" in the direction from O which bisects the plus-x and plus-z axes. The axes would then appear as in Figure 3, and we would have System B, with $X = x - z$. The three centers would merge at the origin and the loci would be coincident filled ellipses, though the equations are inconsistent since the spheres of Figure 2 do not intersect. But if the point of view for Figure 2 were at infinity on the line bisecting the plus-z and minus-x axes, the loci should be separated. This gives us Figure 4, with System A, and the prediction is verified, though the projection boundaries in Figures 3 and 4 are not circles as we might have expected.

Thus the loci of equations on the 3-axes plane may be considered as projections or silhouettes of the surfaces of solid analytic geometry, and knowledge of these surfaces helps us to predict the forms of the loci. The projections of

quadric surfaces will include, for example, curves (from cylindrical surfaces "seen on end"), "windows," (looking endwise through a hyperboloid of one sheet) and in fact all types of closed or open areas which are bounded, if at all, by conic sections. And this, incidentally, completes the list of possible types of quadratic loci on the n-axes plane.

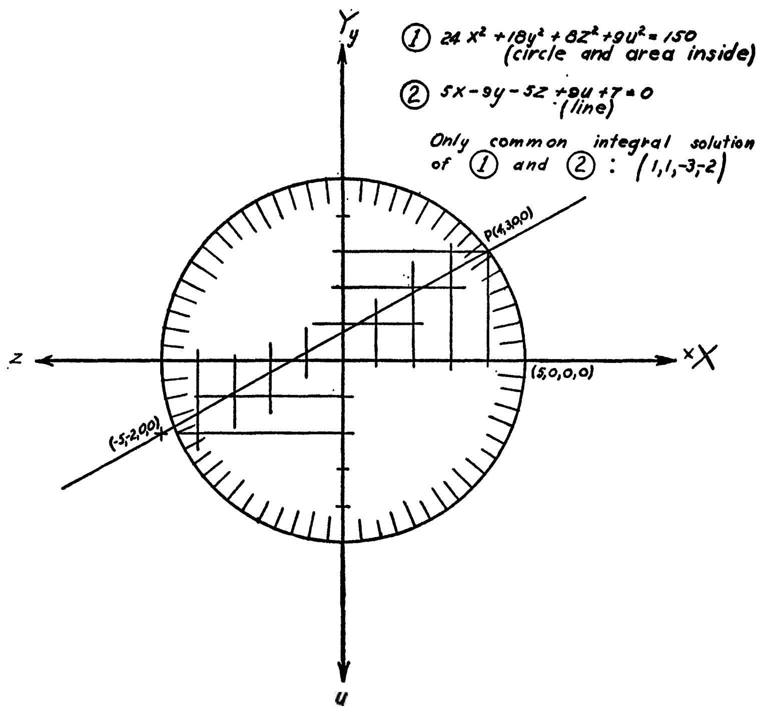


FIG. 5. System B.

In the case of four or more variables to say that "when we change the axes we change the point of view" is of course only a figure of speech, but it is helpful. To illustrate, let us return to equations (3). In System A the two loci, by the discussion preceding (37), are the filled ellipses $X^2/5+(Y-5)^2/7=1$ and $X^2/8+Y^2/8=1$, which certainly overlap. But when we observe the second terms, namely $(y-2)^2/2$ and $(y+1)^2/2$, we note that if

$$(50) \qquad X = x + z + u; \qquad Y = y,$$

the vertical distance 3 between the centers is more than the sum of the semi-minor axes, and the loci cannot overlap. The equations of the boundaries (Fig. 7) are then

$$(51) \qquad \frac{(X-3)^2}{10} + \frac{(Y-2)^2}{2} = 1, \qquad \frac{(X-1)^2}{14} + \frac{(Y+1)^2}{2} = 1.$$

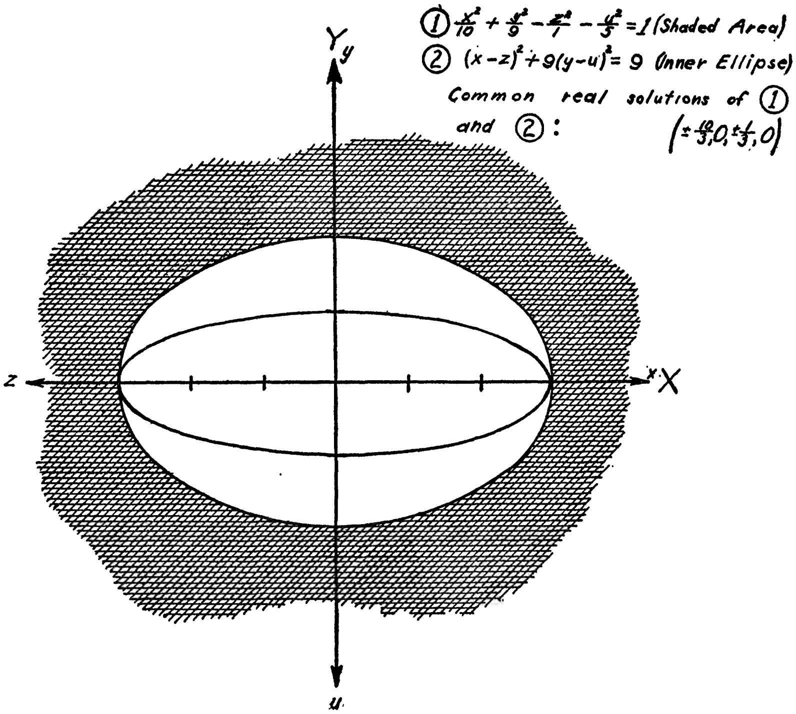


FIG. 6. System B.

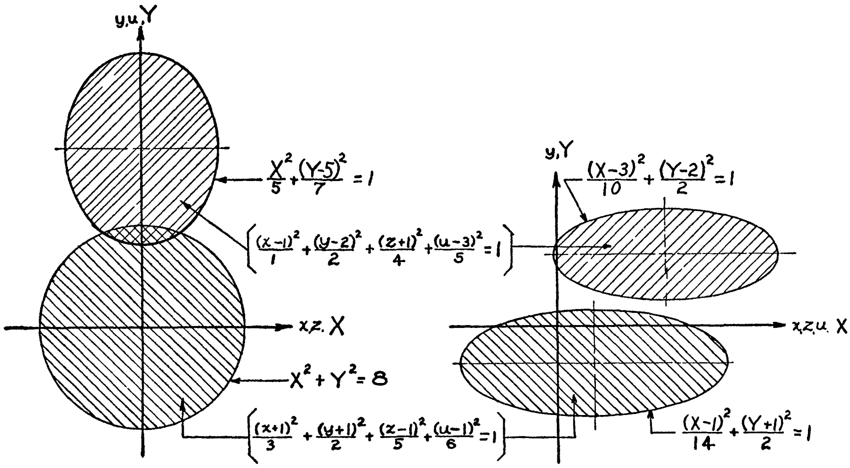


FIG. 7

This illustration points to a generalization, thus:

THEOREM 12. *Necessary conditions for the consistency of*

$$\sum \frac{(x_i - a_i)^2}{c_i^2} = 1, \qquad \sum \frac{(x_i - b_i)^2}{d_i^2} = 1, \qquad c_i > 0,\ d_i > 0 \tag{52}$$

are: (a) $|a_i - b_i| \leqq c_i + d_i$ *for each* i, *and* (b) $|\sum a_i - \sum b_i| \leqq \sqrt{\sum c_i^2} + \sqrt{\sum d_i^2}$ *where all summations are from* $i = 1$ *to* $i = n$.

The second necessary condition appears if we let

$$X = \sum_1^n x_i; \qquad Y = 0, \tag{53}$$

so that the loci are line segments on the X-axis with centers respectively at $(\sum_1^n a_i,\ 0)$ and $(\sum_1^n b_i,\ 0)$.

An application appears even in plane analytic geometry. The ellipses

$$\frac{(x+1)^2}{9} + \frac{y^2}{4} = 1, \qquad \frac{(x-3)^2}{5} + \frac{(y-3)^2}{5} = 1 \tag{54}$$

do not touch by test (b), since $|(-1+0)-(3+3)| > \sqrt{13} + \sqrt{10}$, though the decision is close and tediously attained by the usual methods.

11. A complete solution. We now return to equations (2). The reader will note that the axes-systems thus far discussed will translate an equation of the first degree into a plane usually, and into a line only exceptionally and for special coefficients. The locus of the second equation in (2) is stubbornly a filled ellipse in almost any set-up. But if we let $X = Ax + Cz$ and $Y = By + Du$ (or, in the general case, $X = \sum_1^m A_i x_i$, $Y = \sum_1^k B_i y_i$) we get the line $X + Y = E$, and our general method for finding the other locus still applies. There is still one point (X, Y) to correspond with any given set of coordinates. The routine method gives the result for (2) as already stated, and the details of the proof for the generalization to n variables are similar to those in the proof of Theorem 10.

More generally, we have here a method applying to linear-and-quadratic pairs in n variables. We make the locus of the linear equation a straight line, and then find the other locus as it is constrained to be under the required transformation. The line touches or does not touch the other locus according as the equations are consistent or not. Various extensions of the method are possible.

12. Concluding remarks and summary. It is evident that simultaneous solutions of two or more equations in n unknowns are greatly facilitated if at least one locus is degenerate, so that we may deal with curves crossing areas or each other. Whenever we can gain this end by adjusting the transformation to one of the equations, we can afford to discard canonical forms and let the other

loci be what they may be as found by the general method. For example, consider

$$U \equiv 4x^2 + y^2 + z^2 + u^2 - 4xy + 4xz - 2yz - 1 = 0. \tag{55}$$

Having in mind Theorem 3, with similar extensions to axes-systems other than A, we note first that

$$\begin{aligned} U_x &= 8x - 4y + 4z = 4(2x - y + z); \\ U_y &= 2y - 4x - 2z = -2(2x - y + z); \\ U_z &= 2z + 4x - 2y = 2(2x - y + z); \\ U_u &= 2u. \end{aligned}$$

This method suggests, as it may in less obvious cases, a more useful version of the first equation. We see that (55) can be written thus:

$$(2x - y + z)^2 + u^2 = 1. \tag{56}$$

An efficient transformation appears at once, namely:

$$X = 2x - y + z; \qquad Y = u. \tag{57}$$

In the equations of transformation X and Y may advantageously share some small letters. For example, given

$$(3x + 2y - 4z)^2 + (3x + 2y - 4z)(2x - y + z) = 1, \tag{58}$$

the locus is obviously a hyperbola if we let

$$X = 3x + 2y - 4z, \qquad Y = 2x - y + z. \tag{59}$$

The G.C. of (X, Y) are then $(x, (11x - X - 4Y)/2, (7x - X - 2Y)/2)$. No essential change in the general method is necessary.

Let us consider briefly the "worst possible" case, involving n consistent equations in n unknowns, for which the loci in all convenient systems overlap in a common area. Suppose that we then use System A. By eliminating the $n-2$ small letters from $n-1$ of the equations, leaving only X and Y, we get a "curve of intersection." A second curve is obtained from $n-1$ equations including the previously neglected one, and the two equations in X and Y are solved simultaneously. When the G.C. of a point of intersection are substituted in the original equations two of the latter will be dependent on the others, and then the process is repeated with fewer equations and unknowns. This of course is merely a variation of the usual (often hopeless) algebraic approach; but the geometric leads may suggest shortcuts at any point.

Again, if the common locus area is finite, all integral solutions may be found by a systematic examination of lattice points.

We have seen (Theorem 1) that if one equation is linear its locus may be made a line. If all equations are linear various simplifications have been suggested in previous papers. If two of the equations are inconsistent, this shows

up quickly in many cases (Theorem 12).

In some situations algebra must still do the main job. For example, when one "n-variable ellipsoid" is completely "inside" another, the positions of the plane loci will suggest this, but the simplest proof of inconsistency is likely to be found in the use of inequalities.

To summarize, whereas in the case of simultaneous quadratics in n unknowns the algebra of the solution is always adequate when $n=2$, so that the geometric illustrations found in most algebra texts are incidental and unnecessary, though helpful as checks, the situation is practically reversed when $n>2$. Then geometry takes over the dominant role in many cases, while algebra hangs on merely as the necessary clerical help.

References

Earlier Papers by the Author on Extended Analytic Geometry in this MONTHLY

1. An analytic geometry for N variables, vol. 52, 1945, pp. 253–262.
2. Some applications of extended analytic geometry, vol. 56, 1949, pp. 158–164.
3. Functions of N variables in extended analytic geometry, vol. 59, 1952, pp. 453–460.

THE WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

L. E. BUSH, Kent State University

The following results of the fourteenth William Lowell Putnam Mathematical Competition held on March 6, 1954, have been determined in accordance with the constitution of the Competition. This Competition is supported by the William Lowell Putnam Intercollegiate Memorial Fund left by Mrs. Putnam in memory of her husband and is held under the auspices of the Mathematical Association of America.

The first prize, four hundred dollars, is awarded to the Department of Mathematics of Cornell University, Ithaca, New York. The members of the team were Leonard Evens, D. J. Kleitman and Steven Weinberg; to each of these a prize of forty dollars is awarded.

The second prize, three hundred dollars, is awarded to the Department of Mathematics of Harvard University, Cambridge, Massachusetts. The members of the team were David Lewin, Benjamin Muckenhaupt and Kenneth Wilson; to each of these a prize of thirty dollars is awarded.

The third prize, two hundred dollars, is awarded to the Department of Mathematics of Massachusetts Institute of Technology, Cambridge, Massachusetts. The members of the team were George H. Borrmann, Jr., Max A. Plager, and Kenneth E. Ralston; to each of these a prize of twenty dollars is awarded.

The fourth prize, one hundred dollars, is awarded to the Department of Mathematics of the University of Toronto, Toronto, Ontario. The members of the team were Marcus T. Grisaru, William M. Kahan and Charles B. H. Watson; to each of these a prize of ten dollars is awarded.

The five persons ranking highest in the examination, named in alphabetical order, are James D. Bjorken, Massachusetts Institute of Technology; Leonard Evens, Cornell University; William P. Hanf, University of California (Berkeley); Benjamin Muckenhaupt, Harvard University; and Kenneth Wilson, Harvard University. Each of these will receive a prize of fifty dollars.

The five succeeding persons ranking highest in the examination, named in alphabetical order, are Benito Franqui, University of Puerto Rico; Angus C. Kerr-Lawson, University of Toronto; J. S. Lew, Yale University; David Mumford, Harvard University; and Jack Towber, Brooklyn College. Each of these will receive a prize of twenty dollars.

The following teams, named in alphabetical order, won honorable mention: California Institute of Technology, Pasadena, California, the members of the team being George A. Baker, Jr., David G. Cantor, and Robert D. Ryan; Columbia College, New York, New York, the members of the team being Lee Abramson, Elihu Lubkin, and Richard Wasserman; Kenyon College, Gambier, Ohio, the members of the team being Trevor Barker, Robert G. Busacker, and David Ryeburn; Swarthmore College, Swarthmore, Pennsylvania, the members of the team being John Hopfield, Paul Monsky, and Lisa Steiner.

Twelve individuals were given honorable mention. The names are listed in alphabetical order: George A. Baker, Jr., California Institute of Technology; Trevor Barker, Kenyon College; Edward Curtis, Harvard University; John E. Derwent, University of Notre Dame; Donald Fredkin, New York University; Sidney Kahana, University of Manitoba; William M. Kahan, University of Toronto; Elihu Lubkin, Columbia College; Paul Monsky, Swarthmore College; Kenneth Ralston, Massachusetts Institute of Technology; Steven Weinberg, Cornell University; and Harold Weitzner, University of California (Berkeley).

The following is a list of all colleges and universities which entered teams in the Competition. The list, in alphabetical order, is: Agricultural and Mechanical College of Texas, Arizona State College (Flagstaff), Arizona State College (Tempe), Brooklyn College, Brown University, California Institute of Technology, Carleton College, Carnegie Institute of Technology, College of the Holy Cross, College of Saint Catherine, Columbia College (New York), Cornell University, Doane College, Florida Agricultural and Mechanical University, Harvard University, Iowa State College, Kent State University, Kenyon College, Knox College, Lebanon Valley College, Massachusetts Institute of Technology, McGill University, McMaster University, Queen's College (Flushing, N. Y.), Queen's University (Kingston, Ontario), Stanford University, Swarthmore College, Syracuse University, Tennessee Agricultural and Industrial State College, Texas College (Tyler), The College of the City of New York, The

Cooper Union, United States Naval Academy, Université de Montréal, University of British Columbia, University of California, University of Georgia, University of Manitoba, University of Minnesota, University of Notre Dame, University of Oregon, University of Puerto Rico, University of Rochester, University of the South, University of Toronto, University of Washington, Ursinus College, Yale University.

The following additional colleges and universities entered individual contestants only: Bethel College, Florida Southern College, Huron College, Nebraska State Teachers College, New York University, North Georgia College, Oberlin College, Purdue University, Regis College, Saint Joseph College, Saint Olaf College, University of California (Berkeley), University of Colorado, University of Kentucky, University of Michigan, University of New Mexico, University of Oklahoma, Washington University, Western Washington College of Education.

A total of 231 undergraduates representing 67 institutions took part in the Competition.

The departments of mathematics of any of the competing institutions may obtain the rankings of their individual contestants (except that the relative rankings of the first five will not be divulged) by writing to the Director of the Competition, Room 301 Merrill Hall, Kent State University, Kent, Ohio. These rankings may now be given to the individual contestants by their own departments of mathematics. Any other departments of mathematics may obtain the individual rankings of contestants for the purpose of selecting graduate students.

Participants in the Competition were given the following lists of problems:

Part I

March 6, 1954

MORNING SESSION: 9:00 A.M. TO 12:00 NOON

Answer the questions in any order and by any method. Show all of your work in logical sequence and indicate all answers clearly. No tables or other books may be used. Use the right hand pages of your examination booklet for your solutions, use the left hand pages for scratch work. Cross out any work which you do not wish to have considered. Partial credit may be given on a question, even when the solution is not completed.

Omit one question. You must indicate which question is omitted.

1. Let n be an odd integer greater than 1. Let A be an n by n symmetric matrix such that each row and each column of A consists of some permutation of the integers $1, \cdots, n$. Show that each one of the integers $1, \cdots, n$ must appear in the main diagonal of A.

2. Consider any five points P_1, P_2, P_3, P_4, P_5 in the interior of a square S of side-length 1. Denote by d_{ij} the distance between the points P_i and P_j. Prove that at least one of the distances d_{ij} is less than $\sqrt{2}/2$. Can $\sqrt{2}/2$ be replaced by a smaller number in this statement?

3. Prove that if the family of integral curves of the differential equation

$$\frac{dy}{dx} + p(x)y = q(x) \qquad p(x)\cdot q(x) \neq 0$$

is cut by the line $x=k$, the tangents at the points of intersection are concurrent.

4. A uniform rod of length $2k$ and weight w rests with the end A against a smooth vertical wall, while to the lower end B is fastened a string BC of length $2b$ coming from a point C in the wall directly above A. If the system is in equilibrium, determine the angle ABC.

5. If $f(x)$ is a real-valued function defined for $0<x<1$, then the formula $f(x) = o(x)$ is an abbreviation for the statement that

$$\frac{f(x)}{x} \rightarrow 0 \qquad \text{as } x \rightarrow 0.$$

Keeping this in mind, prove the following: if

$$\lim_{x\to 0} f(x) = 0 \quad \text{and} \quad f(x) - f\left(\frac{x}{2}\right) = o(x),$$

then $f(x) = o(x)$.

6. Suppose that $u_0, u_1, u_2 \cdots$ is a sequence of real numbers such that

$$u_n = \sum_{k=1}^{\infty} u_{n+k}^2 \qquad \text{for } n = 0, 1, 2, \cdots.$$

Prove that if $\sum u_n$ converges then $u_k = 0$ for all k.

7. Prove that there are no integers x and y for which

$$x^2 + 3xy - 2y^2 = 122.$$

Part II

AFTERNOON SESSION: 2:00 TO 5:00 P.M.

Answer the questions in any order and by any method. Show all of your work in logical sequence and indicate all answers clearly. No tables or other books may be used. Use the right hand pages of your examination booklet for your solutions, use the left hand pages for scratch work. Cross out any work which you do not wish to have considered. Partial credit may be given on a question, even when the solution is not completed.

Omit one question. You must indicate which question is omitted.

1. Show that the equation $x^2 - y^2 = a^3$ has always integral solutions for x and y whenever a is a positive integer.

2. Assume as known the (true) fact that the alternating harmonic series

(1) $1 - 1/2 + 1/3 - 1/4 + 1/5 - 1/6 + 1/7 - 1/8 + \cdots$

is convergent, and denote its sum by s. Rearrange the series (1) as follows:

(2) $1 + 1/3 - 1/2 + 1/5 + 1/7 - 1/4 + 1/9 + 1/11 - 1/6 + \cdots$.

Assume as known the (true) fact that the series (2) is also convergent, and denote its sum by S. Denote by s_k, S_k the kth partial sum of the series (1) and (2) respectively. Prove the following statements.

(a) $$S_{3n} = s_{4n} + \frac{1}{2} s_{2n}.$$

(b) $$S \neq s.$$

3. Let a and b denote real numbers such that $a<b$. The symbol (a, b) will denote the closed interval with the end points a, b. Let there be given a collection of closed intervals $(a_1, b_1), \cdots, (a_n, b_n)$ such that any two of these closed intervals have at least one point in common. Prove that there exists then a point which is contained in every one of these intervals.

4. Given the focus f and the directrix D of a parabola P and a line L, describe (with proof) a Euclidean (*i.e.* ruler and compass) construction of the point or points of intersection of L and P. Be sure to identify the case for which there are no points of intersection.

5. Let $f(x)$ be a real-valued function, defined for $-1<x<1$, such that $f'(0)$ exists. Let a_n, b_n be two sequences such that

$$-1 < a_n < 0 < b_n < 1, \qquad \lim_{n\to\infty} a_n = 0, \qquad \lim_{n\to\infty} b_n = 0.$$

Prove that

$$\lim_{n\to\infty} \frac{f(b_n) - f(a_n)}{b_n - a_n} = f'(0).$$

6. Prove that every positive rational number is the sum of a finite number of distinct terms of the series

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots .$$

7. Show that

$$\lim_{n\to\infty} \sum_{s=1}^{n} \left(\frac{a+s}{n}\right)^n \qquad (a > 0)$$

lies between e^a and e^{a+1}.

Solutions of the Problems*

The following solutions are not taken from any of the contestants' papers, but generally embody ideas used by many contestants. The presentation here is intended as a brief sketch of the method of proof rather than as a model of a detailed proof such as is expected from the contestants.

Part I

1. Any integer j such that $1 \leqq j \leqq n$ occurs just once in each row and hence occurs an odd number (n) times in the matrix. By the symmetry of the matrix the number of off-diagonal occurrences of j is even and hence j must occur on the diagonal.

2. Divide the open square S into four congruent half-open squares. Two points, say P_1, P_2, must belong to one of the half-open squares. The components of the vector $\overline{P_1P_2}$ parallel to the sides of the square S are then each numerically less than $1/2$, and thus the length of the vector $\overline{P_1P_2}$ is less than $1/\sqrt{2}$.

Since P_5 may be taken at the center of S and P_1, P_2, P_3, P_4 placed arbitrarily close to the four vertices of S it follows that the theorem is false if a smaller number is substituted for $1/\sqrt{2}$.

3. The equation of the tangent line to the integral curve at the point (k, c) is $y-c=[q(k)-cp(k)](x-k)$ and this equation is satisfied by the point, $(k+1/p(k), q(k)/p(k))$ which does not depend on c.

4. Let F, T, and R be vectors representing the forces acting on the rod due to gravity, the tension of the string, and the reaction of the wall. By the principle of moments these vectors must be concurrent for equilibrium to obtain. If B is not on the wall the point D where these forces concur is the midpoint of BC. Hence $4k^2+4b^2-8bk \cos \theta=(AC)^2=b^2-(AD)^2=b^2-(4k^2+b^2-4bk \cos \theta)$ and so $\cos \theta=2k/3b+b/3k$. If $b<k$ or $b>2k$, this is impossible, and then B is on the wall and $\theta=0$.

5. Given $\omega>0$ there is a $\phi>0$ such that $|f(x)-f(x/2)|<\omega|x|$ for $0<|x|<\phi$. Then for $0<|x|<\phi$ we have $|f(x)-f(x/2^n)| \leqq \sum_{j=0}^{n-1} |f(x/2^j)-f(x/2^{j+1})| < \sum_{j=0}^{n-1} \omega|x|/2^{j+1}<\omega|x|$ for every positive integer n. Since n is arbitrary we have $|f(x)| \leqq \omega|x|$ for $0<|x|<\phi$.

6. Clearly $0 \leqq u_{n+1} \leqq u_n$, $n=1, 2, \cdots$. If $\sum u_j$ converges we may take $k \geqq 1$ so that $\sum_{k+1}^{\infty} u_j<1$. Then $u_{k+1} \leqq u_k=\sum_{k+1}^{\infty} u_j^2 \leqq u_{k+1} \sum_{k+1}^{\infty} u_j \leqq u_{k+1}$. Hence $u_k=u_{k+1}$ and so $u_{k+1}=0$, implying $u_j=0$ for $j>k$ since $u_{k+1}=\sum_{k+2}^{\infty} u_j^2$. By finite induction $u_j=0$ for $j<k+1$.

7. If x and y are integers then $17y^2+488=k^2$ for some integer k. Hence

* These solutions are published solely for the information of interested persons. Neither the editor, nor the director of the competition, nor the paper grader will enter into any correspondence concerning them.

$k^2 \equiv 488 \equiv 12 \pmod{17}$. By examination of a complete residue class, or by the theory of quadratic residues, this is shown to be impossible.

Part II

1. Putting $x+y=a^2$ and $x-y=a$ we have the solutions $x=(a^2+a)/2$, $y=(a^2-a)/2$ which are clearly integers when a is an integer.

2. Evidently $S_3=s_4+s_2/2$. Since

$$S_{3n} = \sum_{j=1}^{2n} \frac{1}{2j-1} - \sum_{j=1}^{n} \frac{1}{2j}$$

then, by induction,

$$\begin{aligned} S_{3n+3} &= S_{3n} + \frac{1}{4n+1} + \frac{1}{4n+3} - \frac{1}{2n+2} \\ &= s_{4n} + s_{2n}/2 + \frac{1}{4n+1} - \frac{1}{4n+2} + \frac{1}{4n+3} - \frac{1}{4n+4} + \frac{1}{4n+2} - \frac{1}{4n+4} \\ &= s_{4n+4} + s_{2n+2}/2. \end{aligned}$$

Since the series are given convergent $S=3s/2$. Since

$$s = \sum_{n=1}^{\infty}\left(\frac{1}{2n-1} - \frac{1}{2n}\right) > 0$$

it follows that $S \neq s$.

3. Since (a_j, b_j) and (a_k, b_k) have a point in common we have $a_j \leqq b_k$, $j, k = 1, 2, \cdots, n$. Thus $m=\max [a_1, \cdots, a_n]$ satisfies $a_j \leqq m \leqq b_j$ for $j=1, \cdots, n$. The property also holds for an infinite collection of intervals if we take $m = \sup (a_j)$.

4. Let p be any point on L but not on D. Construct a circle with center p and tangent to D at m. Construct line F through f and the intersection of L and D (if L and D are parallel take F parallel to D). Let the above circle intersect F in r, s. Take points t, u on L so that ft is parallel to ps and fu is parallel to pr. Then t and u are the required points. If r and s coincide then t and u coincide and the parabola intersects L in just the point t. If the circle does not cut F the parabola does not cut L. The proof follows by consideration of similar triangles.

5. Let $r+s=1$, $r>0$, $s>0$, and $h \leqq k$ be real numbers. Then $h \leqq rh+sk \leqq k$. That is, a weighted mean of two real numbers lies between the numbers. Hence

$$\frac{f(b_n)-f(a_n)}{b_n-a_n} = \frac{f(b_n)-f(0)}{b_n} \cdot \frac{b_n}{b_n-a_n} + \frac{f(0)-f(a_n)}{0-a_n} \cdot \frac{-a_n}{b_n-a_n}$$

and so $(f(b_n)-f(a_n))/(b_n-a_n)$ lies between

$$(f(b_n) - f(0))/b_n \text{ and } (f(0) - f(a_n))/(0 - a_n).$$

Since the latter two fractions approach $f'(0)$ as $n\to\infty$ so then must the fraction

$$(f(b_n)-f(a_n))/(b_n-a_n).$$

6. Let A and B be positive integers. Then by the divergence of the harmonic series there is a unique non-negative integer n_0 such that $\sum_{j=1}^{n_0} 1/j<A/B \leqq \sum_{j=0}^{n_0+1} 1/j$. (The $\sum_{j=1}^{0}$ is taken as 0.) If equality holds the desired representation is at hand and so we assume $A/B<\sum_{j=0}^{n_0+1} 1/j$. Then $A/B-\sum_{j=1}^{n_0} 1/j =C/D<1/(n_0+1)$. Take n_1 as the unique positive integer such that $1/(n_1+1) \leqq C/D<1/n_1$. We again suppose inequality as the problem is otherwise solved, and put $C/D-1/(n_1+1)=E/F>0$. But $E/F=(C(n_1+1)-D)/D(n_1+1)$ and $C(n_1+1)-D<C$ so that with E/F in lowest terms we must have $E<C$. Then $E/F<1/n_1(n_1+1)$ and so the unique integer n_2 such that $1/(n_2+1)\leqq E/F <1/n_2$ satisfies $n_2>n_1$. In a finite number of steps we must obtain the desired representation, since if the equality does not occur before, it must occur when the numerator of the reduced fraction has become 1.

7. For $|x|<n$, $(1+x/n)^n\leqq e^x$. Hence for $n>a$ we have

$$\sum_{s=1}^{n}\left(\frac{a+s}{n}\right)^n=\sum_{r=0}^{n-1}\left(1+\frac{a-r}{n}\right)^n<\sum_{r=0}^{\infty}e^{a-r}=\frac{e^{a+1}}{e-1}$$

and so

$$\overline{\lim}\sum_{s=1}^{n}\left(\frac{a+s}{n}\right)^n\leqq\frac{e^{a+1}}{e-1}\cdot$$

For any positive integer k and $n>a+k$

$$\sum_{s=1}^{n}\left(\frac{a+s}{n}\right)^n\geqq\sum_{r=0}^{k}\left(1+\frac{a-r}{n}\right)^n$$

so that

$$\underline{\lim}\sum_{s=1}^{n}\left(\frac{a+s}{n}\right)^n\geqq\sum_{r=0}^{k}e^{a-r}.$$

Since this is true for all positive integers k we must have

$$\underline{\lim}\sum_{s=1}^{n}\left(\frac{a+s}{n}\right)^n\geqq\sum_{r=0}^{\infty}e^{a-r}=\frac{e^{a+1}}{e-1}\cdot$$

Since

$$e^a<\frac{e^{a+1}}{e-1}<e^{a+1}$$

the proof is complete.

MATHEMATICS FOR SOCIAL SCIENTISTS*

R. R. BUSH, W. G. MADOW, HOWARD RAIFFA, R. M. THRALL†

1. Introduction. (Madow). The 1953 Summer Institute of Mathematics for Social Scientists met for eight weeks in the summer of 1953 under the sponsorship of the Social Science Research Council with the help of a grant from the Behavioral Sciences Division of the Ford Foundation.

Approximately twenty hours weekly were devoted to lectures by the four members of the faculty. In addition, ten hours weekly were devoted to supervised study and supplementary lectures.‡ Finally, eight visitors gave eighteen lectures on applications of mathematics in the social sciences that were not discussed in the regular lectures. The Institute was neither a conference nor a discussion group. It was simply a stiff summer session in which topics in mathematics and applications of mathematics in the social sciences were taught intensively.

Although the nature of the student body and the intensity of the program keep the Institute from being considered to be a model for an undergraduate mathematics curriculum, yet the subject matter and level of the curriculum of the Institute were such that we believe the Institute to be a suitable subject for discussion here.

Of late years considerable dissatisfaction has developed with the present fairly standardized first two years undergraduate mathematics curriculum. The reasons for this dissatisfaction are well known and need no repetition. The dissatisfaction is felt not only by social scientists but by others; perhaps most of all, by mathematicians. The Mathematical Association of America has at least one committee working in this area.

We feel that a two year undergraduate curriculum based largely upon the subject matter of the 1953 Summer Institute can be successfully taught not only to majors in social sciences but also, after minor revision, to all students except engineers and physical scientists. Actually, this curriculum might even meet the needs of engineers and physical scientists, but we think it would be practical first to organize this curriculum as an alternative to, rather than replacement of, the present curriculum.

For admittance to the Institute, students were required to be social scientists and to have had at least one semester of college mathematics or its equivalent in independent study. Over 250 persons applied for admission. Thirty persons were admitted and given grants. Twenty-six persons were admitted without

* This is a report of a round table discussion that occurred at the meeting of the Mathematical Association of America at Johns Hopkins University, December 31, 1953.

† The faculty of the 1953 Summer Institute consisted of W. G. Madow, University of Illinois, director, R. R. Bush, Harvard University, Howard Raiffa, Columbia University, and R. M. Thrall, University of Michigan, assisted by R. L. Davis, University of Michigan and R. E. Priest, University of Illinois.

‡ These lectures were given by Davis and Priest.

grants. Of the 56 persons thus admitted 41 attended the Institute.

Twenty-two of the students had received the Ph.D. degree, 12 of them having academic rank of assistant professor or higher. The other 19 were graduate students in a social science.

Twenty-four of the students were between 22 and 30 years of age; the remaining 17 were between 31 and 42 years of age.

The students came from 23 institutions in 13 states. There were 20 psychologists, 11 sociologists, 7 economists, and 3 political scientists.

Twenty-two of the students had studied some calculus. Nineteen had not. But the training in calculus was often weak and had not been used for some years.

The students were highly selected not only by the fact they were admitted to the Institute but also by the fact that they applied. They were very highly motivated and were mature responsible people who had given evidence of achievements in their own fields.

It is still early to state how successful we were. Of the 41, only two did really poorly and only five did fairly poorly. We were well satisfied with the others. But it is only as our students begin using their work of this summer that we will begin to be able to evaluate this summer's work.

No summer institutes are planned for 1954, but we expect to have two Summer Institutes of Mathematics for Social Scientists in the summer of 1955. At each of these, there will be two curricula, one for those who have studied calculus and one for those who have studied college algebra but not the calculus. It is hoped that detailed announcements will be available in the fall of 1954.

In Sections 2, 3, and 4, Bush, Thrall and Raiffa discuss the following five points:

1. Social scientists as students of mathematics
2. Objectives of the Institute
3. Some considerations in selecting the mathematics to be taught to social scientists
4. The mathematics curriculum of the 1953 Institute
5. Some aspects of teaching mathematics to social scientists

In Section 5 will be found comments.

2. The development of mathematics courses for social scientists. (Bush). I will discuss two central problems that arise in developing mathematics programs for social scientists. First I will comment on the *kinds* of mathematics which I think should be included in such a program, and second, I will express my views on how such mathematics should be taught.

In planning a mathematics curriculum for social scientists, one can adopt many different criteria for including or excluding any particular topic in mathematics. At one extreme, one might argue that the program should be a survey of *all* kinds of mathematics known to man and let the social scientists of a few

decades hence decide which kinds are most useful to their fields. A person cynical about what has been done in quantitative social science might very well take such an extreme position. But the difficulty with this position is that the program is necessarily too lengthy or too superficial.

Another possible view that is held by some people is that social scientists should learn the kinds of mathematics which are proven tools in physics, chemistry, and astronomy. The trouble with this proposal is that it excludes very little and probably makes a faulty assumption that the social sciences are fundamentally similar to the natural sciences.

An attempt to evaluate what kinds of mathematics are most appropriate for social science problems is another possible approach. But here one can find only a small area of agreement among people who have thought about the problem. The difficulty is simple: to make a sound prognosis about the "future social science," one must have a thorough knowledge of the several social sciences and at the same time know a good deal about all fields of mathematics. No such man exists as far as I know. The people who attempt to draft a program of study on this basis are either competent mathematicians who have had little real experience with social science research, or social scientists who have rather limited mathematical skills but who are anxious to use those skills.

The only satisfactory criterion, I maintain, for selecting content of a program in mathematics for social scientists is the following: Select and weight topics in accordance with what has already been done in applying mathematics to social research. Unfortunately, relatively little has been done in this direction outside of statistics, but the list of papers and books is long enough to allow us to make some inferences. One advantage of this criterion is that the teacher has useful illustrations of the mathematics being taught. This is important, as I will argue later.

On the basis of this criterion I would propose the following outline as desirable for any program of the type we are discussing.

1. *Probability theory with special emphasis on stochastic processes.* The argument for this topic hinges upon the importance of statistics and the growing interest in stochastic models for learning, communication, and small group behavior.
2. *Calculus and differential equations.* Not only is calculus used in most treatments of mathematical statistics, but it also arises in many models for social science phenomena. Some work on finite calculus should be included because of its use in stochastic models and in statistics.
3. *Algebra and axiomatics.* Various uses of matrix algebra, set theory, and axiomatics can be found in the social science literature. I would argue that matrix theory is the most important topic in this category.

The three main categories I have listed should be given about equal weight I believe—and this belief is based upon my acquaintance with what has been done in applying mathematics to social research. I should point out that the list of topics just given is roughly the list of topics included in the program at the

Social Science Research Council's Summer Institute on Mathematics for Social Scientists, held at Dartmouth College last summer.

The other major problem I want to discuss is that of *how* mathematics should be taught to social science students. One part of this problem concerns how much detail and rigor should be included in the program, and this is clearly determined mainly by the time available and the capacities of the students. Concerning this latter matter—the capacity of the social science student for learning mathematical material—let me state some observations. First of all, a great many undergraduates in our universities drift into social science departments because they have had trouble with courses in mathematics and the natural sciences or because they know they dislike such subjects. The clinical psychologist can supply many reasons for such behavior, but prominent among those reasons is unpleasant experiences with mathematics in elementary and secondary schools. Hence, to solve this problem we need to improve secondary education in mathematics or recruit our social science students from a different population. For many years to come, I suspect, we will merely have to accept as a boundary condition that students in the social sciences on the whole do not like mathematics, find it difficult, and indeed have serious psychological blocks against learning mathematics. This fact creates some special problems that do not exist in the teaching of mathematics to engineers, chemists, and physicists.

A closely related observation is that few social scientists will bother with mathematics unless they are highly motivated—have found their lack of mathematical training a serious handicap in their own work or have been convinced by their elders that it will become a serious handicap. Therefore in optional programs or courses in mathematics for social scientists, we can depend on a high level of motivation, a real eagerness to learn, even though these same students have serious psychological problems in this connection. But it is very important, I'm convinced, to keep in mind the source and direction of this strong motivation. In a word, it is directed at finding security in his own field. Consequently, the man who teaches mathematics to social scientists must recognize his dual role of teacher and psychotherapist.

Because of the rather special and intense motivation in the student to acquire skills useful in his own field, the method of teaching is critical. One simply cannot teach mathematics to social scientists of today the way he teaches it to mathematics students; the motivation, the system of needs, the goals of these two kinds of students are entirely different. I think the social science student of mathematics must have frequent reinforcement from manipulative skill at working easy problems—problems he previously could not handle. Such reinforcement will maintain his motivation.

My main point on how mathematics should be taught to social scientists is this. It should be problem-oriented. Each topic in mathematics should be introduced by displaying concrete, non-trivial problems in social research and indicating how some mathematical machinery will be useful. Elegance in mathematical presentation must be sacrificed at times. Social science problems should be

stated, some mathematics taught, and the original problems solved by using that mathematics.

To develop a course such as I propose, a mathematician must necessarily learn a good deal about current problems in the several social sciences, and this is not easy. I have one suggestion: mathematicians should collaborate with social scientists in developing programs and giving courses. By such joint efforts, mathematicians can seriously contribute, I believe, to the development of social science.

3. The curriculum of the institute. (Thrall). The 1953 Summer Institute in Mathematics for Social Scientists had four major goals:

A. *Communication.* This is primarily communication with mathematicians; however, experience shows that expressing facts in mathematical language also facilitates communication between social scientists from different areas. One of the bars to communication between mathematician and social scientist is the different meanings attached by them to such words as *relation, function, variable, vector, dimension, continuum.* If the meanings were entirely different the discrepancy would be quickly discovered and cared for; the real difficulty is that the meanings have enough in common to obscure the difference until the damage is done. Since terms such as these are used in social science for essentially mathematical purposes, it seems desirable that the social scientist should learn and then use their mathematical meanings (at least when speaking to mathematicians).

A second bar to communication is lack of technical acquaintance by the social scientist with such basic mathematical concepts as order relation, partial order, real number, continuous functions.

B. *Preparation for Further Courses.* In a single eight-week session one cannot cover all of the mathematics needed by a social scientist. However, a reasonable goal would be to lay a foundation which would enable the student to continue his mathematical education with courses such as advanced calculus, linear algebra, mathematical statistics.

C. *Reading Knowledge.* The amount of mathematics appearing in papers and books in the social sciences is steadily increasing. The mathematical material sometimes appears in appendices and is usually summarized or paraphrased in word form. However, the critical and thorough reader will wish to be able to follow the mathematical arguments. Most of the mathematics used is statistical in nature, but a reading knowledge requires familiarity with such concepts and areas as integral, derivative, order relation, matrix, vector, linear transformation, stochastic process, linear programming, game theory.

D. *Model Building.* The mathematical background needed for model building in the social sciences is much the same qualitatively as for reading knowledge except for a considerably greater emphasis on axiomatics and foundations. Quantitatively, there is a great difference between reading and creating and one cannot expect an eight-week course to produce full-blown experts in the

use of mathematics for model building. However, this goal should be kept in mind in constructing the course.

The goals discussed above would apply to any mathematics course designed for social scientists, but here we are thinking primarily of a coures designed for social scientists who are at or beyond the Ph.D. level when they realize the need for mathematics. At this level it is not practical to undertake the traditional undergraduate mathematics sequence, and even when new undergraduate sequences become available, there will still be a need for special courses to care for mature social scientists. The topics included in the 1953 Summer Institute were selected after consideration of the goals listed above and also keeping in mind the nature of the participants. The topics are not listed in order of importance or in the order in which they should be presented. The letters following each topic indicate the goal (or goals) for which it is important.

1. Set algebra, relations, functions, one-to-one correspondence, equivalence relations, partitions, order relations. A.
2. Axiomatic development of number system including the concept of limit of a sequence. Careful definitions, but proofs limited to heuristic discussions. A, C.
3. Differential and integral calculus. Emphasis on polynomials, but also logarithms and exponential functions. Some analytic geometry included. B, C.
4. Selected topics from advanced calculus, including partial derivatives and multiple integrals. B.
5. Axiomatics, some simple system in detail and the general principles of model building. D.
6. Linear algebra, vector spaces, matrices, linear transformations. B, C.
7. Introduction to probability, sample spaces, stochastic processes, Markov chains. B, C.
8. Models from social science situations; this should include numerous small illustrative examples and also some large scale models such as linear programming, utility theory, learning theory, social choice, game theory, measurement theory. A, C.

4. Two aspects of a mathematics program for social scientists. (Raiffa). I would like to divide my comments into two parts: first, the general teaching approach that I would advocate for a program similar to the one given at Dartmouth this past summer; second, the importance of abstract thought (in contradistinction to mathematical technique) to the social scientist.

The points I raise here are, I believe, non-controversial in broad outline—indeed, in broad outline they are trivial. However, the stress given to these issues is more problematical, and since my own viewpoints have changed over the past summer, I would like to outline my subjective opinions at this point.

It is often said that the art of good lecturing is first to say what you are

going to say, say it, and then say what you have said. The price in subject matter covered for following this advice is well worth it *for the social scientist studying mathematics* (for the mathematics student it is not so necessary). In particular, I interpret this advice as follows: Motivation is well worth the expense —this includes motivation of mathematical subject matter, motivations of definitions, motivation of hypotheses, motivation of proofs (rigorous proofs play an important role in such a course—as I see it!)

As regards motivation of subject matter, I think that broad areas should be outlined before plunging into detail. For example, I would not advocate proceeding from a discussion of real numbers to a definition of limit (assuming, of course, that limits were not introduced formally in defining—or should I say in "talking about"—real numbers). I think it would be more appropriate to start off with a series of problems, perhaps related to the social sciences, to abstract these to problems involving maxima and minima, areas, sums of series, *etc.*, and then to show in turn that these have a common abstraction in terms of limits of sequences. Then one could introduce the limit notion, keeping in mind the diverse examples which motivated the abstract subject matter, and using these examples as test cases for the developing theory. Another point I wish to emphasize for the social scientist is that it is a mistake to build up to a principal theorem or application without disclosing our aims during the development of the theory. Of course it might not be possible to point meaningfully to our goal without the necessary machinery at hand, but I think one should not use this as an excuse for postponing the motivation indefinitely.

As regards motivation of definitions, I think it desirable for the student to know where he is going. Tentative definitions should be tried and shown to be faulty. Thus the ϵ-δ definition of limit should crystallize only after some preliminary fumbling with "definitions" which "really don't capture the idea which we are after." Note that the idea should precede the definition. The non-uniqueness of ways of defining some concept should be indicated.

As regards motivation of hypotheses of a theorem, I think it desirable for the lecturer to check the theorem's validity by means of special cases before discussing its proof. Numerous counter-examples should be given to theorems when hypotheses are altered or relaxed. With this preliminary discussion of the theorem, the student should begin to appreciate its meaning, why certain hypotheses are needed, and what cases are carefully ruled out by the hypotheses. By attempting to find counter-examples to the theorem itself and finding it to no avail the student often gains an insight into the crux of the proof. Incidentally one should not always formulate true theorems; theorems should be thought of as intelligent guesses or conjectures.

As regards motivating the formal proof itself, I wish to point out that understanding each step of a proof is no indication of understanding the proof looked at as a whole. Similarly, if one understands each theorem it does not mean one understands the subject matter as a whole. We should not lose sight of the *Gestalt* by looking too much at details. Learning mathematics by under-

standing in turn each sentence in a formal style of presentation can become a cook-book style of learning. Hence I would stress the "idea" of a proof as much as the formal proof itself, and the "idea" should precede the formal proof. However, one should emphasize that the "idea" falls short of the requirements of a formal proof and these shortcomings should be explicitly pointed out. When the formal proof is completed one should tie in loose ends by checking the places where the hypotheses were needed in the proof. If time is pressing, as it always is, I would much prefer to see the formal proof omitted rather than omitting the motivating remarks and/or the "idea" of the proof (especially since the formal proof can be found in texts while the motivation is not often enough in print). However, if this program is too time-consuming for the lecturer, part of it could be assigned explicitly as a problem for the student.

In the realm of motivation one should stress the creativity of the mathematician and one should indulge in heuristic arguments. It should be pointed out that mathematical creativity is largely a trial-and-error procedure (similar to empirical research) and that the formal proof is usually quite different from the pioneering attack on the problem.

The technique of starting a lecture by summarizing the pertinent information from previous lectures helps the student to see the main trend, but what I consider more important is that it affords an opportunity for the lecturer to repeat the material in a more symbolic and abstract form, unencumbered by motivation and examples, and thus far more compact; and then there is the secondary effect of having the student aware of all the innuendos and the rich meaning involved in a pithy abstract mathematical statement. Hence I would recommend repeated summaries of material covered in order to raise the mathematical maturity level of the students.

I concur wholeheartedly with Bush that, psychologically speaking, the student has to have some sense of accomplishment and should not become overawed with the extensive scope of the material. To this end, there should be numerous graded exercises. We should play along with the game that if one can manipulate and substitute in formulas then one "understands" the theory. We should not destroy this false sense of security. Where one should draw the line is hard to say, but personally, after the summer's experience, I would say that there should be more manipulation than we gave this summer. Any program such as the Summer Institute must face the problem of weighing the emphasis between abstract and concrete presentation of the subject matter (*e.g.*, vectors *vs.* n-tuples, linear transformations *vs.* matrices, stress on limit notions *vs.* more topics in maximization, *etc.*). If approach A is easier to get across than approach B, is it better? Not necessarily. I feel that another indicator that one should take into account is whether A or B is more conducive to the general "mathematical maturity level" of the student, and I would go so far as to introduce material which is primarily intended to increase mathematical maturity. Of course, I admit that I do not know how to do this effectively, and I do not deny that solving

a series of differential equations may contribute to the mathematical maturity of the student.

In order to defend retaining some moderate degree of abstract presentation in programs of this kind for the social scientist, I would like to say something about axiomatics in general. I choose this topic because I think it is defended too often for reasons which I do not consider important; but, on the other hand, I think it is important to include this topic in our program. Some people argue that for approaches to the social sciences to be sophisticated they must be axiomatic in nature; they point to game theory as an example. On the contrary, I think that premature emphasis on the axiomatic development of a subject matter—learning theory, for example—can be detrimental. If applied to large areas it can cause sterility of mathematical ideas. The fact that Von Neumann axiomatized game theory is not the reason that game theory is interesting to some mathematicians. The reason is rather that the theory presents well-formulated unsolved problems which would exist regardless of the axiomatic formulation of the subject matter. Certainly I would agree that it is sometimes beneficial for the social scientist to try to axiomatize an area—mainly because this directs his attention to the structure and to some of the basic notions of the area. However, axiomatic-type thinking has played and will play in the future an increasingly important role in mathematical work in the social sciences. Often in the behavioral sciences, when one grapples with some nebulous material (*e.g.*, cohesiveness of a group, so-called "rational" behavior, "socially desirable" welfare functions, *etc.*), definition after definition is discarded or discredited because it does not fit the bill in certain situations (*i.e.*, does not fulfill the hazy desiderata one has formed in one's own intuition). It then behooves one to pay more attention to these subjective desiderata and to formulate conditions one wishes the definitions to satisfy. These conditions should be viewed as axioms, and one should then determine whether they are consistent, independent, and categorical. If a set of mutually inconsistent conditions is prescribed, then one must review one's intuition. As a case in point, Arrow, in his *Social Choice and Individual Value*, formulates some historically important and intuitively palatable conditions for a social welfare function to satisfy, only to show that these conditions are inconsistent. This plays the valuable role of directing our attention to our inconsistent intuitions rather than to the shortcomings of specific proposals. A similar example can be pointed out in the area of decision making under uncertainty, where a set of reasonable but inconsistent conditions has been given (Milnor, Chernoff) for a rule of inductive behavior. These conditions are then studied further to show that various subsets lead to various principles (minimax principle of Wald, equally likely *a priori* principle of Laplace, *etc.*). Thus the investigations of these conditions serve to resolve certain philosophical difficulties for oneself. Another area in which there has been much thinking in axiomatic terms is in the solution concept of the n-person game. If a set of conditions or requirements turns out to be consistent and categorical in the sense that it uniquely characterizes a concept, then this procedure

implicitly serves as a definition of that concept; if a set of conditions turns out to be consistent but permits a subset of possible interpretations then one has localized one's consideration to this well-defined subset and one hopes that various arguments will hold for all the interpretations of that subset (*i.e.*, have an argument remain invariant over the subset characterized by our initial desiderata). Instances of these variants have turned up in the social sciences, and they are becoming increasingly popular. I thus conclude that the study of axiomatics in general is pertinent to the study of mathematics for social scientists.

5. Discussion. (Madow). On the whole, the curriculum of the 1953 Summer Institute was chosen in the way outlined by Bush, and, indeed, this is reflected by the rather close agreement between the list of topics recommended by Bush and that used in the 1953 Institute.

I do believe that it is an exaggeration to say that one should select and weight topics in accordance with what has already been done in applying mathematics to social research. We all know that certain lines of approach are developed and then an impass reached. For that reason, in reviewing what has been done, one must also take into account what is being done and what is likely to be done in the future. Thus, twenty years ago, one would have said that it was quite unnecessary to study algebra and set theory in economics and statistics. Just about that time, the importance of both of these subjects was beginning to be recognized and today the topics are far more important than they were twenty years ago. On the other hand, the importance of the calculus, particularly in statistics, has been sharply reduced. Thus, as in so many other fields, merely to consider the past would result in a biased program.

On the other hand, to consider only one's own personal research or the research activities of a small group would be likely to be very much biased also. It is a question of judgment and one can not avoid some forecast.

A similar comment applies to Bush's statement that he could argue that matrix theory is the most important topic among matrix algebra, set theory, and axiomatics. All three are needed, and the precise balance will depend on the individuals and needs concerned.

On the whole, I agree with Bush's characterization of social scientists, namely, that if they study mathematics, they will turn out to be highly motivated but also that many will have had some bad experiences with mathematics. A further difficulty faced by the social scientist as an undergraduate, when he starts studying mathematics, will be that he will not find the mathematics used in courses in his own subject matter. This is a disadvantage not faced by the physical scientist or engineer who, after studying some mathematics finds that it is referred to in other courses. Thus, the social scientist comes with high motivation but the knowledge that he has had previous difficulties and the feeling that he could probably get along without the mathematics. Then, too, he is not part of a captive audience, as are the students from the physical sciences

and engineering. The result is that he definitely requires a higher level of teaching than do the usual students of mathematics courses. In this connection, it is important to realize that social scientists believe that learning is better advanced through reinforcement and reward than through challenge, whereas in our mathematics courses, we tend to challenge the student. Here, again, we need better mathematics teaching. It is worth noting that the great interest of mathematicians in the type of curriculum we suggest may well produce a higher level of teaching than now exists.

Bush's discussion of how to teach mathematics to social scientists should be read in conjunction with the comments of Raiffa on the same subject. Perhaps I can summarize the situation by saying that we think that mathematics should be taught, not presented.

In our announcement of the Institute, we listed as the goals in order: model building, reading knowledge, and preparation for further courses. It was our feeling that all but the very poorest students would be able to communicate, that many of the students would be able to take further courses and read mathematical literature in their own fields, and that a few would be able to create models. Of the various goals listed by Thrall, I would say that the most of our students and the faculty were in agreement that the most important single result would be an increase in reading knowledge, and that this was attainable during the eight-week session.

Although the course we gave at Dartmouth was designed primarily for mature students, we believe that with very little change it would be a more suitable mathematics course for undergraduate social scientists than one or two years of the classical undergraduate curriculum in mathematics. I even think that the student would benefit far more from taking this course one year and repeating it the second than he would from the current first two years of college mathematics. Of course, there are administrative reasons why this may not be practicable, but I mention it to stress the point of view.

I think that there are many differences of opinion concerning Thrall's identification of mathematical topics and goals. To mention just one, to the economist and to the statistician, the reasons for studying the partial derivative far transcend the preparation for further courses in mathematics. But the precise identifications are not essential.

Whether the axiomatic development of the number system was needed is doubtful. If needed, it is hard to see why it is important for the goal of communication. It should not be understood that each of the topics by Thrall has equal importance. Thus, far more space would be given to item 6 and to item 7 than, let us say, to item 2.

All of us are in general agreement with the distribution of time mentioned by Bush. The only differences that would occur would be of the order of replacing one-third by one-quarter, or by one-fifth, neither of which would be great changes.

I should like to stress my agreement with Raiffa's comments on the teaching of mathematics, and to add that they really go beyond the teaching of mathematics to social scientists, and apply to teaching of mathematics in general. This is particularly true with respect to his statement that understanding each step of a proof is no indication of understanding the proof looked at as a whole, and that understanding each theorem does not mean that one understands the subject matter as a whole. Of course, it is not easy to achieve the full understanding, but this is a challenge to those who teach mathematics, as well as to those who study it.

It seems likely that axiomatic *versus* non-axiomatic treatments will continue to be a source of controversy. In general, social scientists will not find axiomatic approaches too difficult. Considerable gains to them will result from the consciousness that comes from the axiomatic approach, of the way in which mathematics and the world of experience interact.

After a lapse of many centuries, axiomatic approaches have again become important in many parts of mathematics. Here, as in many other questions of education, we must be certain that our preferences, be they for axiomatic or non-axiomatic approaches, are based on rational reasons rather than on what we happen to have studied.

I have just two more comments. At the present time, when a student has had undergraduate mathematics and decides to continue the study of mathematics beyond the junior year, he often feels as though the subject matter he is studying has changed completely. With the curriculum that we taught in the summer of 1953, this would not be true. I think, therefore, that that curriculum would be superior to the present undergraduate curriculum not only for the social scientist but also for students of mathematics, and indeed, for students from all fields other than those such as the physical sciences and engineering in which it is important that certain technical processes be known at certain times. Even for such students, I think that a curriculum such as we are discussing would be better than the present curriculum, but this needs to be worked out more.

Doubtless, there will be many questions raised concerning the inclusion or exclusion of certain topics. For example, if differential equations are taught, should one include differential equations of the second order? I suggest that we must avoid this type of question at present. The mathematics that is used in social science papers depends on the mathematical background that the researcher happens to have had. It would be impossible to cover all points, and it will be impossible to satisfy all people on the amount of time devoted to different topics. What we need is to have more courses, more experiments. Then, gradually, a curriculum will evolve.

MATHEMATICAL NOTES

EDITED BY F. A. FICKEN, University of Tennessee

Material for this department should be sent to F. A. Ficken, University of Tennessee, Knoxville 16, Tenn.

A NOTE CONCERNING THE QUOTIENT $(r^{p-1}-1)/p$

C. A. NICOL, University of Texas

This note contains two theorems concerning the Fermat Quotient, and a theorem concerning a sum of powers of primitive roots modulo an odd prime. The symbol $q(r)$ will denote the quotient $(r^{p-1}-1)/p$.

THEOREM 1. *If p and r are primes and r is a primitive root modulo p, then*

$$q(r) = (r-1)\{e(r) + (p-1)/2\} \tag{1}$$

where $e(r)$ denotes the exponent of the highest power of r which divides $q(r)!$.

Proof: Consider first of all the identity

$$(r^{p-1}-1)/p = (r-1)\{(r^{p-2}-1)/p + \cdots + (r-1)/p + (p-1)/p\}. \tag{2}$$

For each integral value of α in the interval $1 \leqq \alpha \leqq p-2$ the quotient $(r^{\alpha}-1)/p$ may be written as $I_\alpha + s_\alpha/p$ where I_α is a non-negative integer and the quotient is a proper fraction. Also, the numbers $s_1, s_2, \cdots, s_{p-2}$ are equal to the set $1, 2, \cdots, p-2$ in some order. Thus the identity may be written as

$$(r^{p-1}-1)/p = (r-1)\{I_{p-2} + \cdots + I_1 + (p-1)/2\}.$$

If x is a real number let the symbol $[x]$ denote the largest integer not exceeding x. If c is an integer in the interval $1 \leqq c \leqq p-2$ we wish to establish the following relation:

$$[(r^{p-1}-1)/pr^c] = [(r^{p-c-1}-1)/p] = I_{p-c-1}. \tag{3}$$

The following known lemmas are employed [1].

LEMMA 1. *If x is a non-negative real number, and if a is a rational integer, then*

$$[x/a] = [[x]/a].$$

LEMMA 2. *If n is a positive integer and p is a prime, then the exponent of the highest power of p which divides $n!$ is equal to*

$$[n/p] + [n/p^2] + [n/p^3] + \cdots.$$

If x in Lemma 1 is replaced by $(r^{p-1}-1)/r^c$, relation (3) follows. Furthermore if the quotient $(r^{p-1}-1)/pr^c$ is written as $((r^{p-1}-1)/p)/r^c$ and if n in Lemma 2 is replaced by the numerator of the latter quotient, the theorem is immediate.

THEOREM 2. *If $S(m^k)$ denotes the sum of the k-th powers of the primitive roots modulo an odd prime p, then*

$$S(m^{p-1}) = \{p(p-1)/2\}S(m) + p\sum_{i=1}^{N}(m_i - 1)D(m_i) - N(p+1)(p-2)/2,$$

where N is the number of primitive roots modulo p and $D(m_i) = [q(m_i)/m_i] + [q(m_i)/m_i^2] + \cdots$.

Proof. Since a primitive root need not be a prime we write relation (1) in the form,

$$m^{p-1} = 1 + pD(m)(m-1) + (m-1)p(p-1)/2$$

where

$$D(m) = [q(m)/m] + [q(m)/m^2] + [q(m)/m^3] + \cdots.$$

Summation over the primitive roots mod p yields the theorem.

The following theorem is not a generalization of Theorem 1.

THEOREM 3. *If p and r are primes and r belongs to d modulo p, then*

$$q(r) = (r-1)\{e(r) - (p-1)e'(r)/d\} + (p-1)(r^d - 1)/pd$$

where $e'(r)$ denotes the exponent of the highest power of r dividing $((r^d-1)/p)!$ and $e(r)$ is defined in theorem 1.

Proof. If $d=1$ or $p-1$ the theorem is trivial. Hence suppose that $1<d<p-1$. In relation (2) consider the quotients $(r^\alpha-1)/p = I_\alpha + s_\alpha/\text{p}$. We may write $r^\alpha - 1 \equiv s_\alpha \pmod{p}$. Then we have $s_\alpha \equiv s_\beta \pmod{p}$ if and only if $\alpha \equiv \beta \pmod{d}$ and, since $0 \leqq s_\alpha \leqq p-1$, $0 \leqq s_\beta \leqq p-1$, it follows that $s_\alpha = s_\beta$ if and only if $\alpha \equiv \beta \pmod{d}$. Also since d divides $(p-1)$ we have $(p-1)/d$ integers s_j, $j=1, \cdots, d-1$, and then equation (2) becomes

$$(4)\ q(r) = (r-1)\{I_{p-2} + \cdots + I_1 + (p-1)(s_1 + \cdots + s_{d-1})/pd + (p-1)/p\}.$$

By the argument given in Theorem 1, the sum $I_{p-2} + \cdots + I_1$ is equal to $e(r)$. Also $(s_t)/p = (r^t-1)/p - [(r^t-1)/p]$ for $1 \leqq t \leqq (d-1)$. Hence the sum $(s_1 + \cdots + s_{d-1})/p$ is equal to $(r^d-1)/(r-1)p - d/p - e'(r)$ since from the argument in Theorem 1 we have

$$\sum_{t=1}^{d-1}(r^t - 1)/p = e'(r).$$

When these quantities are substituted in equation (4) the theorem is immediate.

Bibliography

I. Introduction to Number Theory, T. Nagell, 47, Uppsala, 1951.

CYCLOTOMY AND THE CONVERSE OF FERMAT'S THEOREM

MORGAN WARD, California Institute of Technology

The following theorem of A. Hurwitz [1] was stated without proof in answer to a question raised by E. B. Escott [2] regarding a test for the primality of the Fermat numbers 2^k+1 (k a power of two) stated without proof by E. Lucas in his *Theorie des Nombres* [3].

HURWITZ'S THEOREM: *Let $Q_n(x)$ denote the cyclotomic polynomial of order n and degree $\phi(n)$. Then n is a prime number if there exists an integer a such that*

$$Q_{n-1}(a) \equiv 0 \pmod{n}. \tag{1}$$

This theorem is a simple consequence of the converse of Fermat's theorem. For let n be greater than one. Then the factorization of $x^{n-1}-1$ into its irreducible factors over the rational field is

$$x^{n-1} - 1 = \prod_{d\mid n-1} Q_d(x). \tag{2}$$

Here the product extends over all distinct divisors d of $n-1$.

The discriminant $\pm(n-1)^{n-1}$ of $x^{n-1}-1$ is prime to n. Consequently if d and d' are distinct divisors of $n-1$, the resultant of $Q_d(x)$ and $Q_{d'}(x)$ is always prime to n. Hence for any integer a and any proper divisor d of $n-1$, the three numbers $Q_{n-1}(a)$, $Q_d(a)$ and n are co-prime.

Now assume that the congruence (1) is satisfied. By what precedes, $Q_d(a)$ is prime to n for every proper divisor d of $n-1$. But by (2)

$$a^{n-1} - 1 \equiv \prod_{k\mid n-1} Q_k(a) \equiv 0 \pmod{n},$$
$$a^d - 1 \equiv \prod_{k\mid d} Q_k(a) \not\equiv 0 \pmod{n}, \qquad d \mid n-1;\, d < n-1.$$

Therefore n is a prime by the converse of Fermat's theorem. (For this converse see for example, O. Ore, *Number Theory*, Chapter XIV.)

For example, $Q_2(a)=a+1\equiv 0 \pmod 3$ for $a=2$; $Q_6(a)=a^2-a+1\equiv 0 \pmod 7$ for $a=3$. Hence 3 and 7 are primes. But the most interesting application of the theorem is that made by Hurwitz himself to the case when $n=2^k+1$ so that $Q_{n-1}(x)=x^{(n-1)/2}+1$; namely $n=2^k+1$ is a prime number if and only if there exists an integer a such that $a^{(n-1)/2}\equiv -1 \pmod{n}$. For $a=3$, this result is the test quoted by Lucas and used extensively by him and by other arithmeticians for investigating particular Fermat numbers.

References

1. A. Hurwitz, Mathematische Werke, vol. 2, page 747.
2. Intermédiare des Math., vol. II, 1896, pp. 80 and 214.
3. Lucas published proofs elsewhere, but the test is due to T. Pepin. For a history and references to other proofs see Dickson, *History of the Theory of Numbers*, vol. 1, Chap. XV.

CLASSROOM NOTES

EDITED BY G. B. THOMAS, Massachusetts Institute of Technology

All material for this department should be sent to G. B. Thomas, Department of Mathematics, Massachusetts Institute of Technology, Cambridge 39, Mass.

TWO ELEMENTARY FALLACIES

EDWIN HALFAR, University of Nebraska

Because of the similarity to arguments to which students in elementary classes are frequently exposed, the following elementary fallacies seem to have especial appeal.

1. The hyperbola and parabola are identical. The locus of points which are such that the slope of the line joining any one of them to the fixed point $(a, 0)$ is equal to the point's distance from the y-axis is $x^2-ax=y$. On the other hand, if the fixed line is $ax+by+c=0$ and the fixed point (x_1, y_1), then the locus is $(ax+by+c)/(a^2+b^2)^{1/2}=(y-y_1)/(x-x_1)$, a hyperbola. Since, as every analytic geometry student knows, by a translation and a rotation the line $ax+by+c=0$ and the point (x_1, y_1) can be transformed into the y-axis and $(a, 0)$ respectively; and furthermore since the conic sections are invariant under translation and rotation, the parabola is the same as the hyperbola.

2. The length of the ellipse is $\pi(a+b)$. Given the ellipse with semi-axes a and b respectively, increase the axes by ϵ on each end. The difference between the areas enclosed by the two ellipses thus formed is $\pi(a+\epsilon)(b+\epsilon)-\pi ab=L'\epsilon$ where L' is a mean line lying between the two ellipses. After an algebraic simplification, one has $\pi[(a+b)+\epsilon]=L'$. Passing to the limit as ϵ tends to zero, one has $\pi(a+b)=L$, the length of the ellipse, since L' tends to L as ϵ tends to zero.

As a "check" of the "validity" of this result, one may observe that the circle is a special case of the ellipse so that by letting $a=b$, one obtains $L=2\pi a$.

ON A GRAPHICAL SOLUTION OF THE FIRST ORDER LINEAR DIFFERENTIAL EQUATION

M. S. KLAMKIN, Polytechnic Institute of Brooklyn

The linear differential equation

$$\frac{dy}{dx}+P(x)y=Q(x) \tag{1}$$

can always be solved by quadratures to

$$y=e^{-\int P dx}\int Qe^{\int P dx}dx. \tag{2}$$

However, in many cases the quadratures cannot be effected in finite form. For these cases we can use the following graphical method.

One first plots the curves C_1: $[y=F(x)]$, and C_2: $[x=G(y)]$, (assumed to be continuous), as shown in Figure 1. Through an arbitrary point $P(x, y)$, a line is drawn parallel to the y-axis intersecting C_1 at A; through A, a line is drawn parallel to the x-axis intersecting C_2 at B. One then draws the line PB. Since the coordinates of points A and B are $(x, F(x))$, and $(GF(x), F(x))$, respectively, the slope of PB is $(y-F(x))/(x-GF(x))$ [$GF(x)$ is to mean $G(F(x))$]. A small section of the integral curve to

$$\frac{dy}{dx}=\frac{y-F(x)}{x-GF(x)}, \tag{3}$$

passing through P is given by a small segment PP_1 of PB. The procedure is now repeated starting from P_1. The proximity of the successive points, P, P_1, P_2, $\cdots$, taken will determine the accuracy of the construction.

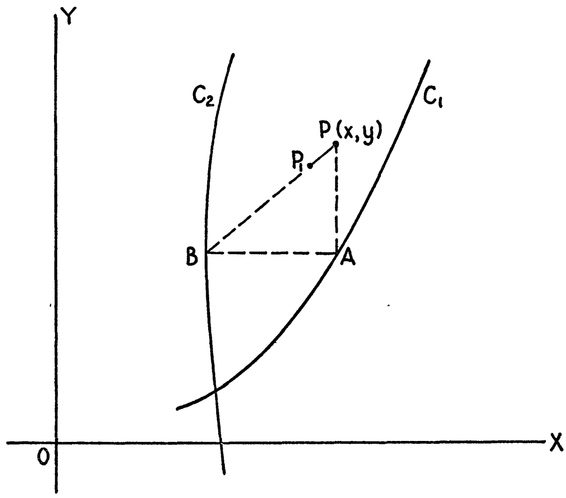


FIG. 1

Now, we have to identify the functions $F(x)$ and $G(x)$. By comparing equations (1) and (3) we must have

$$P(x)=\frac{1}{GF(x)-x}, \tag{4}$$

$$Q(x)=\frac{F(x)}{GF(x)-x}. \tag{5}$$

Solving equations (4) and (5) simultaneously, we get

$$F(x)=\frac{Q(x)}{P(x)}, \tag{6}$$

$$GF(x) = \frac{1}{P(x)} + x, \quad \text{or} \tag{7}$$

$$G(y) = \frac{1}{PF^{-1}(y)} + F^{-1}(y). \tag{8}$$

The above method can also be adapted to solving the Bernouilli equation.

Using a similar procedure, but now plotting four curves instead of two, we can graphically solve the non-linear differential equation:

$$\frac{dy}{dx} = \frac{\phi(x) - G(y)}{\psi(x) - F(y)}. \tag{9}$$

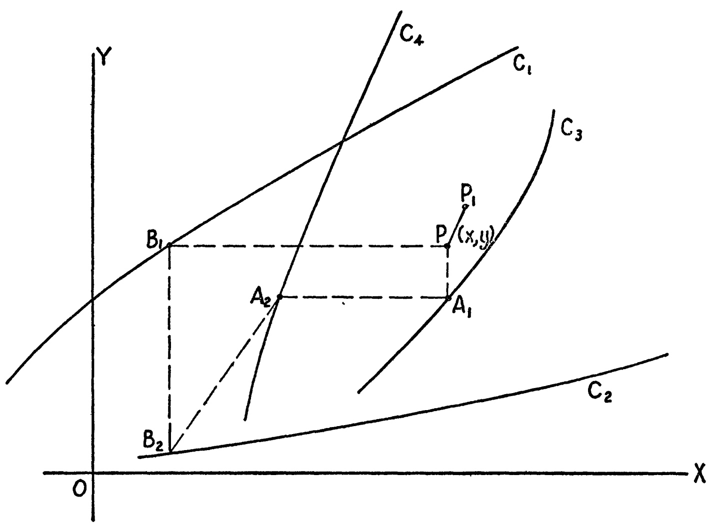


FIG. 2

In Figure 2, the equations of the curves C_1, C_2, C_3, and C_4 (assumed to be continuous) are given by

$$\begin{aligned} C_1&: \quad x = F(y), \\ C_2&: \quad y = GF^{-1}(x) \\ C_3&: \quad y = \phi(x), \\ C_4&: \quad x = \psi\phi^{-1}(y). \end{aligned} \tag{10}$$

PB_1, A_1A_2, B_1B_2, and PA_1 are drawn parallel to the coordinate axes, while PP_1 is drawn parallel to A_2B_2.

It follows from the construction that the slope of PP_1 satisfies equation (9).

This latter construction generalizes the procedure of J. P. Russell and the author (this MONTHLY, vol. 61, 1954, pp. 188–189).

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1131. *Proposed by L. A. Ringenberg, Eastern Illinois State College*

A student solves the differential equation $(dy/dx)^2=x^2$ and reports as follows: "General solution $(2y-2c)^2=x^4$; p-discriminant $x=0$; c-discriminant $x=0$. Since the p-discriminant locus consists of the envelope, cuspidal, and tac loci while the c-discriminant consists of the envelope, cuspidal, and nodal loci, and since the line $x=0$ is not an envelope, it ought to be a cuspidal locus. But the general solution consists of the parabolas $y=x^2/2+c$ and $y=-x^2/2+c$, and parabolas don't have cusps. How can there be a cuspidal locus if there are no cusps?" Resolve the predicament.

E 1132. *Proposed by Leon Bankoff, Los Angeles, California*

A common external tangent of two circles, tangent externally at C, cuts their smallest circumscribed circle in P and Q. The common internal tangent at C intersects the minor arc PQ in E, and the major arc QP in F. PC extended meets the outer circumference in K. Show that arc $QK=$arc KF.

E 1133. *Proposed by M. R. Spiegel, Rensselaer Polytechnic Institute*

Show that a necessary and sufficient condition that the imaginary roots of

$$f(z) = a_0z^n + a_1z^{n-1} + \cdots + a_n = 0, \qquad a_0 \neq 0,$$

occur in conjugate pairs is that the numbers $a_0, a_1, \cdots, a_n$ all lie on a common ray emanating from the origin in the complex plane.

E 1134. *Proposed by Hüseyin Demir, Zonguldak, Turkey*

Prove that a square integer is not a perfect number.

E 1135. *Proposed by Vern Hoggatt, San Jose State College*

If a third order determinant has elements $1, 2, 3, \cdots, 9$, what is the maximum value it may have?

SOLUTIONS

An Optimum Problem

E 804 [1948, 96]. *Proposed by S. H. Gould, Purdue University*

Denote by U the ellipsoid $a_1^2x_1^2+a_2^2x_2^2+a_3^2x_3^2=1$, by E_b the ellipse of intersec-

tion of U with the plane $b_1x_1+b_2x_2+b_3x_3=0$, by (p_1, p_2, p_3) a point variable on E_b, and by E_p the ellipse of intersection of U with the plane $p_1x_1+p_2x_2+p_3x_3=0$. Determine (p_1, p_2, p_3) so as to minimize the major axis of E_p.

Solution by R. Beil and the Proposer, Purdue University. Choose b_1, b_2, b_3 so that the endpoint of the vector $b=(b_1, b_2, b_3)$ lies on U, and consider the two cases: (i) b is shorter than the mean axis, call it $u^{(2)}$, of U, (ii) b is not shorter than $u^{(2)}$.

(i) Take $p=(p_1, p_2, p_3)$ orthogonal to b and $u^{(2)}$. Then $u^{(2)}$ is a principal semi-axis of E_p, since it is orthogonal to the tangent at its endpoint. But $u^{(2)}$ must be the major axis of E_p, since the shorter vector b is also a semi-diameter of E_p. So p solves the problem, since $u^{(2)}$ is the shortest possible major axis for all ellipses cut off from U by a plane through the center.

(ii) Denote by B the plane $b_1x_1+b_2x_2+b_3x_3=0$, by T the tangent plane to U at the endpoint of b, and by S the plane through the center parallel to T. Take p along the intersection of S and B. Then b is a principal semiaxis of E_p, since it is orthogonal to the tangent at its endpoint. But b must be the major axis of E_p, since no minor axis can be longer than $u^{(2)}$. So p solves the problem, since b is a semi-diameter for every possible E_p.

If $a_1^2 \leqq a_2^2 \leqq a_3^2$, we readily calculate: (i) p_1, p_2, p_3 are proportional to $b_3, 0, -b_1$, (ii) p_1, p_2, p_3 are proportional to $b_2b_3(a_2^2-a_3^2)$, $b_3b_1(a_3^2-a_1^2)$, $b_1b_2(a_1^2-a_2^2)$.

This solution is a special case of results obtained for Hilbert space by H. F. Weinberger. See his article, "An optimum problem in the Weinstein method for eigenvalues," *Pacific Journal of Mathematics*, vol. II (1952), p. 413.

Generalization of the Inequality of Bernoulli

E 1101 [1954, 123]. *Proposed by M. S. Webster, Purdue University*

The inequality of Bernoulli is often stated as follows: If $h>-1$ and $h\neq 0$, then $(1+h)^n>1+nh$, where n is an integer greater than unity. What is the generalization if n is an arbitrary real number?

Solution by A. R. Hyde, West Hartford, Conn. Clearly, the line $g(h)=1+nh$ is tangent to the curve $f(h)=(1+h)^n$ at (0, 1). The algebraic sign of $f''(h) =n(n-1)(1+h)^{n-2}$ shows by the direction of curvature that $f(h)<g(h)$ in the range $0<n<1$, $f(h)=g(h)$ for $n=0$ and $n=1$, and $f(h)>g(h)$ if $n<0$ or $n>1$.

Also solved by M. S. Klamkin, R. Z. Vause, and the proposer.

Klamkin pointed out that the Bernoulli inequality is contained in the following more general one, found on p. 37 of *Inequalities* by Hardy, Littlewood, and Pólya: If $x>0$, $0<m<n$, then

$$(1 + x/m)^m < (1 + x/n)^n.$$

The Bernoulli inequality is obtained by setting $x=nh$ and $m=1$.

Commutatively Factorable Matrices

E 1102 [1954, 123]. *Proposed by Harley Flanders, University of California*

Let C be an $n\times n$ matrix such that whenever $C=AB$ then $C=BA$. What is C?

Solution by the Proposer. We first prove that C commutes with every $n\times n$ matrix, A. If A is non-singular we set $B=A^{-1}C$, so that $AB=C$. Then $BA=C$, so that $CA=(AB)A=A(BA)=AC$. If A is singular then, for a suitable x, matrix $A_1=xI+A$ is non-singular, A_1 commutes with C, and hence A commutes with C. Now we use the fact that the only matrices C which commute with every matrix are the scalar matrices $C=cI$. The constant c cannot be zero unless $n=1$, for

$$\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}=0, \quad \text{but} \quad \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}.$$

Conversely, if $C=cI$, with c different from zero, then $C=AB$ implies $C=BA$.

Also solved by Joel Brenner, G. E. Forsythe, D. S. Greenstein, Marvin Marcus, D. C. B. Marsh, L. L. Pennisi, P. P. Saworotnow, and Olga Taussky Todd.

An Ellipsoidal Locus

E 1103 [1954, 123]. *Proposed by Paul Monsky, Brooklyn, N. Y.*

Find the locus of the vertex of a tri-rectangular trihedral angle which moves so that its edges intersect a given circle.

Solution by Martinus Esser, Georgia Institute of Technology. Let the equations of the circle be $x^2+y^2=r^2$, $z=0$. Let the vertex of the trihedral be V (coordinates X, Y, Z), and let the intersections of the edges with the circle be A, B, C. The four vertices V, A, B, C determine a rectangular parallelepiped, with three additional vertices in the plane $z=-Z$ and with eighth vertex in the plane $z=-2Z$. The coordinates of the center Ω of this parallelepiped are 0, 0, $-Z/2$. Equating the distances $|\Omega V|$ and $|\Omega A|$, we obtain the desired equation $X^2+Y^2+2Z^2=r^2$ of the locus of V. The locus is thus an oblate spheroid containing the given circle.

Also solved by Michael Goldberg, C. S. Ogilvy, Roscoe Woods, and the proposer.

A Constrained Function

E 1104 [1954, 124]. *Proposed by J. R. Hatcher, Fisk University*

Let $g(x)$ be a given continuous function satisfying $\int_\alpha^\beta g(x)dx=0$. Find $f(x)$ such that $\int_\alpha^\beta f^2(x)dx$ is a minimum subject to the conditions $\int_\alpha^\beta f(x)dx=\gamma$ and $\int_\alpha^\beta f(x)g(x)dx=\delta$, where α, β, γ, δ are constants.

Solution by Chih-yi Wang, University of Minnesota. The relation

$$\int_\alpha^\beta [f(x) - \lambda g(x) - \mu]^2 dx \geqq 0, \qquad \lambda, \mu \text{ real},$$

implies that

$$\int_\alpha^\beta f^2(x)dx \geqq \delta^2/K + \gamma^2/(\beta - \alpha) - (K\lambda - \delta)^2/K - [(\beta - \alpha)\mu - \gamma]^2/(\beta - \alpha),$$

where

$$K = \int_\alpha^\beta g^2(x)dx.$$

It is obvious that the required minimum value $\delta^2/K+\gamma^2/(\beta-\alpha)$ is attained if and only if $\lambda=\delta/K$ and $\mu=\gamma/(\beta-\alpha)$, whence

$$f(x) = \delta g(x)/K + \gamma/(\beta - \alpha).$$

Also solved by D. W. Allan, M. S. Klamkin, and the proposer.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4603. *Proposed by H. S. Shapiro, New York University*

Given $x_i \geqq 0$, $i=1, 2, \cdots, n$. Establish

$$\frac{x_1}{x_2 + x_3} + \frac{x_2}{x_3 + x_4} + \cdots + \frac{x_{n-1}}{x_n + x_1} + \frac{x_n}{x_1 + x_2} \geqq \frac{n}{2},$$

equality occurring only if all denominators are equal.

4604. *Proposed by Richard Bellman, The Rand Corporation, Santa Monica, California*

Suppose it is necessary to traverse a distance x where the legal speed is v_0. If you travel at a speed v, greater than v_0, there is a probability, $p(v)dt$, of being

stopped in the time interval t to $t+dt$ (and given a ticket). This consumes a fixed time r. (Actually, r should be a function of v.) Let us also assume that $p(v_s)=1$, for $v \geqq v_s$, the suicidal velocity. At what speed should one travel to minimize the expected time required to cover the distance x?

4605. *Proposed by D. J. Newman and W. E. Weissblum, Republic Aviation Corporation, Farmingdale, N. Y.*

Given an open, unbounded set of positive reals. Prove that there exists a real number such that infinitely many integral multiples of it lie in the set.

4606. *Proposed by H. R. Smith, McCoy College, Baltimore, Maryland*

Given a set of n countries, $a_1, a_2, \cdots, a_n$, prove that there exist no more than $3n-6$ distinct boundaries of the form mn between the countries a_m and a_n. By "distinct" is meant that if rs and mn are two such boundaries and a_r is the same as a_m then a_s is not the same as a_n.

4607. *Proposed by H. S. M. Coxeter, University of Toronto*

Evaluate

$$\frac{1 - 2^{-2} + 4^{-2} - 5^{-2} + 7^{-2} - 8^{-2} + \cdots}{1 + 2^{-2} - 4^{-2} - 5^{-2} + 7^{-2} + 8^{-2} - \cdots}.$$

SOLUTIONS

Limit of a Sum

4539 [1953, 336]. *Proposed by J. Gallego-Diaz, Madrid, Spain*

Determine

$$\lim_{n\to\infty} [\tanh 1 + \tanh 2 + \cdots + \tanh n - \log \cosh n].$$

Solution by H. A. Robinson, Agnes Scott College. Let

$$S(n) = \tanh 1 + \tanh 2 + \cdots + \tanh n - \log \cosh n.$$

Now

$$\log \cosh n = n + \log (1 + e^{-2n}) - \log 2,$$
$$\tanh k = 1 - 2(e^{2k} + 1)^{-1}$$

imply

$$S(n) = \log 2 - 2\sum_{j=1}^{n} \frac{1}{e^{2j}+1} - \log (1 + e^{-2n}).$$

As $n\to\infty$ the second item of $\lim S(n)$ is a known convergent series whose value is easily computed as $0.2801 \cdots$, and the last item approaches zero. Hence $\lim S(n) = \log 2 - 0.2801 = 0.413$ approximately.

Also solved by W. E. Briggs, Hermann von Schelling, and O. E. Stanaitis.

The Möbius Function and a Certain Divisor Function

4541 [1953, 336]. *Proposed by Jack Warga, Republic Aviation Corporation, Farmingdale, N. Y.*

Let $d_k(n)$ be defined for integers $k \geqq -1$ and $n \geqq 1$ by:

$$d_{-1}(n) = \delta_{1n} = \begin{cases} 1, & n = 1, \\ 0, & n \neq 1, \end{cases}$$

$$d_{k-1}(n) = \sum_{r_1 r_2 \cdots r_k = n} 1 = \text{the number of}$$

representations of n as a product of k factors, for $k \geqq 1$.

Let $\mu(j)$ be the Möbius function.

Let $\lambda(j) = \alpha_1 + \alpha_2 + \cdots$ for $j = p_1^{\alpha_1} p_2^{\alpha_2} \cdots$, (the decomposition of j into prime powers).

Prove that, for all integers j and t such that $t > \lambda(j)$,

$$\mu(j) = -\sum_{k=1}^{t} (-1)^k \binom{t}{k} d_{k-2}(j).$$

Solution by Leonard Carlitz, Duke University. Define $\Delta_t(m)$ by means of

$$\Delta_t(m) = \sum_{r_1 r_2 \cdots r_t = m} 1 \qquad (r_1, \cdots, r_t > 1),$$

where the order of the factors of m must be taken into account, as also in the definition of $d_k(m)$. Evidently $\Delta_t(m) = 0$ for $t > \lambda(m)$. Then it is clear that

$$(1) \qquad (1 - \zeta(s))^t = \left(-\sum_{m=2}^{\infty} m^{-s} \right)^t = (-1)^t \sum_{m=2^t}^{\infty} \frac{\Delta_t(m)}{m^s},$$

$\zeta(s)$ being the Riemann zeta-function. We have also

$$(2) \qquad (1 - \zeta(s))^t = \sum_{k=0}^{t} (-1)^k \binom{t}{k} \zeta^k(s) = \sum_{k=0}^{t} (-1)^k \binom{t}{k} \sum_{m=1}^{\infty} \frac{d_{k-1}(m)}{m^s},$$

so that comparison with (1) yields

$$(3) \qquad \sum_{k=0}^{t} (-1)^k \binom{t}{k} d_{k-1}(m) = (-1)^t \Delta_t(m).$$

In particular we have

$$(3') \qquad \sum_{k=0}^{t} (-1)^k \binom{t}{k} d_{k-1}(m) = 0 \qquad (t > \lambda(m)).$$

In the same way, multiplying both sides of (2) by $(\zeta(s))^{-1}$, we obtain

$$(4) \qquad \mu(m) + \sum_{k=1}^{t} (-1)^k \binom{t}{k} d_{k-2}(m) = (-1)^t \sum_{rs=m} \mu(r) \Delta_t(s),$$

and in particular

$$\mu(m) + \sum_{k=1}^{t} (-1)^k \binom{t}{k} d_{k-2}(m) = 0 \qquad (t > \lambda(m)), \tag{4'}$$

which is the required result. While (3′) is not stated by the Proposer it is an interesting analog of (4′). It is evident how additional formulas can be obtained without difficulty.

Also solved by T. M. Apostol, and the Proposer.

Editorial Note. Apostol calls attention to the formula

$$\Delta_1(n) - \Delta_2(n) + \Delta_3(n) - \cdots = -\mu(n) \qquad (n > 1) \tag{5}$$

which is equivalent to the above results. See H. S. Zuckerman, On some formulas involving the divisor function, *Bulletin of the American Mathematical Society*, v. 49 (1943), pp. 292–298 for a proof of (5) and several other such formulas. Zuckerman attributes (5) to Viggo Brun.

Circle of Best Fit

4542 [1953, 337]. *Proposed by C. D. Olds, San Jose State College, California*

The n points (x_i, y_i), $i=1, 2, \cdots, n$, should lie on a circle but they fail to do so. What circle shall we take which most nearly fulfills the condition that $\sum d_i^2$ is a minimum, where

$$d_i = [(h - x_i)^2 + (k - y_i)^2]^{1/2} - r?$$

The origin of coordinates is at the centroid of the given points; and (h, k) is the center, r the radius, of the circle to be found. If necessary, assume that d_i is small compared with r.

Solution by Hermann von Schelling, Groton, Connecticut. The difficulty of the problem stems from an artificial mathematical formulation. Requested is the circle which gives the "best" fit to n points, P_i. Let us assume that we know this circle. We denote the distances of P_i from the circle by d_i and D_i, where d_i is the numerically smaller one. The proposer starts from the condition

$$\sum d_i^2 = \text{minimum}. \tag{1}$$

It is artificial since there are two distances both of which should be considered. A more natural condition reads

$$\sum (-d_i D_i)^2 = \text{minimum}. \tag{2}$$

It is equivalent with the problem of finding three constant parameters a, b, c so that

$$\sum (x_i^2 + y_i^2 - 2ax_i - 2by_i - c)^2 = \text{minimum}. \tag{3}$$

The classic method of least squares yields the solution.

If d_i is small compared with $2r$, then D_i becomes practically constant and (2) is reduced to condition (1). In this sense it may be said that the solution of (3) "most nearly" fulfills the condition (1) of the proposer.

Also solved by O. E. Stanaitis.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to E. P. Vance, Oberlin College, Oberlin, Ohio.

Calculus. By G. B. Thomas. Cambridge, Addison-Wesley Publishing Company, Inc. 1953. 614 pages. $6.50.

This text would be suitable for a four-semester introductory course in calculus. In addition it contains several topics usually encountered in a course in advanced calculus. The order of presentation and the manner of exposition are such as to make it fall in the category of "traditional" calculus texts.

The first three chapters are devoted to a study of derivatives and differentials, while the next two introduce the topic of integration. The indefinite integral is encountered first, but emphasis is placed on the definite integral. A short chapter on polar coordinates is followed by one on the transcendental functions with a separate chapter on the hyperbolic functions. Chapter nine contains a systematic study of the technique of integration. In the next chapter vectors are introduced and are used in the subsequent chapters on solid geometry and partial differentiation. The remaining topics in the book are multiple integrals, infinite series, and differential equations.

The book has many good features. Throughout there are numerous examples to illustrate the important features of definitions and theorems. In addition there is an ample supply of drill problems and also some of greater difficulty to challenge the better student. The diagrams in the book are good; in fact the whole format is pleasing. The use of vectors in the chapters on solid geometry and partial differentiation appealed to the reviewer, as did the chapter on infinite series.

Misprints appeared to be few, only one causing any difficulty. This was in the index where a reference to *acceleration* is given on page 19, but the word does not occur on that page.

Those who like a text to be concise might not care for the way in which everything is spelled out in great detail. But to the reviewer it seemed like a good book, and compares favorably with texts well-established in the field.

W. FRASER
Toronto

Higher Transcendental Functions, Vols. I and II based, in part, on notes left by Harry Bateman. By the staff of the Bateman manuscript project (Arthur Erdélyi, Director; Wilhelm Magnus, Fritz Oberhettinger, Francesco G. Tricomi). New York, McGraw-Hill Book Company, 1953. Vol. I, 26+302 pages. $6.50. Vol. II, 17+396 pages. $7.00.

A review could hardly improve on what Erdélyi himself has written in description of this work (pp. xv–xvii). "The work of which this book is the first volume might be described as an up-to-date version of *Part II. The Transcendental Functions* of Whittaker and Watson's celebrated 'Modern Analysis'. Bateman (who was a pupil of E. T. Whittaker) planned his 'Guide to the Function' on a gigantic scale. In addition to a detailed account of the properties of the most important functions, the work was to include the historic origin and definition of, the basic formulas relating to, and a bibliography for *all* special functions ever invented or investigated. These functions were to be catalogued and classified under twelve different headings according to their definition by power series, generating functions, infinite products, repeated differentiations, indefinite integrals, definite integrals, differential equations, difference equations, functional equations, trigonometric series, series of orthogonal functions, or integral equations. Tables of definite integrals representing each function and numerical tables of a few new functions were to form part of the 'Guide'. An extensive table of definite integrals and a Guide to numerical tables of special functions were planned as companion works.

"The great importance of such a work hardly needs emphasis. Bateman's unparalleled knowledge of the mathematical literature, past and present, and his equally exceptional diligence, would have made the book an authoritative account of its vast subject, and in many respects a definitive account; a Greater Oxford Dictionary of special functions.

"A realistic appraisal of our abilities and of the time at our disposal led to a drastic revision of Bateman's plans. Only Bateman himself had the erudition to give a reliable and accurate history of special functions, and the man power available to us was insufficient for the inclusion of all functions. Thus we restricted ourselves to an account (probably far less detailed than that planned by Bateman) of the principal properties of those special functions which we considered the most important ones. The loss thus caused to mathematical scholarship is great, regrettable, and final, but we venture to hope that it will be counterbalanced in some measure by the considerable reduction in size of the book, and by the gain in the clarity of its organization. We can only hope that although the scope of the present work is much narrower than that envisaged by Bateman, in its humbler sphere the book will be more useful. . . .

"For the most part we were unable to make extensive use of Bateman's voluminous notes: we found it easier to compile our account of the various functions from our knowledge of these functions supplemented by routine search in the available literature."

A few mathematicians will be disappointed by this work. To them I can remark that, having known Bateman toward the end of his life, I think it unlikely he would have finished either of his planned works no matter how long he had lived. As can be seen from the sequence of books he did finish, in him the characteristic British passion for disorder grew with age to monstrous intensity and the huge task of organization before actually beginning to write these two works he was reluctant to face. On the other hand, the staff which has written the work under review has taken up a large responsibility, for to them alone was given any opportunity of making use of the great mass of material left by Bateman, an opportunity, as they tell us, they have decided to decline. It seems unfair both to Bateman and to the distinguished authors themselves that Bateman's name, not theirs, appears on the title page, which is cluttered besides with government gobbledegook.

The analysts of the twenties and thirties turned away from the "classical" approach with its formulae and explicit calculation, preferring instead to seek generality, method, and idea. To many trained in this "modern" line of thought, works such as Bateman planned are a voice from the past, of interest only to "applied mathematicians." Perhaps there was a trace of this view behind the decision to write a handbook instead of a treatise. If, as one often hears, special functions are used only by numerical practitioners who do not happen to have a large XXXAC at their disposal, the compactness and selection of the present work are advantages. We note that on one of the back covers the publisher advertises engineering handbooks. But, while the children of the "classical" analysts tend to refer to them with a shade of patronism, sometimes grandchildren are closer than children. With the resurgence of applied *mathematics* (not merely *application* of mathematics), so noticeable a trait of the mathematical scene of the fifties, it is possible that exhaustive treatises in analysis may come into the regard that the classical exhaustive treatises on hydrodynamics and elasticity have again and rightly been granted. That the handbook scheme permits a reduction in *size*, to which the introduction several times refers, may not always be a recommendation. It is to be hoped that Bateman's notes remain intact for possible future use.

Turning now to what these books are rather than what they are not, we find them in every respect admirable examples of what a handbook should be. The authors have written a first rate Webster's Collegiate without the benefit of a Greater Oxford as source. In making this project possible the Office of Naval Research has done service to many parts of science. Most of the material is available in no other book. While it is easy to find misprints in the text and bibliography, I have not searched for errors in the formulae themselves, since I doubt if there are any. The work has been carried out in the spirit that reliability comes first, and in a project of this type and size the labor of checking must be very heavy indeed.

The organization is by functions: gamma, hypergeometric, Legendre, generalizations of the hypergeometric (several types, two chapters), confluent

hypergeometric, Bessel, parabolic and paraboloidal, incomplete gamma, orthogonal polynomials, spherical and hyperspherical, orthogonal polynomials in several variables, elliptic. Most of the chapters begin with theory, followed by a table of formulae. There is an attempt to compromise between a pure list of results and an exposition of the subject. More difficult theorems are stated without proof, but when the steps in deriving a formula are fairly easy they are indicated. References tend to be to relatively recent works, especially when there exists a standard treatise for the function in question. There is a brief index of subjects, but no index of names; in the text, the discoverer of a formula is sometimes noted, and if so, accurately, but usually without a reference. There is little if any attempt to provide the reader with methods of attack for new problems. All kinds of readers will welcome heartily the careful and precise statement of range of validity which accompanies every formula.

Unfortunately it is necessary to complain about the format. The book is planographed, not printed. While the result is readable, it is not pretty. The typewriter was a good one as typewriters go, but the typing is ragged. For example, on the dedication page the first letter is out of line, as are many of the capitals throughout, and on nearly any page one sees letters run together or stretched apart. These books will certainly remain the standard in their field, which has a wider audience than most in mathematics, for a quarter century or longer. Every serious student of analysis or of any branch of applied mathematics will wish to own them, and the publisher had all reasonable assurance of a steady moderate sale for many years. To issue these monuments in a format suggesting a government progress report seems to me not only an abuse of the copyright privilege and a virtual insult from the publisher to the authors but also a major underestimate of a commercial opportunity.

C. TRUESDELL
Indiana University

NEW BOOKS RECEIVED

Henri Poincaré. By Tobias Dantzig. New York, Charles Scribner's Sons, 1954, xi+149 pages. $3.00.

College Algebra. By H. G. Apostle. New York, Henry Holt and Company, 1954, xiii+432 pages. $4.50.

Intermediate Algebra. By L. J. Adams. New York, Henry Holt and Company, 1954, xiv+370 pages. $3.40.

Introduction to Elliptic Functions. By F. Bowman. New York, John Wiley and Sons, Inc., 1954. 115 pages. $2.50.

Mathematics of Statistics, Part One. Third Edition. By J. F. Kenney and E. S. Keeping. New York, D. Van Nostrand Company, Inc., 1954. xiii+346 pages. $5.00.

Geometrical Mechanics and De Broglie Waves. By J. L. Synge. New York, Cambridge University Press, 1954. ix+167 pages. $4.75.

Introduction to Mathematical Statistics. Second Edition. By P. G. Hoel. New York, John Wiley and Sons, Inc., 1954. xi+331 pages. $5.00.

Transactions of the Symposium on Fluid Mechanics and Computing. Volume I. Sponsored by The American Mathematical Society and the Office of Ordnance Research, U. S. Army. New York, Interscience Publishers, Inc., 1954. 243 pages. $5.00.

Mathematics in Agriculture. Second Edition. By R. V. McGee. New York, Prentice-Hall, Inc., 1954. x+208 pages. $5.35.

Analytic Geometry. By Gordon Fuller. Cambridge, Massachusetts, 1954. ix+205 pages.

Theory of Games and Statistical Decisions. By David Blackwell and M. A. Girshick. New York, John Wiley and Sons, Inc., 1954. ix+355 pages. $7.50.

Calculus. Third Edition. By G. E. F. Sherwood and A. E. Taylor. New York, Prentice-Hall, Inc., 1954. xv+579 pages. $7.65.

Calculus. By G. M. Merriman. New York, Henry Holt and Company, 1954. xii+625 pages. $6.50.

Differential Equations With Applications. By Herman Betz and P. B. Burcham and G. M. Ewing. New York, Harper and Brothers Publishers, 1954. x+310 pages. $4.50.

Introductory Calculus With Analytic Geometry. By E. G. Begle. New York, Henry Holt and Company, 1954. x+304 pages. $4.50.

Elementary Theory of Numbers. By Harriet Griffin. New York, McGraw-Hill Book Company, 1954. ix+203 pages. $5.00.

Differential and Integral Calculus. Fifth Edition. By C. E. Love and E. D. Rainville. New York, The Macmillan Company, 1954. 14+526 pages. $5.75.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

MATHEMATICS DIVISION OF ASEE

The Mathematics Division of the American Society for Engineering Education met on June 16–18, 1954, at the University of Illinois, Urbana, Illinois. Four well-attended sessions of the Division were held. The following new officers of the Division were elected at the annual business meeting: Chairman, Professor C. O. Oakley, Haverford College; Secretary, Professor W. E. Restemeyer, University of Cincinnati; Director, Dr. R. S. Burington, Bureau of Ordnance, Navy Department. Professor H. M. Gehman, University of Buffalo, and Dr. C. V. Newsom, State University of New York, will continue to serve as Directors

and Professor H. K. Justice, University of Cincinnati, continues as Council Member of ASEE.

The next annual meeting of the ASEE Mathematics Division will be held in June 1955 at Pennsylvania State University. For further information write to Professor W. E. Restemeyer, University of Cincinnati.

REPORT OF MANPOWER RESOURCES IN MATHEMATICS

Manpower Resources in Mathematics, a report on the professional characteristics, employment, and earnings of mathematicians in the United States, has been issued by the National Science Foundation. The report, prepared jointly by the Foundation and the Bureau of Labor Statistics of the United States Department of Labor, is based on information supplied to the National Scientific Register in 1951 by about 2,400 mathematicians.

The report can be obtained from the Superintendent of Documents, United States Government Printing Office, Washington 25, D. C., at a cost of twenty cents.

RESEARCH FELLOWSHIPS IN PSYCHOMETRICS

The Educational Testing Service is offering for 1955–56 its eighth series of research fellowships in psychometrics leading to the Ph.D. degree at Princeton University. Open to men who are acceptable to the Graduate School of the University, the two fellowships each carry a stipend of $2,500 a year and are normally renewable. Fellows will be engaged in part-time research in the general area of psychological measurement at the offices of the Educational Testing Service and will, in addition, carry a normal program of studies in the Graduate School.

Suitable undergraduate preparation may consist either of a major in psychology with supporting work in mathematics, or a major in mathematics together with some work in psychology. However, in choosing fellows, primary emphasis is given to superior scholastic attainment and demonstrated research ability rather than to specific course preparation.

The closing date for completing applications is January 13, 1955. Information and application blanks will be available about November 1 and may be obtained from: Director of Psychometric Fellowship Program, Educational Testing Service, 20 Nassau Street, Princeton, New Jersey.

CONFERENCE ON THE ROLE OF WOMEN'S COLLEGES IN THE PHYSICAL SCIENCES

A conference on the role of the women's colleges in the physical sciences was held on the Bryn Mawr College campus on June 17–18, 1954. It was attended by professors of chemistry, mathematics, and physics from seventeen independent women's colleges, by twelve research and development officials from large industries, and by representatives of government agencies, hospitals, and educational associations.

A detailed report of the conference is being printed. It may be obtained on

request from the Chairman, Dr. W. C. Michels, Bryn Mawr College, Bryn Mawr, Pennsylvania.

PERSONAL ITEMS

Dr. E. O. Lovett, first president of Rice Institute, has been awarded the honorary degree of LL.D. by Princeton University, where he was formerly Professor of Mathematics.

Cornell University announces the following: Dr. Ilse Novak Gal and Dr. I. S. Gal have been promoted to assistant professorships; Professor H. L. Hamburger of the University of Cologne, Germany, has been appointed Visiting Professor of Mathematics for the year 1954–55; Miss Evelyn M. Bender, Mr. H. K. Flesch, Mr. H. F. G. Gardner, and Mr. M. E. Muller have been appointed to instructorships; Mr. G. E. Collins and Mr. Elliott Mendelson have been awarded National Science Fellowships; Professor R. J. Walker is on leave of absence; Professor W. A. Hurwitz has retired with the title of Professor Emeritus.

Georgia Institute of Technology announces: Assistant Professors W. B. Evans, J. H. Wahab, and R. A. Willoughby have been promoted to associate professorships; Dr. R. H. Kasriel, formerly aeronautical research scientist for the National Advisory Commission for Aeronautics, and Dr. W. M. Perel, previously a teaching assistant at Indiana University, have been appointed to assistant professorships; Professor D. M. Smith has retired.

At Massachusetts Institute of Technology: Mr. D. G. Aronson, Mr. F. G. Brauer, Mr. N. J. Hicks, Mr. J. E. Kimber, Jr., and Mr. Gustave Solomon have been appointed to instructorships.

Michigan State College makes the following announcements: Professor R. D. James of the University of British Columbia has been appointed Visiting Professor for the year 1954–55; Dr. Edward Silverman of Sandia Corporation has been appointed to an assistant professorship; Dr. J. H. McKay of the University of Washington and Dr. Andre Laurent, a postdoctoral fellow at the University of Chicago, have been appointed to instructorships; Assistant Professor Henry Parkus has been promoted to an associate professorship; Dr. J. G. Hocking and Dr. J. B. Kelly have been promoted to assistant professorships; Assistant Professor Mary H. Payne has resigned; Dr. H. A. Hanson has resigned to accept an associate professorship at Upsala College; Dr. C. H. Kraft has resigned to accept an assistant professorship at the University of California.

State Teachers College at Montclair, New Jersey, announces: Professor D. R. Davis has been appointed Chairman of the Department of Mathematics, replacing Professor V. S. Mallory who has retired; Assistant Professor B. E. Meserve of the University of Illinois has been appointed to an associate professorship.

University of Colorado reports that Professor B. W. Jones of the University and Professor K. A. Hirsch of Queen Mary College of London have exchanged positions for the academic year of 1954–55.

University of North Carolina announces that a symposium on multivariate statistical analysis was held at the University on April 21–24, 1954. About 35 statisticians, psychologists and economists attended. The formally announced participants were M. G. Kendall, C. R. Rao, Harold Hotelling, S. N. Roy, R. C. Bose, R. L. Anderson, Aleyamma George, Seymour Geisser and Earl Diamond, but numerous others contributed also. Considerable unpublished material and many new ideas were brought to light in fields such as factor analysis, multicollinearity, discriminant functions and related classification problems, basic criteria for multipleparameter estimation, canonical correlation, serial correlation, T statistics, and multivariate confidence bounds.

University of South Carolina announces the following: Professor Tomlinson Fort of the University of Georgia has been appointed to a professorship; Dr. Eckford Cohen, previously at the Institute for Advanced Study, and Dr. D. D. Strebe of the University of Buffalo have been appointed to assistant professorships.

Mr. R. A. Barnett, previously a teaching assistant at the University of Southern California, is teaching now in the Oakland Public Schools, California.

Dr. Archie Blake, formerly head of the I.B.M. Computing Section, Cornell Aeronautical Laboratory, Buffalo, New York, is now an advisory engineer for the Westinghouse Electric Corporation, Baltimore, Maryland.

Associate Professor R. C. Boles of State Teachers College, Florence, Alabama, has been appointed to an associate professorship at Mercer University.

Mr. E. N. Brandt, Jr., previously a student at the University of Oklahoma, has been appointed to a graduate assistantship at Oklahoma Agricultural and Mechanical College.

Assistant Professor J. R. Byrne of San Jose State College has been appointed to an instructorship at Portland State Extension Center, Oregon.

Mr. T. E. Cheatham, Jr. and Mr. J. W. Smith, graduate students at Purdue University and participants in a fellowship program established by ElectroData Corporation, were given a ten-week computer instruction course at ElectroData's computing center, Pasadena, California.

Mr. D. R. Childs, recently a student at the University of New Hampshire, has been appointed to a graduate assistantship at Vanderbilt University.

Mr. K. J. Cohen, former fellow at Cornell University, is now National Science Foundation Predoctoral Fellow at Carnegie Institute of Technology.

Assistant Professor K. L. Cooke of State College of Washington is on leave of absence as a research associate at Massachusetts Institute of Technology for the year 1954–55.

Mr. E. L. Dubowsky, previously a graduate assistant at Kansas State College, is teaching now in Colby Community High School, Kansas.

Professor Howard Eves of Harpur College has accepted a position as Visiting Professor for the academic year 1954–55 at the University of Maine.

Associate Professor E. L. Godfrey of Defiance College has a position as a mathematician at Wright-Patterson Air Force Base, Dayton, Ohio.

Mr. G. R. Grainger, recently a graduate student at the University of Notre Dame, is employed now as a research engineer by Consolidated Aircraft Corporation, San Diego, California.

Mr. J. E. Guinane, previously a graduate student at California Institute of Technology, is now with the Raytheon Manufacturing Company, Waltham, Massachusetts.

Professor T. R. Hollcroft, chairman of the Department of Mathematics of Wells College, has retired with the title of Professor Emeritus; he has been appointed Historian of the College.

Assistant Professor S. L. Jamison of Florida State University is now Applied Science Representative for International Business Machines, Los Angeles, California.

Mr. A. T. Kovitz, former instructor at Polytechnic Institute of Brooklyn, is now with the Arma Corporation, Roosevelt Field, New York.

Mr. Rolando Lara, formerly with the Seismograph Service Corporation, Tulsa, Oklahoma, is employed now by the Lorac Service Corporation, Morgan City, Louisiana.

Mr. Elmer Latshaw, recently a technical engineer with ACF Brill Motors Company, Philadelphia, Pennsylvania, has a position as a mechanical engineer at Naval Air Material Center, Philadelphia.

Dr. C. E. Lemke of Carnegie Institute of Technology has accepted a position as a research associate at the Knolls Atomic Power Laboratory, General Electric Company, Schenectady, New York.

Dr. M. D. Marcus, who has been a research assistant at the University of California, has been appointed to an instructorship at the University of British Columbia.

Dr. A. D. Martin of Oberlin College is now at the Institute for Advanced Study.

Mr. R. D. Mayer, recently a student at Purdue University, has been appointed to a teaching assistantship at the University of Washington.

Mr. R. J. Mercer has accepted a position at the Naval Electronics Laboratory, San Diego, California.

Mr. R. B. Merrill, formerly of the Bureau of Ships, Washington, D. C., is now a sales correspondent with the Eagle-Picher Company, Chicago, Illinois.

Assistant Professor D. G. Miller of the University of Louisville is now Research Associate, Department of Chemistry, Brookhaven Laboratory, Upton, New York.

Mr. Leonard Miller, who has been associated with the New York Bottle & Closure Company, New York City, has accepted a position with the Abbey Northern Glass Corporation, Brooklyn, New York.

Professor C. N. Mills has retired from his position as Head of the Department of Mathematics of Illinois State Normal University and has joined the faculty of Augustana College.

Miss Mardell S. Miskowski, formerly a graduate assistant at Marquette

University, has a position as a mathematician at Aberdeen Proving Ground.

Mr. Robert E. Montgomery of Trinity College, Hartford, Connecticut, is now at the United States Naval Research Laboratory, Washington, D. C.

Dr. F. R. Olson of Duke University has been appointed to an assistant professorship at the University of Buffalo.

Dr. A. M. Peiser, who has been engaged in private consulting in Wantagh, New York, is now Head of the Electronic Computer Section, M. W. Kellogg Company, Jersey City, New Jersey.

Mr. D. W. Pounder has been transferred from deHavilland Propellers Limited, England, to deHavilland Aircraft of Canada Limited.

Professor Edward Rayher of Bergen Junior College, Teaneck, New Jersey, has been appointed to an instructorship at Fairleigh Dickinson College.

Mr. F. A. Raymond, recently a student at Harvard University, has been appointed to a teaching fellowship at the University of Michigan.

Dr. Charles Roth, formerly guidance counselor at City College of the City of New York, is engaged now as a consulting psychologist by John R. Martin Associates, New York City.

Associate Professor A. J. Sachs has been promoted to the position of Professor of the History of Mathematics at Brown University.

Mr. L. A. Schmidt has been promoted to the position of Supervisor, Aerodynamics and Compressible Flow Section, Armour Research Foundation of Illinois Institute of Technology.

Dr. Wladimir Seidel, who has been at the Institute for Advanced Study, has been appointed to a professorship at the University of Notre Dame.

Assistant Professor E. L. Stanley of Clemson Agricultural College has been promoted to an associate professorship.

Mr. C. E. Stewart, previously a research assistant at the Armour Research Foundation of Illinois Institute of Technology, has accepted a position at the Boeing Airplane Company, Seattle, Washington.

Mr. C. R. Strain, who has been a teaching fellow at the University of Michigan, is employed now as a mathematician by Engineering Research Associates, Arlington, Virginia.

Dr. J. L. Ullman of the University of Michigan has been promoted to an assistant professorship.

Assistant Professor R. Z. Vause, Jr., of Clemson College has been appointed to an assistant professorship at the University of North Carolina.

Professor V. H. Wells of Williams College has retired.

Mr. Nathan Wetrogan, formerly with the United States Navy Hydrographic Office, Washington, D. C., has a position at the Lockheed Aircraft Corporation, Marietta, Georgia.

Mr. L. H. Williams, recently a graduate assistant at the University of Georgia, is now on active duty in the United States Navy and is at the United States Naval Proving Ground, Dahlgren, Virginia.

Dr. W. E. Wilson of the Office of Ordnance Research, Durham, North

Carolina, has been appointed to a professorship in the College of Engineering, Pennsylvania State University.

Mr. F. W. Darling, who had retired from his position with the United States Coast and Geodetic Survey, Washington, D. C., died on April 21, 1954. He had been a member of the Association for thirty-four years.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 81 persons have been elected to membership by the Board of Governors on applications duly received.

G. L. ALEXANDERSON, Student, University of Oregon.

MARY N. APPLEGATE, Student, University of Oklahoma.

D. D. AUFENKAMP, Doctorat d'Universite (Paris) Instr., Reed College.

C. W. BARNETT, M.S. (Louisiana S. U.) Jr. Mathematician, Lockheed Aircraft Co., Marietta, Ga.

J. C. BARTLETT, Student, Kentucky Wesleyan College.

J. A. BEANE, M.A. (Buffalo) Professor and Head of Department of Drawing, Mechanics and Design, School of Engineering, University of Buffalo.

D. C. BENSON, M.S. (Iowa S. C.) Instr., Iowa State College.

W. H. BEYER, M.S. (V.P.I.) Mathematician, Goodyear Aircraft Corp., Akron, Ohio.

KURT BING, Ph.D. (Harvard) Asst. Professor, Rensselaer Polytechnic Institute.

NORBERT BISCHOF, Student, Syracuse University.

J. H. BLAU, Ph.D. (North Carolina) Asso. Professor, Antioch College.

A. A. BLUNDI, B.A. (LaSalle) Computing Systems Engineer, Burroughs Corp.

C. E. BUIE, M.A. (U.C.L.A.) Teacher, Pasadena City College.

ARTHUR BURGER, Student, New Jersey State Teachers College, Upper Montclair.

MARIA CASTELLANI, Ph.D. (Rome) Asso. Professor, University of Kansas City.

D. A. COPE, Engineering Asst., Glenn L. Martin Co., Baltimore, Maryland.

JOSEPH DEMILIA, B.S. in E.E. (C.C.N.Y.) Electrical Engineer, U. S. Naval Gun Factory.

J. E. DERWENT, Student, University of Notre Dame.

DIANA DODGE, B.A. (Smith) Grosse Pointe Farms, Michigan.

A. H. EVANCIC, M.A. (Pittsburgh) University of Pittsburgh.

C. H. FINNIE, JR., B.S. (Southwest Texas S.) Lt. USAF, Student, University of Calif.

ANNE J. FLANAGAN, B.A. (Seton Hill) Assistant, Florida State University.

H. J. FLETCHER, Ph.D. (Utah) Instr., Brigham Young University.

R. M. GARDNER, Student, Arizona State College at Flagstaff.

R. L. GAY, M.S. (North Carolina S. C.) Asst. Professor, Wake Forest College.

JOHN GURLAND, Ph.D. (California) Asso. Professor, Iowa State College.

J. R. HAMILTON, M.S. (N.Y.U.) Asst. Professor, Long Island University.

J. H. HAMMER, Studienreferendar, Oberschule, Bremen, Germany.

W. J. HARROLD, Student, Polytechnic Institute of Brooklyn.

R. M. HENDRICKS, B.A. (Santa Barbara C.) Lt. (j.g.), United States Navy.

J. T. HINELY, JR., Student, University of Georgia.

G. C. HOLT, M.A. (Tennessee) Aerophysics Engineer, Consolidated-Vultee Aircraft Co., Fort Worth, Texas.

J. E. HOULE, JR., M.A. (Catholic) Instr., Georgetown University.

MRS. MARY R. HUDSON, M.A. (Peabody) Asso. Professor, Florence State Teachers College.

H. L. HUNZEKER, M.S. (Iowa S. C.) Instr., DePauw University.

KENJI INOUYE, M.A. (Columbia) Teacher, Hilo High School, Hawaii.

WALTER JAMES, B.S.Ch.E. (Minnesota) Instr., University of Minnesota.

S. R. JONES, Student, Harvard University; Design Engr., Barkley & Dexter Labs., Boston, Mass.

R. E. KELLY, B.S. (Duquesne) Mathematician, U. S. Bureau of Mines, Pittsburgh, Pa.

MAURICE KENNEDY, M.S. (Ireland) Grad. Student, California Institute of Technology.

N. K. KIM, Student, University of Oregon.

O. M. KLOSE, S.M. (Chicago) Asst. Professor and Head of Mathematics Department, Seattle University.

L. A. KOKORIS, Ph.D. (Chicago) Asst. Prof., University of Washington.

M. L. KROM, B.A. (Iowa) Radio Repairman, United States Army.

AARON LIEBERMAN, M.S. (N.Y.U.) Mathematician, Republic Aviation Corp., Farmingdale, N. Y.

J. P. LIPP, Student, University of Oklahoma.

L. H. LOOMIS, Ph.D. (Harvard) Asso. Professor, Harvard University.

ELIHU LUBKIN, Student, Columbia University.

L. C. MARSHALL, M.A. (Pennsylvania) Mathematician, White Sands Proving Ground, New Mexico.

PEREA MCCANE, B.A. (Seton Hill) Mathematician, Aberdeen Proving Ground, Maryland.

R. V. MCGEE, M.S. (Texas A.&M.) Asso. Professor, Agricultural and Mechanical College of Texas.

D. T. MITCHELL, Student, Wabash College.

C. B. MOORE, Ph.D. (Kentucky) Senior Aerophysics Engr., Consolidated-Vultee Aircraft Corp., Fort Worth, Texas.

BARBARA A. O'CONNELL, Student, Regis College.

L. N. ORLOFF, M.A. (California) Draftsman, Pacific Gas and Electric Co., San Francisco, California.

G. W. PEGLAR, Ph.D. (Iowa) Asst. Professor, Iowa State College.

J. H. PIERSON, B.S. (New Mexico) Albuquerque, New Mexico.

R. L. PLUNKETT, Ph.D. (Virginia) Asst. Professor, Florida State University.

L. B. PORTER, M.S. (Michigan) Actuarial Asst., North Carolina Mutual Life Insurance Co., Durham.

K. E. RALSTON, Student, Massachusetts Institute of Technology.

JOHN RAUSEN, M.A. (Columbia) Instr., University of Connecticut.

BERNARD REIMANN, Studienrat, Wilhelm-Gymnasium, Hamburg, Germany.

P. M. RINEARSON, B.S. (Washington) Research Asst., University of Washington.

L. C. ROBBINS, JR., M.A. (Pennsylvania) Supervisor of Programming, Burroughs Corp., Pa.

C. A. ROGERS, M.S. (Washington) Colorado Agricultural and Mechanical College, Fort Collins, Colorado.

MARILYN A. ROGERS, Student, Carleton College.

W. H. SAWYER, Student, University of Oklahoma.

S. A. SIMS, M.S. (Texas A.&M.) Asst. Professor, Agricultural and Mechanical College of Texas.

SISTER MARION BEITER, M.S. (St. Bonaventure) Instr., Rosary Hill College.

L. M. SOULE, B.A. (Alfred) Jr. Instr., Agricultural and Technical Institute, Alfred, N.Y.

J. C. SPRADLING, B.S. (Missouri) Teacher, Wisconsin Institute of Technology.

R. W. STEWART, Student, Arizona State College at Tempe.

M. L. TOMBER, Ph.D. (Pennsylvania) Instr., Amherst College.

E. A. TRAYNOR, Student, Holy Cross College.

FRANCES L. WALKER, B.A. (Oklahoma C. for Women) Grad. Asst., University of Oklahoma.

J. N. WALLACE, M.A. (Duke) Long Branch, New Jersey.
C. B. H. WATSON, Student, University of Toronto.
H. J. WEISS, D.Sc. (Carnegie) Asst. Professor, Iowa State College.
B. G. WILLIS, Student, Oklahoma Agricultural and Mechanical College.
C. E. YINGST, Student, Lebanon Valley College.
P. W. ZEHNA, A.M. (Colorado S. C.) Acting Instr., Colorado State College of Education.

THE MAY MEETING OF THE ALLEGHENY MOUNTAIN SECTION

The twenty-eighth meeting of the Allegheny Mountain Section of the Mathematical Association of America was held on May 1, 1954, at Marshall College, Huntington, West Virginia. Professor Frederick H. Steen, Chairman of the Section, presided at the morning and afternoon sessions.

There were twenty-eight in attendance including the following twenty-one members of the Association:

A. N. Aheart, A. G. Anderson, H. W. Baeumler, J. J. Barron, E. J. Cogan, A. B. Cunningham, H. A. Davis, A. R. Erskine, Mary A. Goins, Hunter Hardman, E. E. High, J. C. Knipp, L. T. Moston, B. H. Mount, Jr., Pauline E. Mount, E. F. Myers, Ruth E. O'Donnell, L. A. Ondis II, Morris Ostrofsky, F. H. Steen, J. K. Stewart.

Dr. Frank Bartlett, Dean of the College of Arts and Sciences, welcomed the people to Marshall College. Then the following papers were presented.

1. *Operation of the analytical group in industry*, by Dr. B. H. Mount, Jr., Westinghouse Electric Corporation.

The author described the personnel and the computing devices used to solve the engineering problems in a certain industrial analytical group. Certain mechanical and electrical problems were described and their solutions on analog and digital computers were outlined.

2. *An asymptotic expansion of a certain infinite integral*, by Professor Nelson Yeardley, Thiel College, introduced by the Chairman.

An asymptotic expansion of the integral

$$\int_0^\infty \exp\left(-x[v(1-2^\alpha/x)/4+4q/vx]\right)v^{-3/2}(1-e^{-v})^{-1/2}dv, \qquad x>0,\ \alpha>-1,\ v \text{ complex},$$

is obtained by using an adaption of Van der Waerden's method as described in his paper, *On the method of saddle points*, Applied Scientific Research, v. B2, no. 1B, 1951. The fact that the exponent is a function of x as well as v produces complications which do not occur in Van der Waerden's problem. These complications are resolved.

3. *Survey courses in mathematics for the liberal arts student*, by Mr. E. J. Cogan,, Pennsylvania State University.

The purposes of survey courses in mathematics were discussed. Among them are: to provide closure in communication; to explain the terms: truth, validity, proof; to introduce the ability to construct mathematical models for various systems; to furnish a background of basic ideas to the student of mathematics and science. Special techniques of presenting material were suggested as well as certain lists of topics. Among the course ideas were: basic analysis; the reading of data numerically presented; logic and foundations; mathematical games. A list of available textbook material was given.

4. *Hypercomplex systems in genetics*, by Professor A. G. Anderson, Duquesne University.

The manner in which various linear non-associative algebras lend themselves to problems involving diverse types of inheritance was discussed, with particular emphasis upon a very simple hypercomplex system which finds applications throughout the field of genetics. The commutative algebra having the multiplication table $XX=X$, $YY=X/4+Y/2+Z/4$, $ZZ=Z$, $XY=X/2+Y/2$, $XZ=Y$, $YZ=Y/2+Z/2$ is the zygotic algebra of simple Mendelian inheritance, but also finds use in connection with the prediction of quantitative characteristics in polygenic systems. Genetic algebras for asymmetrical inheritance also exist.

5. *Scoring problems in algebra*, by Dr. G. R. Lewis, Clarion State Teachers College, introduced by the Chairman.

An algebra test consisting of five problems of varying difficulty was administered to 101 students in five high schools. The tests were scored on a right-wrong basis. Then, each problem was analyzed and credit assigned to the steps in the solution, scores being obtained on this basis. Coefficients of correlation were computed and differences in mean scores determined and tested for significance. It was discovered that the differences in means were non-significant. The coefficients of correlation were high enough to warrant concluding that a student's rank was predicted by a raw score as adequately as by a computed score.

L. T. MOSTON, *Secretary*

THE MAY MEETING OF THE ILLINOIS SECTION

The thirty-third annual meeting of the Illinois Section of the Mathematical Association of America was held at Knox College, Galesburg, Illinois, on Friday afternoon and Saturday forenoon, May 14 and 15, 1954. Professor M. C. Hartley, Chairman of the Section, presided at all sessions.

There were forty-three in attendance, including the following thirty-two members:

Beulah M. Armstrong, H. G. Ayre, J. W. Beach, Max Beberman, Imogene C. Beckemeyer, H. R. Beveridge, A. H. Black, A. O. Boatman, J. R. Brown, John Christopher, M. D. Eulenberg, A. E. Gault, J. S. Georges, A. E. Hallerberg, M. C. Hartley, E. C. Kiefer, Evelyn K. Kinney, A. O. Lindstrum, Jr., W. C. McDaniel, A. W. McGaughey, B. E. Meserve, E. B. Miller, W. L. Miser, G. E. Moore, M. G. Moore, Gordon Pall, L. L. Pennisi, Annette Sinclair, S. Grace Smyth, Rothwell Stephens, C. J. Stowell, and Arnold Wendt.

At the business meeting on Saturday morning the following officers were elected for the coming year: Chairman, Professor Rothwell Stephens, Knox College; Vice-Chairman, Professor Hugh Beveridge, Monmouth College; Secretary-Treasurer, Professor A. W. McGaughey, Bradley University. The Secretary-Treasurer's report was given and approved. The Section Governor, Professor E. B. Miller, gave a short report concerning the business of the Board of Governors during the past year. Business meeting adjourned.

The following program was presented:

1. *An integrated high school mathematics program*, by Professor Max Beberman, University of Illinois High School.

Since it is almost impossible at the ninth grade level to separate college-potential students

from terminal students, there is need for a course which can serve as common ground for all students who are ready for high school mathematics. The integrated course satisfies this requirement because it takes concepts from many areas of mathematics and of general educational value and places them in the early years of high school. More specialized needs can be met in the later years. The first course at University High includes equations, inequalities, loci, inductive and deductive reasoning, computations with measurements, trigonometry, and a set theoretic development of the cardinal numbers.

2. *Progress report on the ICP mathematics examination*, by Professor H. Glenn Ayre, Western Illinois State College.

A state-wide committee in Illinois working on college admission requirements set as one of five criteria to be used by a college or university to provide the best prediction for probable college success the following: "Score on a simple mathematical test." The Illinois Council of Teachers of Mathematics assumed the responsibility for constructing such a test.

The first task of the committee was to determine the nature and scope of the test. It was agreed to attempt to build a test that would assist in determining a student's ability to do college work and measure competency in simple mathematics, but it would not attempt to measure achievement in high school mathematics nor indicate a student's preparation for college mathematics courses. The content should, for the most part, cover arithmetic and general mathematics with emphasis on basic concepts, principles, and applications.

The test items are now in the hands of the Associate Director of the Illinois Curriculum Program who has assumed responsibility for providing the professional work necessary to put the test in final form and establish norms.

3. *The small college and the new engineering requirements*, by Professor M. Anice Seybold, North Central College.

The University of Illinois School of Engineering has new entrance requirements which amount essentially to four years of high school mathematics. The speaker suggests a freshman course in analytic geometry and the calculus combined for students who satisfy these requirements. These students would probably need some review of high school mathematics. Textbooks are already available for the combined course and have been in use in the eastern part of the United States for some time. Most colleges would want their students to have about two or three more hours of calculus. Such a course in Intermediate Calculus would be relatively easy for the small college to arrange. At the end of the freshman year, students would have the necessary calculus prerequisite for most junior and senior courses now requiring calculus.

The speaker closed with an appreciation of the benefits mathematics departments will reap from the new engineering requirements and a warning that performance of students in new mathematics courses, such as combined analytic geometry and calculus, should be checked carefully and often against the performance of students in traditional courses.

4. *Report of the committee on the strengthening of teaching mathematics in secondary schools*, by Professor B. E. Meserve, University of Illinois.

This year the Committee has been sponsored by both the Illinois Section of the Mathematical Association of America and the Illinois Council of Teachers of Mathematics. Most of the Committee's activities may be considered as implementations of its report last year (See this MONTHLY, vol. 60: pp. 652–661). It has been primarily concerned with (1) strengthening the State Department of Education's forthcoming "Guide to Supervision, Evaluation, and Recognition of Illinois Schools (Kindergarten through Junior College)" and (2) sponsoring the contest prepared by the Metropolitan New York Section of the Mathematical Association of America. Fifty Illinois high schools and over two thousand students are participating in the contest this year.

5. *Straight lines and curved lines*, by Professor W. M. Johnson, Department of Architecture, University of Illinois. (By invitation.)

A non-technical presentation offering a few suggestions on rhythm, balance, harmony and color as they apply to photography. In making the presentation the speaker reviewed the basic rules of composition mentioned above and illustrated them by using straight and curved lines in a chalk-talk which was followed by a showing of 2×2 Kodachrome slides.

6. *A generalization of a theorem in schlicht function theory*, by Professor Arnold Wendt, Western Illinois State College.

Generalize the hypotheses of Lemma III, which is a special case of the uniformization principle, on page 15 of the 1950 Colloquium Publication, *Coefficient Regions for Schlicht Functions*, so that the closed arcs I_ν', I_ν'' of $|z|=1$ are identified by $g_\nu(\theta)$ such that when $e^{i\theta}$ lies in I_ν', $e^{ig_\nu(\theta)}$ lies in I_ν'. Here $g_\nu(\theta)$ is a real valued, continuous, monotonic decreasing function of the real variable θ, $g_\nu'(\theta)$ exists except perhaps for finitely many θ, and $-M<g_\nu'(\theta)<-N, 0<N<M<\infty$. It can be shown that there exists a function $w=F(z)$ having the properties of the function in the lemma referred to except that it may be only continuous on the identified arcs.

7. *The role of amateurs in number theory*, by Professor Gordon Pall, Illinois Institute of Technology.

Some past contributions by amateurs and examples of problems which have interested amateurs were given. To illustrate the need for proof, examples were given of properties of numbers which hold for small numbers, and whose first exception occurs indefinitely far out. Characteristics of amateurs were discussed.

8. *A method for representing the sum of certain series*, by Professor L. L. Pennisi, University of Illinois, Chicago Undergraduate Division.

The main result of this paper is contained in the following:

THEOREM. *If $z\neq 0$ and $R\beta>-1$, then*

$$\sum_{n=0}^{\infty}\frac{z^{n+1}}{(1+\beta)\cdots(n+\beta+1)}=\frac{e^z}{z^\beta}\int_0^z e^{-w}w^\beta dw, \tag{1}$$

and if $0<|z|<1$ and $R\beta>-1$, then

$$\sum_{n=0}^{\infty}\frac{n!z^{n+1}}{(1+\beta)\cdots(n+\beta+1)}=\frac{1}{z^\beta}\int_0^z\frac{w^\beta dw}{(1-z)+w}, \tag{2}$$

where $R\beta$ denotes the real part of the complex number β; w^β denotes $\exp[\beta L(w)]$, *$L(w)$ being an arbitrary branch of* $\log w$ *on the segment $[0, z]$; z^β denotes* $\exp[\beta L(z)]$, *and the integrand is taken along the line segment joining 0 to z.*

In particular, when $z\to 0$ and β real in (1) and (2), one obtains the sums of many interesting series.

9. *Parabolic functions*, by Professor J. S. Georges, Wright Junior College, Chicago.

Just as the circular functions are associated with the unit circle and the hyperbolic functions with the rectangular hyperbola, even so the parabolic functions may be associated with the unit parabola. The area parameter t is defined to be twice the area of the parabolic sector, namely, $t=4\int_1^x\sqrt{1+x}dx-2x\sqrt{1+x}=\text{cosp}^{-1}x=\text{sinp}^{-1}y/2$.

The properties of the parabolic functions and of the inverse parabolic functions are discussed

in terms of continuity, differentiability, and integrability. Relations between circular, hyperbolic, and parabolic functions are pointed out in terms of the particular parameters involved.

A. WAYNE McGAUGHEY, *Secretary*

THE MAY MEETING OF THE INDIANA SECTION

The thirty-first annual meeting of the Indiana Section of the Mathematical Association of America was held at Rose Polytechnic Institute, Terre Haute, Indiana, on May 8, 1954. Two sessions were held at which Professor C. P. Sousley of Rose Polytechnic Institute, Chairman of the Section, presided.

There were fifty-one in attendance including the following thirty-six members of the Association:

H. W. Alexander, W. C. Arnold, A. P. Boblett, Stanley Bolks, G. E. Carscallen, W. W. Chambers, K. W. Crain, D. E. Deal, M. W. DeJonge, W. E. Edington, P. D. Edwards, R. E. Ekstrom, H. E. H. Greenleaf, J. R. Hadley, J. W. Hamblen, Ralph Hull, E. L. Klinger, L. H. Lange, G. T. Miller, Vera T. Morris, P. A. Nurnberger, Gloria Olive, T. P. Palmer, J. C. Polley, Tibor Rado, R. M. Ross, A. R. Schmidt, M. E. Shanks, W. O. Shriner, Aubrey H. Smith, C. P. Sousley, Anna K. Suter, R. O. Virts, M. S. Webster, K. P. Williams, G. N. Wollan.

The following officers were elected: Chairman, Professor H. W. Alexander, Earlham College; Vice-Chairman, Mr. R. O. Virts, Central High School, Fort Wayne, Indiana.

Professor P. D. Edwards of Ball State Teachers College reported that the Committee on Awards had awarded Association medals to five high school seniors on the basis of excellence in mathematics demonstrated in the Indiana Science Talent Search competition.

Professor Edwards also reported on the results of a study of mathematical preparation for college in Indiana, Michigan, and Illinois in which he, representing the Indiana Section, had collaborated with Professor P. S. Jones of the University of Michigan and Professor B. E. Meserve of the University of Illinois. A report on the study had been published under the heading, "Mathematical Preparation for College," in *The Mathematics Teacher*, Vol. XLV, May, 1952.

In accordance with a recent suggestion of the Board of Governors that the Section Governor be made an officer of the Section, the Constitution was amended so as to include the Section Governor as a member of the Executive Committee.

Professor L. H. Lange reported on the mathematics competition for undergraduate students in Indiana schools which has been sponsored in recent years by Valparaiso University. Following the discussion, and in part on his suggestion, a motion was passed instructing the chairman to appoint a committee to study the value of, need for, and interest in such a competition, and consider the desirability that the Section assume its sponsorship.

The following papers were presented:

1. *Global structure of the family of integral curves of differential equations*, by

Professor M. E. Shanks, Purdue University.

The speaker discussed, in an intuitive way, the problem of topologizing the integral curves of the system $\dot{x}=f(x, y)$, $\dot{y}=g(x, y)$. When the domain of the functions f and g is a closed rectangle containing no critical point of the system, then the natural topology on the integral curves gives a dendrite. Some special curves were considered and the fact noted that the presence of limit cycles renders such topologization impossible. Mention was made of the recent work of L. Markus, in which the notion of separatrix is fundamental, and its relation to the above problem.

2. *A note on the triangular inequality*, by Professor Gloria Olive, Anderson College.

The triangular inequality was discussed from the standpoint of the various mathematical topics it can motivate in an undergraduate seminar on this subject. The inequality was taken from one-dimensional real space into Hilbert space; on the way, a brief geometrically motivated proof, readily extended to n-dimensional space, was presented.

3. *An algebraic proof of the central limit theorem*, by Professor H. W. Alexander, Earlham College.

The usual proof of the central limit theorem is based on the use of moment generating functions, a device which properly belongs in graduate mathematics. The present proof makes use of the multinomial theorem, and in this connection introduces the idea of *similar terms*, that is, terms which have the same exponents in a different order. Expressions are obtained for the higher moments of the quantity $Y_n=n^{-1/2}(X_1+X_2+\cdots+X_n)$, where $X_1, X_2, \cdots, X_n$ are from a common population with zero mean. The moments are shown to approach those of the normal distribution as $n\to\infty$.

4. *Continuity and discontinuity in analysis and geometry*, by Professor Tibor Rado, Ohio State University. (By invitation.)

Professor Rado discussed a series of examples of discontinuous functions and functionals, selected from various areas in analysis and geometry, which may be used in the classroom to throw more light upon the concept of continuity itself.

5. *An application of geometric series with two ratios*, by Professor L. H. Lange, Valparaiso University.

A geometric series with two ratios converges under certain conditions and the problem of finding these conditions and the resulting sum was solved by the author subsequent to its statement by F. Watkins as E 981 in this MONTHLY. He had found, and here discussed, an application of this type of series to a problem of obscure origin: that about the bird which flies back and forth between two cars which are following a crash course. Even when generalized this problem has a trivial solution and he is searching for an application to a problem not admitting the trivial solution.

6. *Nomography from the similar triangle viewpoint*, by Professor T. P. Palmer, Rose Polytechnic Institute.

The determinantal method of proof usually employed to establish the validity of nomographic techniques obscures the simplicity of the basic ideas in the usual alignment charts. This paper develops the parallel-line type, the concurrent-line type, and the N type, employing nothing more advanced than the properties of similar triangles. It also develops the two-parallel-line-and-one-curve type, in this case with the aid of elementary analytic geometry. The improved simplicity should render nomographic methods accessible to many who have previously avoided them.

J. C. POLLEY, *Secretary*

THE MAY MEETING OF THE KENTUCKY SECTION

The May meeting of the Kentucky Section of the Mathematical Assocation of America was held at the University of Kentucky, Lexington, Kentucky, on May 8, 1954. Professor Sallie Pence, Chairman of the Section, presided at the meeting.

Sixty persons attended the meeting, including the following thirty-four members of the association:

M. C. Brown, H. W. Burnette, E. A. Cameron, Esther A. Compton, J. B. Cornelison, V. F. Cowling, H. H. Downing, J. C. Eaves, R. I. Fields, Clarence Ford, A. W. Goodman, Beulah Graham, Charles Hatfield, Aughtum S. Howard, Tadeusz Leser, A. G. McGlasson, W. L. Moore, R. S. Park, Mary Pettus, D. W. Pugsley, Sara L. Ripy, G. G. Roberts, W. J. Robinson, D. C. Rose, M. I. Rose, Sister Mary Charlotte, R. H. Sprague, Guy Stevenson, Louise C. Stolle, J. T. Vallandingham, J. A. Ward, R. H. Wilson, Jr., T. M. Wright, W. M. Zaring.

The following officers were elected for the year 1954–55: Chairman, Professor Charles Hatfield, Georgetown College; Secretary-Treasurer, Professor A. W. Goodman, University of Kentucky; Traveling Lecturer, Professor Aughtum S. Howard, Kentucky Wesleyan College.

The following papers were presented:

1. *Number systems*, by Professor J. A. Ward, University of Kentucky.

Number systems to various bases are discussed. The study of such systems makes clear the properties and methods of arithmetic in our ordinary decimal system. Special attention is given to the binary system and its use in electronic digital computers.

2. *Upper limits to real roots of polynomial equations*, by Mr. Eugene Bradley, Georgetown College, introduced by Professor Charles Hatfield.

In locating the real roots of a polynomial equation, it is frequently valuable to use theory of upper limits to aid in the solution. This is a discussion of several theorems on upper limits and a presentation of examples to illustrate their application.

3. *A problem in weighing*, by Professor Guy Stevenson, University of Louisville.

In general find, for a given positive integer N, the minimum number of positive integers $w_1, w_2, \cdots, w_n$ such that $w_1+\cdots+w_n=N$ and such that each integer from 1 to N can be expressed as an algebraic sum of some subset of the w's. It was shown that if $(3^{n-1}-1)/2<N\leq(3^n-1)/2$, the minimum number of w's is n, and if $N=(3^n-1)/2$, there is only one set of w's and they are $1, 3, 3^2, \cdots, 3^{n-1}$. A method was discussed for finding how many sets of n integers may be determined for a given number N. In general, quite a few sets are possible.

4. *Women in mathematics*, by Miss Mary A. Eschrich, Nazareth College, introduced by Sister Mary Charlotte.

Among the many minor mathematicians, there are six women who have been recognized by historians and tradition as most interesting contributors to the mathematical heritage. The lives of fourth century philosopher and astronomer Hypatia, Mary Gaetana Agnesi, eighteenth century Italian student of calculus, Sophie Germain, the scientist who experimented with mathematical physics, England's distaff mathematician, Mary Somerville, and the friend and pupil of Herman

Weierstrass, Sonya Kovalevski, clearly demonstrate the drama and novelty found in the long history of mathematics and mathematicians.

5. *Equation skeletons*, by Professor J. C. Eaves, University of Kentucky.

The equation $AX=K$, where X is the column vector having elements $X_1, X_2, \cdots, X_n$ and A is an m by n matrix, is solved for the x_i by using only the augmented matrix as the skeleton of the system and performing row transformations to obtain equivalent systems and eventually the equivalent system $CX=H$, where C is the appropriate rank canonical form or supplementary rank canonical form. From this the solution is immediately read or the conditions under which a solution exists are apparent.

6. *The mathematics a student should know to enter college*, by Professor Beulah Graham, Campbellsville College.

The widespread lack of background for college freshman mathematics and the accompanying poor study habits due to the lack of required homework in high school are cited. There is obviously no improvement in the readiness of freshmen for college mathematics in the past ten years. However, the colleges are trying to help the situation by introducing a course in basic mathematics and by giving help to the most promising students.

7. *A useful transformation in the theory of binary star orbits*, by Professor R. H. Wilson, Jr., University of Louisville.

The classical orbital elements for binary stars are polar coordinate parameters; equations using them are trigonometric and ill-adapted to machine computation. New "Theele-Innes" analytical geometrical elements A, B, F, and G are direction numbers of the major and minor axes of the true orbit referred to the south-north and west-east lines. Necessary tables of elliptical rectangular coordinates (X, Y) referred to the major axis and latus rectum of the true orbit have been published. Resulting equations for the apparent rectangular coordinates $a=AX+FY$ and $y=BX+GY$ are simply a transformation of coordinates and are convenient for use with calculating machines.

8. *Analytic extension of functions defined by Dirichlet series*, by Mr. D. C. Rose, University of Kentucky.

The author gave some remarks contrasting the analysis of Taylor series with that of Dirichlet series. He then sketched the method employed to obtain very general results on the analytic extension of functions defined by general Dirichlet series.

9. *The conference at Boulder*, by Professor D. W. Pugsley, Berea College.

General, interesting, and personal remarks concerning the author's stay at the conference made up this paper.

10. *Summations*, by Mr. J. B. Cornelison, University of Kentucky.

Several theorems in sums of products of binomial coefficients such as

$$\sum_{k=p}^{k=n} (-1)^k \binom{k}{p}\binom{n}{k} = 0 \qquad \text{for } p < n,$$

were given. A different method of proof was given for each theorem in order to illustrate a few of the many different methods available.

11. *Some questions of current interest concerning the teaching of mathematics*, by Professor E. A. Cameron, University of North Carolina. (By invitation.)

This invited lecturer discussed each of the elementary college mathematics courses, raising

questions as to content and method for each course. Throughout the paper, new subject matter and new aims and purposes for the courses were stressed.

W. J. ROBINSON, *Secretary*

THE MAY MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The spring meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America was held at the University of Maryland, College Park, Maryland, on May 1, 1954. Professor Daniel C. Lewis, Jr., Chairman of the Section, presided at the morning and afternoon sessions.

There were ninety-six persons in attendance, including the following sixty-eight members of the Association.

J. C. Abbott, D. F. Atkins, R. P. Bailey, David Blackwell, R. O. Blummer, Jr., J. W. Brace, B. H. Buikstra, J. M. Cameron, H. H. Campaigne, B. R. Cato, Jr., H. J. Cheston, Jr., G. R. Clements, E. W. Coffin, Elizabeth H. Cuthill, Melchior Di Carlo-Cottone, R. P. Eddy, Gloria C. Ford, C. H. Frick, S. I. Gass, B. C. Getchell, Mildred W. Going, Michael Goldberg, R. A. Good, E. S. Grable, E. C. Gras, Donald Greenspan, D. W. Hall, A. J. Hoffman, E. C. Hubbard, M. A. Hyman, S. B. Jackson, F. E. Johnston, R. H. Kasriel, L. M. Kells, F. B. Key, Rev. C. F. Koehler, Karl Kozarsky, D. C. Lewis, Jr., D. B. Lloyd, M. M. Lotkin, Elizabeth C. Lukacs, Carol V. McCamman, M. H. Martin, Joseph Milkman, L. I. Mishoe, R. W. Moller, T. W. Moore, Morris Newman, W. H. Norris, P. L. Oglesby, M. W. Oliphant, Walter Penney, G. W. Petrie, III, W. W. Proctor, O. J. Ramler, R. W. Rector, Conrad Rennemann, Jr., J. N. Rice, W. G. Rouleau, J. W. Sawyer, C. H. Sisam, W. A. Soaper, Jr., W. J. Strange, Choy-Tak Taam, J. A. Tierney, P. M. Whitman, J. W. Wrench, Jr., D. M. Young, Jr.

The following officers were elected to serve for a period of one year: Chairman, Professor C. H. Frick, Mary Washington College of the University of Virginia; Vice-Chairmen, Miss Carol V. McCamman, Calvin Coolidge High School, Washington, D. C., and Professor M. W. Oliphant, Georgetown University; Secretary, Professor R. P. Bailey, United States Naval Academy. The section voted that a registration fee of twenty-five cents be charged at future meetings to cover costs of future activities of the section. Professor W. H. Norris reported that approximately 1700 students from 46 high schools were scheduled to take part in the section's first high school contest the following week. The secretary announced that at least six lectures had been given in the lecture series for undergraduates sponsored by the section.

The following papers were presented:

1. *Some practical and theoretical applications of the transportation problem*, by Dr. A. J. Hoffman, National Bureau of Standards, Washington, D. C.

The linear programming problem of minimizing $\sum_{i,j} c_{ij}x_{ij}$, $C=(c_{ij})$ being a given m by n matrix and $X=(x_{ij})$ being a variable m by n matrix with non-negative entries and prescribed row and column sums, is known as the transportation problem. Practical applications include: (1) the cheapest shipment of a commodity from various sources to required destinations, (2) most effective assignment of personnel to tasks, (3) evaluation of bids in the awarding of contracts. Theoretical applications include: (1′) various theorems on eigenvalues and singular values of matrices, (2′) alternative proofs of theorems about distinct representatives of subsets. Some of the problems being solved for government agencies were discussed.

2. *Random numbers for high speed computers*, by Mr. J. M. Cameron and Dr. Morris Newman, National Bureau of Standards, Washington, D. C., presented by Dr. Newman.

The use of random numbers in Monte Carlo methods and the solution of statistical problems by sampling experiments has led to a search for simple methods of generating sequences of random numbers on high speed computers. To be useful, such methods must possess simplicity and speed while satisfying certain statistical criteria. Tests for various methods have been programmed by the Bureau of Standards for their electronic digital computer in Washington. These tests provide checks on frequency distribution, moments, serial correlations, and randomness properties such as number of ascending and descending runs. One method tested, involving additions only, is particularly fast.

3. *On the limit of the coefficients of a certain eigenfunction series*, by Professor Luna Mishoe, Morgan State College.

The problem of expanding a function $f(x)$ of bounded variations over the interval (0, 1) in terms of u_n, the eigenfunctions of $u''+q(x)u+\lambda(p(x)u-u')=0$, $u(0)=u(1)=0$, was studied. The resulting series together with conditions for convergence were given. It was shown that the coefficients of u_n in the series approach zero as n approaches infinity if and only if $f(0)=f(1)=0$ for p and q constant.

4. *The propagation of error in numerical integrations*, by Dr. M. M. Lotkin, Ballistic Research Laboratories, Aberdeen Proving Ground.

From the theoretical as well as practical point of view it is of importance to be aware of the stability properties of numerical methods of integration, and growth of error in the large. Proceeding along lines similar to those employed by H. Radamacher, H. Rutishauser, L. H. Thomas, and others, criteria have been developed for stability of a number of well known methods of integrating ordinary differential equations. Using proper values of the pertinent characteristic equations, expressions were derived for the propagation of error due to truncation and rounding.

5. *Arcs of monotone curvature with prescribed end conditions*, by Professor S. B. Jackson, University of Maryland.

Let A and B be two points of the plane which lie respectively on two directed circles $\mathcal{A}$ and $\mathcal{B}$, a line being considered a special case of a circle. The problem is that of drawing an arc AB whose curvature is monotone and which has the given directed circles $\mathcal{A}$ and $\mathcal{B}$ as osculating circles at the endpoints. Special attention was devoted to the question of the existence of such an arc with, in some sense, a minimum of angular measure. Necessary and sufficient conditions were obtained for the existence of such an arc.

6. *Controlled random walks*, by Professor David Blackwell, Howard University.

This was the invited lecture. In a sequence of plays of a two person finite game with a payoff matrix whose elements are vectors in k-space, the extent to which either player can control the average payoff vector in a long series of plays was investigated. In particular every closed convex set S in k-space has one of two properties: either (1) player I can force the average payoff to approach S with probability 1 as the number of plays becomes infinite, or (2) there is a closed convex set T disjoint from S such that player II can force the average payoff to approach T with probability 1 as the number of plays becomes infinite.

C. H. FRICK, *Secretary*

THE MAY MEETING OF THE MISSOURI SECTION

The annual spring meeting of the Missouri Section of the Mathematical Association of America was held jointly with the Missouri Council of Teachers of Mathematics at the University of Missouri, Columbia, Missouri, on May 7, 1954. Professor W. R. Utz, Vice-Chairman of the Section, and Professor Margaret F. Willerding, Associate Secretary of the Section and Chairman of the Missouri Council of Teachers of Mathematics, presided in turn at the morning session. Professor G. H. Jamison, Chairman of the Section, presided at the afternoon session.

There were eighty persons in attendance, including the following thirty-eight members of the Association:

L. W. Akers, J. J. Andrews, S. Louise Beasley, L. M. Blumenthal, P. B. Burcham, W. A. Couch, Mary L. Cummings, C. H. Dalton, J. F. Daly, H. Margaret Elliott, D. H. Erkiletian, Jr., G. M. Ewing, C. V. Fronabarger, Nola L. A. Haynes, F. F. Helton, N. Q. Hubbard, C. A. Hutchinson, G. H. Jamison, C. A. Johnson, L. O. Jones, C. E. Kelley, R. E. Lee, Adele Leonhardy, F. H. Lloyd, Jessie H. McLean, R. J. Michel, R. R. Middlemiss, Marie A. Moore, L. E. Pummill, R. M. Rankin, Francis Regan, Lois J. Roper, Sister M. Pachomia, J. H. Skelton, Sarah D. Springer, W. A. Vezeau, Margaret F. Willerding, J. L. Zemmer.

At the business meeting the following officers were elected for the coming year: Chairman, Professor Maria Castellani, University of Kansas City; Vice-Chairman, Professor C. H. Dalton, Southeast Missouri State College; Secretary-Treasurer, Sister M. Pachomia, College of St. Teresa. Professor Margaret F. Willerding, Harris Teachers College, was elected to serve as Associate Secretary for a period of five years.

It was voted that the Sectional Governor become a member of the Executive Committee. It was decided that the committee appointed last year to study the requirements of the State Department of Education with respect to mathematics continue to serve and follow up the work done during the past year.

The following recommendations were made by the above committee:

"We recommend the following requirements for certifying teachers of mathematics in high school:

1. At least fifteen semester hours of college mathematics, including five hours of college algebra and trigonometry. The college algebra course shall have a prerequisite of one and one-half units of high school algebra or the equivalent. Five hours of analytic geometry or analytic geometry and calculus may be substituted for the college algebra and trigonometry;

2. Teachers of plane geometry and/or solid geometry shall have at least two and one-half hours of college geometry, modern geometry or advanced geometry;

3. Teachers of trigonometry shall have at least five hours of analytic geometry or analytic geometry and calculus."

The following program was presented:

1. *Some remarks concerning tetrahedra,* by Professor L. M. Blumenthal, University of Missouri.

A sextuple of positive numbers are the lengths of the edges of at most thirty tetrahedra, no two of which are metrically congruent. Procedures for obtaining sextuples for which the maximum is realized (maximal tetrahedral sextuples) were presented. Three problems arise concerning tetrahedral-producing operations on a given pair $T_3(p_1, p_2, p_3, p_4)$, $T_3(q_1, q_2, q_3, q_4)$ of tetrahedra.

(1) Determine functions f (called of Type I) such that four points r_1, r_2, r_3, r_4 of euclidean three-space exist with $r_ir_j = f(p_ip_j, q_iq_j)$, $(i, j = 1, 2, 3, 4)$.

(2) Determine functions f of Type II such that the numbers $f(p_ip_j, q_iq_j)$, $(i, j = 1, 2, 3, 4)$ are tetrahedral.

(3) Determine functions f of Type III such that $f(p_ip_j, \gamma(q_iq_j))$ $(i, j = 1, 2, 3, 4)$ are tetrahedral, where γ is a given correlation between the two sextuples p_ip_j, q_iq_j, $(i, j = 1, 2, 3, 4)$.

It was shown that $(x^2+y^2)^{1/2}$ and $(x+y)^{1/2}$ are functions of Type I, while if $T_3(q_1, q_2, q_3, q_4)$ is regular, then $f(x) = x+k$ is of Type I, where $q_iq_j = k > 0 (i \neq j)$. The function $x+u$ is not of Type I for arbitrary tetrahedras. These results are all valid for n-simplices.

2. *Mathematics in general education,* by Professor J. A. Seeney, Lincoln University, introduced by the secretary.

There are many concepts regarding the nature of General Education. Underlying all these ideas is the notion that we must educate the whole man. Many conflicts are evident today as to what this education should be.

General Education is defined as that education needed for the cohesion of a democratic society. It is concerned with the social needs of the layman. High schools and colleges must carry out this function in addition to others imposed by the nature of the society.

We can best meet the mathematics needed by the layman in a required course in general mathematics. The teachers and experts must determine what mathematics is needed in everyday life and teach it so that it has meaning to the student and stresses the important mathematical relationships.

General mathematics should be required of all students. We have no assurance that mathematical skills will function in everyday life unless we teach it that way.

3. *The Missouri traveling exhibit* by Miss Frances Story, St. Charles High School, introduced by the secretary.

The Missouri Council of Teachers of Mathematics is unique in having prepared a traveling exhibit of teaching aids and ideas that would be helpful to the teaching of mathematics. These aids consist of teaching aids for all levels of mathematics instruction—from charts for teaching the number concept in the primary grades to devices for use in plane geometry, trigonometry and advanced algebra in the secondary school. These aids are either student or teacher made and can be copied with little or no expense by using materials at hand.

A teacher has access to the traveling exhibit by requesting it and paying the mailing expense.

The teachers who have compiled the exhibit do not advocate that teaching aids and ideas, such as those in the exhibit, take the place of the normal classroom methods, but that they be used in addition to the regular teaching procedure.

4. *General mathematics—pupil attitude and teacher understanding,* by Mr. E. J. Jackson, St. Louis Public Schools, introduced by the secretary.

The data from a survey of a large number of ninth grade pupils in St. Louis high schools indicate that General Mathematics is looked upon as a valuable vocational tool as well as an important part of one's general education. These data have implications for teachers of the subject with respect to their techniques and philosophy of the subject.

A study of teacher understanding of arithmetic in the Metropolitan Area of St. Louis reveals a serious lack of understanding of the processes, concepts and short-cuts in the mathematics they teach. The evidence is strong that teachers have learned their arithmetic by rote and teach arithmetic concepts as they learned them.

5. *Mathematics in Missouri schools*, by Mr. I. F. Coyle, State Department of Education, introduced by the secretary.

Curricular requirements in mathematics for pupils in elementary and secondary schools are discussed, as are the requirements for the certification of teachers of mathematics in the public schools.

Comparison is noted between Missouri secondary school mathematics enrollments in 1952–53 and those of 1948–49. The comparison shows a slight falling in mathematics on an elective basis.

Possibilities for improving instruction in mathematics in elementary and high schools are described. The purposes and policies of teacher certification are enumerated, and the procedures for determining certification requirements in Missouri are discussed.

6. *Report of the committee appointed to study improvement of mathematical education* by Professor R. J. Michel, Southeast Missouri State College.

7. *A serious look at college mathematics*, by Professor C. A. Hutchinson, University of Colorado, Boulder, Colorado. (By invitation.)

The speaker discussed informally the present day difficulties of college mathematics programs, with respect to the preparation of students, the courses, the text books, using observations of experience at the University of Colorado as basis.

MARGARET F. WILLERDING, *Associate Secretary*

THE MAY MEETING OF THE WISCONSIN SECTION

The twenty-second annual meeting of the Wisconsin Section of the Mathematical Association of America was held at Wisconsin State College, Eau Claire, Wisconsin, on May 8, 1954. Professor L. P. Wahlstrom, Chairman of the Section, presided.

There were fifty-three present including the following twenty-five members:

R. H. Bing, Leonard Bristow, G. L. Bullis, W. L. Duren, Jr., J. V. Finch, C. E. Flanagan, Harold Glander, W. A. Golomski, E. G. Harrell, R. C. Huffer, E. R. Johnston, A. P. Loomer, C. C. MacDuffee, J. R. Mayor, J. J. McLaughlin, Elli Otteson, C. L. Rich, Sister Mary Corona, Sister M. Domitilla, Sister Mary Felice, Sister Mary Petronia, E. W. Swokowski, C. J. Vanderlin, Jr., R. D. Wagner, and L. F. Wahlstrom.

The following officers were elected for the coming year: Chairman, Dr. A. E. May, Racine Extension of the University of Wisconsin; Vice-Chairman, Mr. C. E. Flanagan, Wisconsin State College, Whitewater; Secretary-Treasurer, Sister Mary Felice, Mount Mary College; Program Committee: Professor R. C. Buck, University of Wisconsin, Professor J. C. Stewart, Lawrence College.

The principal item on the agenda of the business meeting was action on the revision of the by-laws of the section. A committee under the chairmanship of Professor H. P. Pettit presented a proposal for such a revision and the report, read by the secretary in the absence of Professor Pettit, was adopted with a slight amendment.

The program was as follows:

1. *Orientation*, by Dr. Leonard Haas, Dean of Instruction, Wisconsin State College, Eau Claire.

2. Panel discussion: *What is going on in Wisconsin colleges?* by Professor J. R. Mayor, University of Wisconsin, Professor R. C. Huffer, Beloit College, and Dr. J. J. McLaughlin, Wisconsin State College, River Falls.

Professor Mayor stated that the major in mathematics in the School of Education, The University of Wisconsin, is required to complete a year of calculus and four mathematics courses (of three semester hours each) at the junior and senior level. A year's course in physics is also required. The student must maintain a grade-point average of 1.3 (A, 3 points; B, 2 points; and C 1 point) in mathematics courses counted on the major. The methods course is given as an education course. The supervision of student teaching and the methods course are in charge of a staff member with academic rank in both the Department of Mathematics and the Department of Education.

On the basis of questionnaires sent to seventeen private colleges and universities in Wisconsin, Professor R. C. Huffer described a "modal" private college. Deviations from the mode and the variety of course offerings were then discussed. Particular attention was given to introductory and integrated courses in the freshman year, and to courses in statistics and the mathematics of finance.

Questionnaires, similar to those sent by Dr. Huffer, were sent to the State colleges, and the results of these were summarized by Dr. J. J. McLaughlin.

3. Panel discussion: *Articulating the high school and college programs in mathematics*, moderated by Miss Dorothy Sward, Roosevelt Junior High School, Racine; Leaders: Mr. Sidney Ainsworth, Wisconsin High School, Madison; Mr. Laurence Cook, Eau Claire High School; Miss Laura Wagner, Fort Atkinson, Wisconsin; Miss Elli Otteson, Eau Claire High School.

4. *Boundary conditions for any new college mathematics*, by Professor W. L. Duren, Jr., Tulane University. (By invitation.)

The most important of the boundary conditions, which apply to the problem of finding an appropriate form of college mathematics for freshmen, is that the subject must be interesting to young mathematicians, who we hope will teach it, as well as to students. In order to survive in the curriculum, our subject must become a part of the student's working mental equipment. These conditions affecting the cultural approaches to mathematics are compatible with the highest cultural aims.

Educational conditions applying to mathematics are: school-college articulation, accommodation to the abilities of average students, advancement of the more able ones, the training of teachers, and the criterion that it is unwise to force immature students upon entering college to make choices which later will restrict their freedom of development. Science and engineering impose conditions of support for the existing technology and for the attack on unsolved problems of science, particularly in social science, which has been relatively neglected by mathematicians.

All this leads us to try to formulate one universal freshman course which could be supplemented by technical "laboratory" work for particular scientific requirements. The wisdom, insight, writing and teaching of the best mathematicians are needed to guide this movement along lines which are true to the development of mathematics.

5. *Summary of the discussions*, by Professor R. H. Bing, University of Wisconsin.

SISTER MARY FELICE, *Secretary*

EMPLOYMENT OPPORTUNITIES

Continental Oil Company, Ponca City, Okla. *Mathematical Analysis Group* has openings for TWO mathematicians or physicists with advanced degrees and background in applied mathematics, numerical analysis and digital computing. One position involves planning and supervision of numerical analysis and digital computing on reservoir engineering and petroleum problems. The other position involves mathematical analysis of refinery, pipe line, and exploration problems, including linear programming for optimum operations. New laboratory building. Excellent benefits and working conditions. Address letter giving education, experience and salary requirement to Personnel Records Division, Continental Oil Company, Ponca City, Okla.

The MONTHLY is devoting this space to paid announcements of employment opportunities for mathematicians. The text of such announcements should be in want-ad form and must be in the hands of the editor (C. B. Allendoerfer, Mathematics Department, University of Washington, Seattle 5, Wash.) before the first day of the month preceding the issue in which the notice is to appear. In order to conserve space and achieve uniformity, the privilege is reserved to rearrange advertisements. Advertisers will be billed by the Association at the rate of $1.50 per line. Rates for display advertising may be obtained from the Advertising Manager.

CALENDAR OF FUTURE MEETINGS

Thirty-eighth Annual Meeting, University of Pittsburgh, Pittsburgh, Pennsylvania, December 30, 1954.

Thirty-sixth Summer Meeting, University of Michigan, Ann Arbor, Michigan, August 29–30, 1955.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Pittsburgh, Pennsylvania, May, 1955.

ILLINOIS, Monmouth College, Monmouth, May 13–14, 1955.

INDIANA, Butler University, Indianapolis, May, 1955.

IOWA, St. Ambrose College, Davenport, April 15–16, 1955.

KANSAS

KENTUCKY

LOUISIANA-MISSISSIPPI, Buena Vista Hotel, Biloxi, Mississippi, February 18–19, 1955.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Georgetown University, Washington, D.C. December 4, 1954.

METROPOLITAN NEW YORK, Queens College, Flushing, New York, April 30, 1955.

MICHIGAN, Michigan State College, East Lansing, March 26, 1955.

MINNESOTA, University of Manitoba, Winnipeg, October 16, 1954.

MISSOURI, University of Kansas City, Spring, 1955.

NEBRASKA

NORTHERN CALIFORNIA, University of California, Berkeley, January 15, 1955.

OHIO

OKLAHOMA, Oklahoma City University, October 29, 1954.

PACIFIC NORTHWEST, University of British Columbia, Vancouver, June 17, 1955.

PHILADELPHIA, Princeton University, Princeton, New Jersey, November 27, 1954.

ROCKY MOUNTAIN, University of Wyoming, Laramie, Spring, 1955.

SOUTHEASTERN, Tennessee Polytechnic Institute, Cookeville, March 11–12, 1955.

SOUTHERN CALIFORNIA, Santa Monica City College, March 12, 1955.

SOUTHWESTERN, University of New Mexico, Albuquerque, Spring, 1955.

TEXAS, Abilene Christian College, Abilene, April, 1955.

UPPER NEW YORK STATE, University of Buffalo, May 14, 1955.

WISCONSIN, Cardinal Stritch College, Milwaukee, May, 1955.

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 61 NUMBER 9

CONTENTS

NOVEMBER 1954

The AMERICAN MATHEMATICAL MONTHLY

(Founded in 1894 by Benjamin F. Finkel)

EDITORIAL CORRESPONDENCE should be addressed to the Editor, C. B. Allendoerfer, Department of Mathematics, University of Washington, Seattle 5, Washington.

ADVERTISING CORRESPONDENCE should be addressed to J. F. Randolph, Morey Hall, University of Rochester, Rochester 3, N. Y.

NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the Monthly should be sent to the Secretary-Treasurer, H. M. Gehman, University of Buffalo, Buffalo 14, N. Y.

THIS IS THE OFFICIAL JOURNAL OF THE

MATHEMATICAL ASSOCIATION OF AMERICA, INC.

(Devoted to the Interests of Collegiate Mathematics)

Entered as second class matter at the post office at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

Subscription Price: To Members, $4 a Year, To Others, $5 a Year.

Published by the Association at Menasha, Wisconsin, and Seattle, Wash. during the months of January, February, March, April, May, June–July, August–September, October, November, December.

PRINTED IN THE UNITED STATES OF AMERICA

AN ELEMENTARY ANALOGUE TO THE GAUSS-BONNET THEOREM

G. PÓLYA, Stanford University

1. We consider a polyhedral surface P of the topological type of a circular disk, bounded by a (skew) closed polygon R (the rim) without double points. We shall appropriately define S, the *area of the spherical image* of P, and T, *the total change of direction* of R on P, and prove that

$$S + T = 2\pi.$$

This is an elementary analogue to, or an elementary limiting case of, the Gauss-Bonnet theorem which itself can be considered as a limiting case of our elementary theorem (when the faces become infinitely small and their number infinitely large).

Let F, E, and V denote the number of faces, edges, and vertices of P, respectively. Since P is a (simply connected) open polyhedral surface, Euler's well-known theorem assumes here the form

$$F - E + V = 1.$$

Euler's theorem is closely connected with our analogue of the Gauss-Bonnet theorem. In fact, we shall kill two birds with one stone, and derive both theorems together.

2. Our proof needs two well-known lemmas and the introduction of appropriate notation.

LEMMA I. *The sum of the angles in a plane polygon with n sides is $(n-2)\pi$.*

LEMMA II. *We consider the polyhedral angle H and its polar polyhedral angle H'. We call L the sum of the face angles of H, and A' the measure of the solid angle included by H'. Then*

$$A' = 2\pi - L.$$

According to the well-known definition of the polar polyhedral angles, each edge of H' corresponds, and is perpendicular to, a face of H. If we assume for a moment that H and H' have the same vertex and describe a unit sphere about this common vertex as center, each polyhedral angle intersects the sphere in a spherical polygon: the spherical polygon associated with H has the perimeter L and that associated with H' the area A'.

Let F_3, F_4, F_5, $\cdots$ denote the number of the triangular, quadrilateral, pentagonal, $\cdots$ faces of P, respectively. Obviously

$$F_3 + F_4 + F_5 + \cdots = F. \tag{1}$$

Some of the edges of P do not, and others do, belong to the boundary R of P. Let E_i denote the number of the former (interior) edges, and E_b that of the latter (boundary) edges. Obviously

(2) $$E_i + E_b = E.$$

Some of the vertices of P do not, and others do, belong to the boundary R. Let V_i denote the number of the former, V_b that of the latter. Obviously

(3) $$V_i + V_b = V.$$

Let α stand for any angle of any of the F faces of P, $\sum\alpha$ for the sum of all such angles, $\sum_i\alpha$ for the sum of those angles whose vertex does not belong to R, and $\sum_b\alpha$ for the sum of those whose vertex does belong to R. Obviously

(4) $$\sum_i \alpha + \sum_b \alpha = \sum \alpha.$$

It is easily seen that

(5) $$3F_3 + 4F_4 + 5F_5 + \cdots = 2E_i + E_b,$$

(6) $$E_b = V_b,$$

(7) $$\sum \alpha = \pi F_3 + 2\pi F_4 + 3\pi F_5 + \cdots .$$

In deriving (7), we use Lemma I.

3. The total change of direction T of the boundary R of P is the sum of the changes of direction of R at its several vertices. If one of these vertices lies on the boundary of just one of the F faces of P, and the angle of that face at that vertex is α, the change of direction is $\pi-\alpha$. If, however, one of the vertices of R belongs to several contiguous faces of P and these faces have at that vertex the angles α', α'', α''', $\cdots$ respectively, the change of direction is $\pi-\alpha'-\alpha''-\alpha'''-\cdots$. Therefore, the total change of direction is

(8) $$T = \pi V_b - \sum_b \alpha.$$

4. In taking the next step, we restrict ourselves to the most intuitive case. We assume that we started from a convex polyhedron C, drew the not self-intersecting closed polygonal line R on its surface, and obtained P as one of the two portions into which R divides the surface of C. We draw to each of the faces of P a normal (an *outer* normal to C). Then we take a unit sphere about a fixed point O of space, and draw to each normal a parallel radius. These radii intersect the surface of the unit sphere in F points; we shall use these F points in constructing the spherical image of P. In fact, let H be the polyhedral angle (interior to C) included by those faces of P that meet in one of the V_i interior vertices of P. Let us draw H', the polar polyhedral angle to H, with O as vertex. The vertices of the spherical polygon associated with H' are among the F points that we have just constructed. The full spherical image of P consists of such spherical polygons, each corresponding to an interior vertex of P. Computing the area of each of these V_i polygons on the basis of Lemma II, we find that the total area of the spherical image is

$$S = 2\pi V_i - \sum_i \alpha. \tag{9}$$

5. We use first (9), (8), and (4), and we pass to the following lines by using (7), then (5) and (1), and finally (6):

$$\begin{aligned} S + T &= 2\pi V_i + \pi V_b - \sum \alpha \\ &= \pi[2V_i + V_b - (F_3 + 2F_4 + 3F_5 + \cdots)] \\ &= \pi[2V_i + V_b - (2E_i + E_b - 2F)] \\ &= \pi[2F - 2E_i - 2E_b + 2V_i + 2V_b]. \end{aligned}$$

Using (2) and (3) we obtain:

$$S + T = 2\pi[F - E + V]. \tag{10}$$

Now let us observe an important point. There is a "flat" polyhedral surface P_f that has exactly F faces, E edges, and V vertices, all contained in the same plane. (If we have obtained P from a convex polyhedron C by dividing the surface of C into two portions by the closed line R, let us choose any point p on the "other side" of R; that is, p belongs to the surface of C, but does not belong to P or R. Now, P is "fully visible" from p; that is, we can project P from p as center of projection onto a plane in one-one fashion: the projection so obtained can be taken as P_f, which is, in fact, the division of a plane polygon with E_b sides into F polygons. Obviously, we could flatten P into P_f by and by, that is, pass from one to the other by a continuous deformation.) Our foregoing derivation and its result (10) are valid not only for P, but also for P_f. Yet, obviously, the spherical image of P_f reduces to a point, its area S_f is nil, and T_f the total change of direction of the boundary of P_f, is 2π (by Lemma I), and so the left hand side of (10) reduces to $0+2\pi$: yet the right hand side is the same for P_f as for P. Therefore, we obtain at one stroke

$$S + T = 2\pi, \qquad F - E + V = 1.$$

6. In order to apply the variant just proved of the Gauss-Bonnet theorem, let us consider each of the F faces of P as a rigid plate; these plates are joined along the E_i interior edges as with hinges. If some of the V_i interior vertices have more than three edges, the polyhedral surface P, although consisting of rigid plates, may be flexible, of variable shape. When P varies, also the shape of its spherical image may vary. Yet T, the total change of direction along the rim of P, remains unchanged, and so must remain also the *area* $S=2\pi-T$ of the spherical image. This is the elementary counterpart of the *theorema egregium* of Gauss according to which the (Gaussian) curvature of a surface remains unchanged when the surface is bent.*

The scope of the theorem proved would be considerably widened by a clear discussion of the validity of Lemma II for non-convex polyhedral angles.

* *Cf.* D. Hilbert and S. Cohn-Vossen, Anschauliche Geometrie, pp. 172–173.

TWISTED CURVES AND THE MEAN-VALUE PROPOSITION

DAVID DEKKER, University of Washington

1. Introduction. It is well known that the mean-value proposition is true for differentiable plane curves; that is, between any pair of points of a plane differentiable curve there exists a third point at which the tangent is parallel to the chord joining the pair of points. Now for twisted curves no such conclusion can be reached since, for example, twisted cubics and circular helices have no tangents parallel to any chord. Since any twisted sufficiently differentiable curve is locally approximately a twisted cubic, one might expect that the existence of a parallel tangent for each chord implies that the curve is plane.

In what follows a general space curve is represented by a vector $\mathbf{x}=\mathbf{x}(t)$ whose components are given by

$$x^i = x^i(t), \qquad i = 1, 2, 3,$$

where the x^i are rectangular cartesian coordinates in three-dimensional Euclidean space, and the functions $x^i(t)$ are real and single-valued on D, a real domain $a \leqq t \leqq b$. To say that $\mathbf{x}(t)$ belongs to class C^n means that the functions $x^i(t)$ have continuous derivatives of order n.

2. Space curves for which a mean-value proposition in the small is true.

THEOREM. *If*

(1) $\mathbf{x}(t)$ *belongs to* C^3,

(2) *the cross product* $\mathbf{x}' \times \mathbf{x}''$ *does not vanish for any value of* t *in* D,

(3) *for each* t *in* D *there exists a positive* ϵ *such that for* t_1 *and* t_2 *in* D *with* $t_1 < t_2$ *and* $|t_i - t| < \epsilon$ *there exist real numbers* τ *and* λ *for which*

$$\mathbf{x}(t_2) - \mathbf{x}(t_1) = \lambda \mathbf{x}'(\tau) \quad \text{and} \quad t_1 < \tau < t_2,$$

then $\mathbf{x}(t)$ *represents a plane curve.*

Condition (3), the mean-value proposition in the small, simply states that for any pair of points of the curve sufficiently close together there is a point between them on the curve at which the tangent is parallel to their chord.

The theorem is proved by use of the Taylor formulas

$$\mathbf{x}(t) = \mathbf{x}(0) + \mathbf{x}'(0)t + \mathbf{x}''(0)\frac{t^2}{2} + \mathbf{x}'''(0)\frac{t^3}{6} + \mathbf{a}(t)$$

and

$$\mathbf{x}'(t) = \mathbf{x}'(0) + \mathbf{x}''(0)t + \mathbf{x}'''(0)\frac{t^2}{2} + \mathbf{b}(t),$$

where

$$\lim_{t\to 0} \frac{\mathbf{a}(t)}{t^3} = 0 \quad \text{and} \quad \lim_{t\to 0} \frac{\mathbf{b}(t)}{t^2} = 0.$$

Assuming $\mathbf{x}(0)$ vanishes, we have from (3) that

$$\mathbf{x}(t) = \lambda \mathbf{x}'(\tau), \qquad 0 < \tau < t < \epsilon, \tag{2.1}$$

or

$$\mathbf{x}'(0)t + \mathbf{x}''(0)\frac{t^2}{2} + \mathbf{x}'''(0)\frac{t^3}{6} + \mathbf{a}(t) = \lambda\left[\mathbf{x}'(0) + \mathbf{x}''(0)\tau + \mathbf{x}'''(0)\frac{\tau^2}{2} + \mathbf{b}(\tau)\right]. \tag{2.2}$$

Since (2) implies that $\mathbf{x}'(t)$ does not vanish for any value of t in D, it follows from (2.1) that

$$\lim_{t\to 0} \frac{\lambda}{t} = 1.$$

The cross product of both members of (2.2) with $\mathbf{x}'(0)$ gives

$$\begin{aligned} &\mathbf{x}'(0) \times \mathbf{x}''(0)\frac{t^2}{2} + \mathbf{x}'(0) \times \mathbf{x}'''(0)\frac{t^3}{6} + \mathbf{x}'(0) \times \mathbf{a}(t) \\ &\quad = \lambda\left[\mathbf{x}'(0) \times \mathbf{x}''(0)\tau + \mathbf{x}'(0) \times \mathbf{x}'''(0)\frac{\tau^2}{2} + \mathbf{x}'(0) \times \mathbf{b}(\tau)\right]. \end{aligned} \tag{2.3}$$

From condition (2), the fact that τ/t is bounded, and (2.3), we find that

$$\lim_{t\to 0} \frac{\tau}{t} = \frac{1}{2}\cdot$$

The dot product of both members of (2.2) with $\mathbf{x}'(0) \times \mathbf{x}''(0)$ yields

$$\begin{aligned} &(\mathbf{x}'(0)\mathbf{x}''(0)\mathbf{x}'''(0))\frac{t^3}{6} + (\mathbf{x}'(0)\mathbf{x}''(0)\mathbf{a}(t)) \\ &\quad = \lambda\left[(\mathbf{x}'(0)\mathbf{x}''(0)\mathbf{x}'''(0))\frac{\tau^2}{2} + (\mathbf{x}'(0)\mathbf{x}''(0)\mathbf{b}(\tau))\right]. \end{aligned} \tag{2.4}$$

If we suppose that the triple scalar product $(\mathbf{x}'(0)\ \mathbf{x}''(0)\ \mathbf{x}'''(0))$ does not vanish, then from (2.4)

$$\lim_{t\to 0} \frac{\tau}{t} = \frac{1}{\sqrt{3}} \neq \frac{1}{2},$$

a contradiction; therefore, $(\mathbf{x}'(0)\mathbf{x}''(0)\mathbf{x}'''(0)) = 0$. The fact that

$$(\mathbf{x}'(t)\mathbf{x}''(t)\mathbf{x}'''(t)) \equiv 0 \tag{2.5}$$

is verified by repeating the above procedure after making the appropriate translation of coordinates and change of parameter t. From (1), (2), and (2.5) we conclude that $\mathbf{x}(t)$ represents a plane curve [1].

That condition (2) is needed is clear from the example of a curve consisting of the boundary of a surface formed by three mutually perpendicular faces of a cube with the portions near the vertices replaced by plane arcs sufficiently smooth to force the curve to satisfy (1). Such a curve satisfies conditions (1) and (3) but not (2). We also observe that for points far enough apart on this curve the chord has no parallel tangent. Actually if condition (3) is changed to include all chords, then (1) and (2) may be replaced by considerably weaker conditions.

3. Space curves for which a mean-value proposition in the large is true.

THEOREM. *If*

(1) $\mathbf{x}(t)$ belongs to C^1,

(2) $\mathbf{x}'(t)$ *does not vanish for any value of* t *in* D,

(3) *for every* t_1 *and* t_2 *in* D *there exist real numbers* τ *and* λ *for which* τ *is in* D *and*

$$\mathbf{x}(t_2) - \mathbf{x}(t_1) = \lambda \mathbf{x}'(\tau),$$

(4) $\mathbf{y} = \mathbf{x}'(t)/|\mathbf{x}'(t)|$ *for all* t *in* D *gives a subspace* T *of the unit sphere with* $\dim T \leqq 1$,

then $\mathbf{x}(t)$ *represents a plane curve.*

Condition (3), the mean-value proposition in the large, simply states that any chord has a parallel tangent at some point of the curve, while condition (4) states that the spherical indicatrix T of tangents has dimension less than or equal to 1. The definition of dimension used here is the topological invariant defined in [2].

We show first that a curve satisfying (1) and (2) with no two of its tangent lines skew is a plane curve. If no two tangents are skew, then there are only three possibilities. The tangents all lie in the same plane (and the curve is a plane curve) or the tangents all pass through the same point or they are all parallel. In the last two cases with conditions (1) and (2) it can be demonstrated by examining the solutions of certain differential equations that the curve is a straight line segment and hence a plane curve.

If we suppose a curve which satisfies (1), (2), (3), and (4) is not a plane curve, then there exists a pair of points X_0, Y_0 of the curve at which the tangents $\underline{X_0X_0}$, $\underline{Y_0Y_0}$ to the curve are skew. Let arcs $\widehat{PQ}$ and $\widehat{RS}$ of the curve be small and disjoint, with $X_0 \in \operatorname{int} \widehat{PQ}$ and $Y_0 \in \operatorname{int} \widehat{RS}$. Let X be any point of $\widehat{PQ}$ and let X' be the point common to the line $\underline{X_0X_0}$ and the plane on X normal to $\underline{X_0X_0}$. Let Y be any point of $\widehat{RS}$ and let Y' be the point common to the line $\underline{Y_0Y_0}$ and the plane on Y normal to $\underline{Y_0Y_0}$. If XY denotes the point of the unit sphere in the direction from the center parallel to the segment $\overline{XY}$, then the set

H of all XY is a subset of the spherical indicatrix K of chords of the curve. The arcs $\widehat{PQ}$ and $\widehat{RS}$ are closely approximated by the segments $\overline{P'Q'}$ and $\overline{R'S'}$ of the tangents at X_0 and Y_0, since any tangent has a contact of order one (at least) with the curve. Furthermore, any point XY of H will be close to the corresponding point $X'Y'$. The set of all points $X'Y'$ is a subset H' of the unit sphere bounded by four great circle arcs with no interior points in common and with no three of the end points on a single great circle. The four curves on the unit sphere traversed by the points XR, PY, XS, QY, as X and Y traverse the arcs $\widehat{PQ}$ and $\widehat{RS}$, will be close to the four great circle arcs bounding H'. From continuity it is clear that H will cover completely a region bounded by the four curves and will closely approximate the region bounded by the four great circle arcs. Since dim H' is clearly 2, we obtain dim $H=2$. But $H\subset K\subset T$ with dim $T \leqq 1$, which is impossible [**2**, p. 26]. Thus the curve must be plane.

References

1. P. W. Gifford, Jr., Some refinements in the theory of specialized space curves, this MONTHLY, vol. 60, 1953, pp. 384–393.

2. W. Hurewicz and H. Wallman, *Dimension theory*, Princeton University Press, 1941.

UNITARY MULTIPLES OF A MATRIX

B. E. MITCHELL, Louisiana State University

Canonical forms for one-sided multiples of a matrix have been considered from time to time. Examples are Schmidt's Theorem (Turnbull and Aitken [**7**], p. 96), the Hermite form (MacDuffee [**4**], p. 35), and Brenner [**1**]. A slightly changed version of an old theorem is given here. Only $n\times n$ matrices over the complex field and vectors over the n-dimensional complex space are considered, although results similar to those in the first part of this paper could be obtained for $m\times n$ matrices.

Given any two non-zero vectors ξ, η, there exists a non-singular matrix P such that $P\xi=\eta$. To see this let Q be a non-singular matrix with ξ as its first column and R be a non-singular matrix with η as its first column. Then $(RQ^{-1})Q=R$, and so $(RQ^{-1})\xi=\eta$. If in addition ξ and η have the same norm, *i.e.*, $\sqrt{\xi^*\xi}=\sqrt{\eta^*\eta}=1/a$, then $a\xi$ and $a\eta$ are normalized and so we may choose unitary matrices R and Q such that $(RQ^{-1})a\xi=a\eta$. In particular for any vector ξ there exists a unitary matrix U such that $U\xi=[a, 0, \cdots, 0]^T$ where $\xi^*\xi=a^2$ and $a>0$ if $\xi\neq 0$.

McCoy [5]. A more elementary proof of McCoy's Theorem is given by Drazin, Dungey, and Gruenberg [2].

References

1. Brenner, J. L., Post-multiplication by a unitary matrix, Math. Notae 9, 1949, pp. 130–132.
2. Drazin, M. P., Dungey, J. W., and Gruenberg, K. W., Some theorems on commutative matrices, Journ. London Math. Soc., vol. 26, 1951, pp. 221–228.
3. *Lecture notes on a course in matric theory given by Professor C. C. MacDuffee*, Univ. of Wis., S. S., 1950.
4. MacDuffee, C. C., *Vectors and matrices*, Carus Math. Monographs, No. 7, Math. Assoc. of Amer., 1943.
5. McCoy, N. H., On the characteristic roots of matric polynomials, Bull. Amer. Math. Soc., vol. 42, 1936, pp. 592–660.
6. Rademacher, H., *On a theorem of Frobĕnius*, *Studies and Essays*, Interscience Publishers, New York, 1948, pp. 301–305.
7. Turnbull, H. W. and Aitken, A. C., *An introduction to the theory of canonical matrices*, Blackie and Son, Ltd., Glasgow, 1948.

APPROXIMATION FORMULAS FOR ELLIPTIC INTEGRALS

E. C. KENNEDY*

1. Elliptic integrals of the second kind. k^2 near one and ϕ small.

Consider the elliptic integral of the second kind

$$E(\phi, k) = \int_0^{\phi} \sqrt{1 - k^2 \sin^2 \phi}\, d\phi = \int_0^{\phi} \cos \phi \sqrt{1 + h \tan^2 \phi}\, d\phi \tag{1}$$

where $h = 1 - k^2$. Expanding the radical by the Binomial Theorem we get for the integrand

$$\cos \phi \left[1 + \frac{h}{2} \tan^2 \phi - \frac{h^2}{8} \tan^4 \phi + \frac{h^3}{16} \tan^6 \phi - \frac{5h^4}{128} \tan^8 \phi + \frac{7h^5}{256} \tan^{10} \phi - \cdots \right]. \tag{2}$$

For $h \tan^2 \phi < 1$ this series converges absolutely and uniformly and may be integrated termwise, obtaining

$$\begin{aligned} E(\phi, h) = {} & A \sin \phi + B \log_e \tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right) + C \tan \phi \sec \phi \\ & + D \tan \phi \sec^3 \phi + E \tan \phi \sec^5 \phi \cdots \end{aligned} \tag{3}$$

where

* Senior Research Engineer. Consolidated Vultee Aircraft Corporation, Ordnance Aerophysics Laboratory, Daingerfield, Texas.

$$A = 1 - \frac{h}{2} - \frac{h^2}{8} - \frac{h^3}{16} - \frac{5h^4}{128}$$

$$B = \frac{h}{2} + \frac{3h^2}{16} + \frac{15h^3}{128} + \frac{175h^4}{2048}$$

$$C = -\frac{h^2}{16} - \frac{9h^3}{128} - \frac{145h^4}{2048}$$

$$D = \frac{h^3}{64} + \frac{95h^4}{3072}$$

$$E = -\frac{5h^4}{768}.$$

Let E_i be the sum of the first i terms of (3). Let R_i be the numerical value of the remainder associated with E_i. That is, $|E(\phi, h) - E_2| = R_2$, *etc.* From (2) it is obvious that

$$R_2 < \frac{h^2}{8}\int_0^\phi \cos\phi \tan^4\phi\, d\phi = \frac{h^2}{8} I_2$$

$$R_3 < \frac{h^3}{16}\int_0^\phi \cos\phi \tan^6\phi\, d\phi = \frac{h^3}{16} I_3$$

$$R_4 < \frac{5h^4}{128}\int_0^\phi \cos\phi \tan^8\phi\, d\phi = \frac{5h^4}{128} I_4$$

$$R_5 < \frac{7h^5}{256}\int_0^\phi \cos\phi \tan^{10}\phi\, d\phi = \frac{7h^5}{256} I_5.$$

Evidently for k^2 near unity and ϕ small E_2, E_3, E_4, E_5 give increasingly accurate approximations to $E(\phi, k)$. Note in particular the simple and compact formula

$$E_2 = \left(1 - \frac{h}{2}\right)\sin\phi + \frac{h}{2}\log_e \tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right) \tag{5}$$

which gives surprisingly good results for h and ϕ small.

To illustrate the degree of accuracy that might be expected of our formulas we shall approximate $E(\phi, k)$ for $\phi = 30°$ and $k^2 = .92$. Here $h = .08$ and we get

$$E_2 = .501972$$

$$E_3 = .50196475$$

$$E_4 = .5019648160$$

$$E_5 = .50196481508.$$

The last result is in error by about one in the last decimal.

Table I below is useful in estimating the error involved in using E_2, E_3, E_4, and E_5.

TABLE I

ϕ	I_2	I_3	I_4	I_5
18°	.00057	.000046	.000004	.000000
24°	.00279	.000398	.000061	.000011
30°	.00929	.002171	.000555	.000150
36°	.02532	.009199	.003690	.001572
42°	.06111	.037872	.020422	.013273

This table tells us that for $\phi=30°$, say, and any acceptable value of h, the error involved in using E_5 as an approximation to $E(\phi, k)$ is measured by

$$R_5 < \frac{7h^5}{256} I_5 = \frac{7}{256}(.000150)h^5 = .000004h^5.$$

Thus in our problem E_5 gives a result in error by less than $(.000004)(.08)^5 = .000\,000\,000\,02$. In fact, $|E_5 - E(\phi, k)| < .000012$ over the wide range of values: $0 \leqq \phi \leqq 42°$, $.5 \leqq k^2 \leqq 1.0$.

Form (3) is easy to evaluate since very accurate tables of the trigonometric functions are quite common and extensive tables of log tan $(\frac{1}{4}\pi + \frac{1}{2}\phi)$ have been constructed by Legendre and Gudermann and are available, though not exactly common.

Another useful approximation to $E(\phi, k)$ may be obtained by expanding the terms in (2) into a Taylor's series and integrating termwise obtaining

$$E(\phi, h) = \phi + A\phi^3 + B\phi^5 + C\phi^7 + D\phi^9 + E\phi^{11} + \cdots \tag{6}$$

where the coefficients are defined by

$$A = -\frac{1}{6} + \frac{h}{6}$$

$$B = \frac{1}{120} + \frac{h}{60} - \frac{h^2}{40}$$

$$C = -\frac{1}{5040} + \frac{31h}{5040} - \frac{5h^2}{336} + \frac{h^3}{112}$$

$$D = \frac{1}{362880} + \frac{173h}{90720} - \frac{23h^2}{2880} + \frac{h^3}{96} - \frac{5h^4}{1152}$$

$$E = \frac{-1}{39916800} + \frac{25261h}{39916800} - \frac{1063h^2}{266112} + \frac{181h^3}{21120} - \frac{65h^4}{8448} + \frac{7h^5}{2816}.$$

The above form is well adapted to a CPC-IBM machine since the coefficients can be evaluated and the series summed in one operation. Forms (3) and (6)

both give quite accurate approximations to $E(\phi, k)$, for small ϕ and k^2 near one, *i.e.*, h small.

2. Elliptic integrals of the first kind. k^2 near one and ϕ small.

Elliptic integrals of the first kind may be treated in a similar manner. Thus

$$F(\phi, k) = \int_0^\phi \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}} = \int_0^\phi \sec \phi (1 + h \tan^2 \phi)^{-1/2} d\phi, \tag{7}$$

where again, $h = 1 - k^2$.

Expanding the integrand by the Binomial Theorem we get the convergent power series

$$\begin{aligned} \sec \phi \Big[1 - \frac{h}{2} \tan^2 \phi + \frac{3h^2}{8} \tan^4 \phi - \frac{5h^3}{16} \tan^6 \phi \\ + \frac{35h^4}{128} \tan^8 \phi - \frac{63h^5}{256} \tan^{10} \phi + \cdots \Big] \end{aligned} \tag{8}$$

which may be integrated termwise in the region of convergence giving

$$\begin{aligned} F(\phi, h) = A \log_e \tan \left(\frac{\pi}{4} + \frac{\phi}{2} \right) + B \tan \phi \sec \phi + C \tan \phi \sec^3 \phi \\ + D \tan \phi \sec^5 \phi + E \tan \phi \sec^7 \phi + \cdots \end{aligned} \tag{9}$$

where

$$\begin{aligned} A &= 1 + h/4 + 9h^2/64 + 25h^3/256 + 1225h^4/16384 \\ B &= - h/4 - 15h^2/64 - 55h^3/256 - 3255h^4/16384 \\ C &= 3h^2/32 + 65h^3/384 + 5705h^4/24576 \\ D &= - 5h^3/96 - 875h^4/6144 \\ E &= 35h^4/1024. \end{aligned}$$

Letting F_i represent the sum of the first i terms of (9) and R_i^* the associated remainders we have

$$R_2^* < \frac{3h^2}{8} \int_0^\phi \sec \phi \tan^4 \phi d\phi = \frac{3h^2}{8} I_2^*$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$R_5^* < \frac{63h^5}{256} \int_0^\phi \sec \phi \tan^{10} \phi d\phi = \frac{63h^5}{256} I_5^*.$$

A fairly accurate approximation to $F(\phi, h)$ is given by the abbreviated formula

$$F_2 = \left(1 + \frac{h}{4}\right) \log_e \tan \left(\frac{\pi}{4} + \frac{\phi}{2}\right) - \frac{h}{4} \tan \phi \sec \phi. \tag{10}$$

If the F_i are evaluated for $\phi = 30°$, $k^2 = .92$ we get

$$F_2 = .54696$$
$$F_3 = .5469867$$
$$F_4 = .54698621$$
$$F_5 = .5469862140.$$

The table below is useful in getting a bound for the error involved in using F_i as an approximation to $F(\phi, k)$.

TABLE II

ϕ	I_2^*	I_3^*	I_4^*	I_5^*
18°	.00061	.00005	.000004	.000000
24°	.00319	.00046	.000071	.000012
30°	.01146	.00273	.000706	.000193
36°	.03452	.01289	.005263	.002270
42°	.09460	.05391	.033695	.022273

Thus for $\phi = 30°$, say, we have

$$R_2^* < \frac{3h^2}{8} (.01146) = (.0043)h^2$$

and

$$R_5^* < \frac{63h^5}{256} I_5^* = (.000046)h^5.$$

For the relative wide range ($0 \leqq \phi \leqq 36°$, $.7 \leqq k^2 \leqq 1.0$) the table shows that $|F(\phi, k) - F_5| < .000015$.

If (8) is expanded into a Taylor's series and integrated termwise we get

$$F(\phi, h) = \phi + A\phi^3 + B\phi^5 + C\phi^7 + D\phi^9 + \cdots \tag{11}$$

where

$$A = (1/2 - h/2)/3$$
$$B = (5/24 - 7h/12 + 3h^2/8)/5$$
$$C = (61/720 - 331h/720 + 11h^2/16 - 5h^3/16)/7$$
$$D = (277/8064 - 3071h/10080 + 249h^2/320 - 25h^3/32 + 35h^4/128)/9.$$

Both (9) and (11) give good results when h and ϕ are small.

3. Elliptic integrals of the second kind. k^2 small.

If k^2 is small (say, $k^2<.5$) and ϕ large (say, $\phi>\pi/4$) the above approximations may be unsatisfactory. In such a case we transform (1) as follows:

$$
\begin{aligned}
E(\phi, k) &= \int_0^\phi \sqrt{1 - k^2 \sin^2 \phi}\, d\phi = \int_0^\phi \sqrt{1 - \frac{k^2(1 - \cos 2\phi)}{2}}\, d\phi \\
&= \frac{1}{\sqrt{1+M}} \int_0^\phi \sqrt{1 + M \cos 2\phi}\, d\phi \\
&= \frac{1}{2\sqrt{1+M}} \int_0^{x=2\phi} \sqrt{1 + M \cos x}\, dx,
\end{aligned} \tag{12}
$$

where $M=k^2/(2-k^2)$.

Expanding the radical by the Binomial Theorem and integrating termwise we get

$$
\begin{aligned}
E(x, M) = \frac{1}{2\sqrt{1+M}} \Big[& I_0 + \frac{M}{2} I_1 - \frac{M^2}{2^3} I_2 + \frac{M^3}{2^4} I_3 - \frac{5M^4}{2^7} I_4 + \frac{7M^5}{2^8} I_5 \\
& - \frac{21M^6}{2^{10}} I_6 + \frac{33M^7}{2^{11}} I_7 - \frac{429M^8}{2^{15}} I_8 + \frac{715M^9}{2^{16}} I_9 - \frac{2431M^{10}}{2^{18}} I_{10} \\
& + \frac{4199M^{11}}{2^{19}} I_{11} - \frac{29393M^{12}}{2^{22}} I_{12} + \frac{52003M^{13}}{2^{23}} I_{13} \\
& - \frac{185725M^{14}}{2^{25}} I_{14} + \cdots \Big]
\end{aligned} \tag{13}
$$

where

$$
\begin{aligned}
I_0 &= \int_0^x dx, \qquad I_1 = \int_0^x \cos x dx = \sin x \\
I_2 &= \int_0^x \cos^2 x dx = \frac{\sin x \cos x}{2} + \frac{x}{2}, \\
I_3 &= \int_0^x \cos^3 x dx = \frac{\sin x \cos^2 x}{3} + \frac{2}{3} I_1, \\
&\cdots\cdots\cdots\cdots \\
I_n &= \int_0^x \cos^n x dx = \frac{\sin x \cos^{n-1} x}{n} + \frac{n-1}{n} I_{n-2}.
\end{aligned}
$$

Since I_n approaches zero monotonically, it is obvious that (13) converges rapidly for $M \ll 1$. The integrals are easy to evaluate since they depend on a recursion formula.

In the case of the complete elliptic integral we have

$$
\begin{aligned}
E(M) = \frac{\pi}{2\sqrt{1+M}} [1 &- .0625M^2 - .0146484375M^4 - .0064086914M^6 \\
&- .0035798550M^8 - .0022821575M^{10} - .0015808695M^{12} \\
&- .0011594388M^{14} - \cdots].
\end{aligned} \tag{14}
$$

Series (14) converges rapidly for relatively large values of k^2. Thus for $k^2=.5$ $(M=1/3)$ we get $E_2=1.3509$, $E_3=1.35066, \cdots E_7=1.3506438818$, a result in error by possibly one in the last decimal place. For $k^2=.75$ $(M=.6)$ the series begins to converge rather slowly. It gives $E_8=1.2110564$, a result in error by .0000004.

4. Elliptic integrals of the first kind. k^2 small.

In this case

$$
F(\phi, k) = \int_0^{\phi} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}} = \frac{\sqrt{1+M}}{2}\int_0^{x=2\phi} \frac{dx}{\sqrt{1+M\cos x}}, \tag{15}
$$

where again $M=k^2/(2-k^2)$.

Proceeding as before we get

$$
\begin{aligned}
F(x, M) = \frac{\sqrt{1+M}}{2}\Bigg[I_0 &- \frac{M}{2}I_1 + \frac{3M^2}{2^3}I_2 - \frac{5M^3}{2^4}I_3 + \frac{35M^4}{2^7}I_4 \\
&- \frac{63M^5}{2^8}I_5 + \frac{231M^6}{2^{10}}I_6 - \frac{429M^7}{2^{11}}I_7 + \frac{6435M^8}{2^{15}}I_8 \\
&- \frac{12155M^9}{2^{16}}I_9 + \frac{46189M^{10}}{2^{18}}I_{10} - \frac{88179M^{11}}{2^{19}}I_{11} \\
&+ \frac{676039M^{12}}{2^{22}}I_{12} - \cdots \Bigg]
\end{aligned} \tag{16}
$$

where the I_n are exactly those given in series (13). For the complete integral we have

$$
\begin{aligned}
F(M) = \frac{\pi\sqrt{1+M}}{2} [1 &+ .1875M^2 + .1025390625M^4 + .070495605M^6 \\
&+ .0536978M^8 + .043361M^{10} + .03636M^{12} + \cdots].
\end{aligned} \tag{17}
$$

Formulas (14) and (16) converge far more rapidly than do formulas (524) and (525) in *A Short Table of Integrals* by Peirce.

MATHEMATICAL NOTES

EDITED BY F. A. FICKEN, University of Tennessee

Material for this department should be sent to F. A. Ficken, University of Tennessee, Knoxville, Tenn.

AN INEQUALITY FOR CONVEX FUNCTIONS

E. M. WRIGHT, University of Aberdeen, Scotland

In what follows, m and n are positive integers, x a non-negative real number and $f(x)$ a real function of x. We call the set of real numbers $a_1, \cdots, a_m$ an m-set if

$$a_1 \geqq a_2 \geqq a_3 \geqq \cdots \geqq a_m \geqq 0.$$

All our subsequent statements are true if no further restrictions are imposed on x and a's or if, on the other hand, for some fixed $b>0$, we insist that $x \leqq b$ and $a_1 \leqq b$; that is, we may work indifferently in a fixed finite interval or in the semi-infinite interval $x \geqq 0$.

We use $P(m)=P(m,f)$ to denote the proposition (true for some f, false for others):

For every m-set

$$f(a_1) - f(a_2) + \cdots + f(a_m) \geqq f(a_1 - a_2 + \cdots + a_m) \; (m \text{ odd}),$$
$$f(a_1) - f(a_2) + \cdots - f(a_m) \geqq f(a_1 - a_2 + \cdots - a_m) - f(0) \; (m \text{ even}).$$

Payne and Weinstein conjectured and Weinberger [*Proc. Nat. Acad. Sci.*, vol. 38, 1952, pp. 611–613] recently proved the special case of $P(m,f)$ in which $f(x)=x^r$ and $r \geqq 1$. In an abstract of Weinberger's paper [*Math. Reviews*, vol. 14, 1953, p. 24], Bellman supplied a simple proof that $P(m,f)$ (in a slightly weakened form) is true for any continuously differentiable convex f. Both Weinberger's and Bellman's proofs depend on differentiation. In fact, $P(m,f)$ for any continuous convex function f is a special case of Theorem 108 of Hardy, Littlewood and Pólya's *Inequalities* (Cambridge, 1934, hereafter referred to as HLP).

My object here is to show that the $P(m,f)$ for any given f have a very simple logical relationship among themselves, independent of any ideas of differentiation or of continuity. We use capital letters to denote propositions. $A+B \rightarrow C$ means "if A and B are both true, then C is true," and $A \equiv B$ means "$A \rightarrow B$ and $B \rightarrow A$."

All our statements refer to a fixed function f, which need not be continuous. We observe that nothing is changed in $P(m,f)$ if we replace every f by $f-c$ for some c independent of x. Hence, without loss of generality, we may suppose $f(0)=0$. For such an f, $P(m,f)$ reads

For every m-set,

$$\sum_{k=1}^{m} (-1)^{k-1} f(a_k) \geqq f\left(\sum_{k=1}^{m} (-1)^{k-1} a_k \right).$$

$P(1, f)$ is trivially true. If we put $a_{m+1}=0$, we see that

$$P(m+1, f) \to P(m, f). \tag{1}$$

Next, if we assume $P(m, f)$ and $P(3, f)$ and use them in succession, we have

$$\begin{aligned} \sum_{k=1}^{m+2} (-1)^{k-1} f(a_k) &\geqq f(a_1) - f(a_2) + f\left(\sum_{k=3}^{m+2} (-1)^{k-1} a_k \right) \\ &\geqq f\left(\sum_{k=1}^{m+2} (-1)^{k-1} a_k \right), \end{aligned}$$

and this is $P(m+2, f)$. Hence

$$P(3, f) + P(m, f) \to P(m+2, f). \tag{2}$$

Using (2) to establish an obvious induction we have $P(3, f) \to P(2n+1, f)$ and $P(3, f)+P(2, f) \to P(2n, f)$, for every positive integer n. By (1), however, we see that $P(3, f) \to P(2, f)$ and so

$$P(3, f) \to P(m, f), \qquad (m \geqq 1). \tag{3}$$

By repeated use of (1), we see that

$$P(m, f) \to P(3, f), \qquad (m \geqq 3) \tag{4}$$

and so

$$P(m, f) \equiv P(3, f), \qquad (m \geqq 3). \tag{5}$$

Finally, while $P(3, f) \to P(2, f)$, the converse is false. For example, let $f(x)$ be defined as

$$0 \ (0 \leqq x < 1), \qquad x - 1 \ (1 \leqq x < 2), \qquad 1 \ (2 \leqq x < 3), \qquad x - 2 \ (x \geqq 3).$$

Then $P(2, f)$ is true, as may be easily seen from a figure. But $P(3, f)$ is false; for example,

$$f(3) - f(2) + f(1) = 1 - 1 + 0 = 0, \qquad f(3 - 2 + 1) = f(2) = 1.$$

Hence $P(3, f)$ is the fundamental inequality. If we put $a_1 = x_1 + \delta$, $a_2 = x_1$, $a_3 = x_2$, we see that $P(3, f)$ is equivalent to

$$f(x_1 + \delta) - f(x_1) \geqq f(x_2 + \delta) - f(x_2) \tag{6}$$

for all $\delta > 0$ and all $x_1 \geqq x_2 \geqq 0$. This may be taken as a definition of convexity, in which case we have shown that, for any $m \geqq 3$, the truth of $P(m, f)$ is a necessary and sufficient condition for f to be convex.

HLP take

$$f(x_3) + f(x_4) \geqq 2f\left(\frac{x_3 + x_4}{2}\right) \tag{7}$$

for all $x_3 \geqq 0$, $x_4 \geqq 0$ as the definition of convexity. Relation (6) implies (7) and, for continuous $f(x)$, (7) implies (6) (HLP Theorem 86). If there is any $f(x)$ which satisfies (7) and not (6), it must be discontinuous and so (HLP Theorem 111) unbounded in any finite interval. The only known examples of discontinuous convex functions depend for their construction on Zermelo's Axiom of Choice (HLP p. 96). These examples satisfy both (6) and (7), so that whether functions satisfying (7) and not (6) exist is unknown, even if Zermelo's axiom is true.

A NOTE ON COMPLETE RESIDUE SYSTEMS

W. J. Coles and F. R. Olson, Duke University

The following is known [1]:

Theorem. *If m is an integer $\geqq 3$, and if $\{a_i\}$, $\{b_i\}$, $i=1, \cdots, m$, are two complete residue systems (mod m), then $\{a_i\, b_i\}$ is not a complete residue system (mod m).*

We here offer a simpler proof.

The theorem is well-known [2] for $m=p$ an odd prime. Indeed, excluding the zero element, by Wilson's Theorem the product of the remaining elements of a complete residue system is congruent to -1 (mod p); since this is true for $\{a_i\}$, $\{b_i\}$, it must fail for $\{a_ib_i\}$.

Evidently the theorem holds for $m=4$. We proceed by multiplicative induction. We assume the theorem for m arbitrary, $m \geqq 3$. Let p be an arbitrary prime; suppose that $\{a_ib_i\}$, $i=1, \cdots, mp$, is a complete residue system (mod mp). Now, every complete residue system (mod mp) contains exactly m multiples of p. Hence $p \mid a_i$ if and only if $p \mid b_i$, else $\{a_ib_i\}$ will contain more than m multiples of p. We may suppose $a_j = a_j' p$, $b_j = b_j' p$, $j=1, \cdots, m$, where $\{a_j'\}$, $\{b_j'\}$ form complete residue systems (mod m). Thus the set $\{a_ib_i\}$ contains the set $\{a_j' b_j' p^2\}$. The incongruence of the elements $a_j' b_j' p^2$ (mod mp) implies the incongruence of the elements $a_j' b_j' p$ (mod m), which implies the incongruence of the elements $a_j' b_j'$ (mod m). This is contrary to our hypothesis on m; hence the supposition that $\{a_ib_i\}$ is a complete residue system (mod mp) is false, and the theorem is established.

References

1. S. Chowla and T. Vijayaraghavan, On complete residue systems, Quart. J. Math., vol. 19, pp. 193–194.

2. G. Pólya and G. Szegö, Aufgaben und Lehrsätze aus der Analysis, vol. 2, chapter 8, problem 245, pp. 158, 379.

SOME TRIGONOMETRIC, HYPERBOLIC AND ELLIPTIC APPROXIMATIONS

J. S. FRAME, Michigan State College

In this paper we shall present some close inequalities involving trigonometric and hyperbolic functions, and some estimates for the incomplete elliptic functions of first and second kind.

When the prismoid formula is used to evaluate the average value of $f(x) = \cos x$ on the interval $-\theta \leqq x \leqq \theta$ we obtain

$$\frac{\sin \theta}{\theta} = \frac{1}{2\theta}\int_{-\theta}^{\theta} \cos x dx \sim \frac{\cos \theta + 4 + \cos \theta}{6} = \frac{2 + \cos \theta}{3}. \tag{1}$$

This approximation is equivalent to approximating a small angle θ in a right triangle by $3 \sin \theta/(2+\cos \theta)$, and thus leads to the formula [4]

$$A^\circ = \frac{a}{2c + b} 172^\circ \tag{2}$$

for the smallest angle A° in a right triangle with sides $a<b<c$. W. A. Hurwitz [7] pointed out that this remarkable formula (2) was published in the works of Nicolaus Cusanus [3] in 1514, and in 1699 by Ozanam [8]. Many approximations for arcs and chords of circles, such as those given by J. M. Bruce [1, 2], are based either directly on (1) or are equivalent to it.

The accuracy of (1) can be considerably improved by taking account of its error of about $\theta^4/180$ in the following way. We write

$$1 - \frac{\sin \theta}{\theta} = \frac{1 - \cos \theta}{3 - q\theta^2/10} \tag{3}$$

where q is a function of θ. We find that q varies monotonically from 1 to $10/\pi^2 = 1.013212$ as θ varies from 0 to π, and thus obtain the strong inequalities

$$\frac{\sin \theta(3 - \theta^2/10)}{2 + \cos \theta - \theta^2/10} < \theta < \frac{\sin \theta(3 - \theta^2/\pi^2)}{2 + \cos \theta - \theta^2/\pi^2}, \qquad \text{for } 0 < \theta < \pi. \tag{4}$$

For $\theta = \pi/2$ and $\theta = \pi/3$ these inequalities for $\theta/\sin \theta$ become

$$\frac{3 - \pi^2/40}{2 - \pi^2/40} = \frac{11.1304}{7.1304} < \frac{\pi}{2} < \frac{11}{7}, \tag{4a}$$

$$\frac{3 - \pi^2/90}{2.5 - \pi^2/90} = \frac{52.2608}{43.2608} < \frac{2\pi}{3\sqrt{3}} < \frac{52}{43}. \tag{4b}$$

Actually the approximation $78/43$ for $\pi/\sqrt{3}$ is the best rational approximation for this number with denominator less than 100, and its relative error is less than 0.0001.

To establish the fact that the function $q(\theta)$ in (3) increases monotonically as θ varies from 0 to π, and then decreases as θ varies from π to 2π, we may use the

following device. We first differentiate the expression

$$\frac{q}{10}=\frac{3}{\theta^2}-\frac{1-\cos\theta}{\theta(\theta-\sin\theta)} \tag{5}$$

and then multiply by the factor $\theta^2(\theta-\sin\theta)^2/\sin\theta$ in order to simplify the denominator and remove the root of $dq/d\theta$ at $\theta=\pi$. We then expand the resulting function $\phi(\theta)$ as an infinite sum of functions that are each positive on the interval $0<\theta<2\pi$, as follows:

$$\begin{aligned}\phi(\theta) &= \frac{\theta^2(\theta-\sin\theta)^2}{10\sin\theta}\frac{dq}{d\theta} \\ &= (\cos\theta-1+\theta^2/2)+6\{1-(\sin\theta)/\theta\} \\ &\quad +6\{1-(\theta/2)\cot(\theta/2)\}-3\theta^2/2 \\ &= (48-\theta^2)\theta^8/10!+\sum_{n=3}^{\infty}p_n(\theta),\end{aligned} \tag{6}$$

where the even polynomials $p_n(\theta)$ defined by

$$\begin{aligned}p_n(\theta) &= \frac{\theta^{4n}}{(4n)!}-\frac{\theta^{4n+2}}{(4n+2)!}+\frac{6\theta^{4n-2}}{(4n-1)!}-\frac{6\theta^{4n}}{(4n+1)!} \\ &\quad +6B_{2n-1}\frac{\theta^{4n-2}}{(4n-2)!}+6B_{2n}\frac{\theta^{4n}}{(4n)!}\end{aligned} \tag{7}$$

are positive on the interval $0<\theta<2\pi$, for $n>1$. The positive coefficients B_n that appear in the expansion of $1-(\theta/2)\cot(\theta/2)$ are the Bernoulli numbers 1/6, 1/30, 1/42, 1/30, 5/66, *etc.* Since $\phi(\theta)$ is positive, it follows that $dq/d\theta$ has the same sign as $\sin\theta$ on the interval $0<\theta<2\pi$.

In the second inequality of (4) we set $\theta=\pi x$ and examine the error by studying the function $\epsilon(x)$:

$$\epsilon(x)=\frac{\sin\pi x}{\pi x}+\frac{1-\cos\pi x}{3-x^2}-1\sim\frac{(x^2-x^4)^2}{143-20x^2}, \qquad 0<x^2<1. \tag{8}$$

The right member of (8), like $\epsilon(x)$, vanishes to the fourth order for $x=0$, and to the second order for $x=\pm1$, and when the ratio $(x^2-x^4)^2/\epsilon(x)$ is plotted against $u=x^2$, it is closely fitted by the line $143-20u$ for values of u between 0 and 1. Thus the maximum value of $\epsilon(x)$ on $[0, 1]$ is only about 0.0004.

Similar inequalities involving hyperbolic functions [6] arise when we change the sign of θ^2 in (3). We have

$$\frac{\sinh u}{u}-1=\frac{\cosh u-1}{3+qu^2/10}, \qquad 0<q\leqq 1, \tag{9}$$

where $q\to1$ when $u\to0$; but here q decreases from 1 to about 0.91 as u increases

from 0 to 5. Hence

$$\frac{\sinh u(3 + u^2/11)}{2 + \cosh u + u^2/11} < u < \frac{\sinh u(3 + u^2/10)}{2 + \cosh u + u^2/10}, \text{ for } 0 < u < 5. \tag{10}$$

The substitution $u = \ln z$ gives inequalities for estimating logarithms:

$$\frac{(z^2 - 1)(3 + u^2/11)}{z^2 + 4z + 1 + 2zu^2/11} < \ln z < \frac{(z^2 - 1)(3 + u^2/10)}{z^2 + 4z + 1 + 2zu^2/10}, \qquad u = \ln z < 5. \tag{11}$$

Thus for example, for $z = 2$ we have

$$\frac{9 + 3u^2/11}{13 + 4u^2/11} < \ln 2 < \frac{9 + 3u^2/10}{13 + 4u^2/10}, \quad u = \ln 2 = .693147, \tag{11a}$$

and the approximation $9/13 = 0.6923$ is too small. But if we substitute $u = 9/13$ in the right member of (11a) we obtain the value 0.693146, with error about 10^{-6}.

We now attempt to extend an approximation like (3) to elliptic functions F and E of the first and second kinds:

$$F = \int_0^\theta \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}}, \qquad E = \int_0^\theta \sqrt{1 - k^2 \sin^2 \phi}\, d\phi. \tag{12}$$

Clearly for $k^2 = 0$, both F and E reduce to θ. When the elliptic functions $\sin \theta = \operatorname{sn} F$, $\cos \theta = \operatorname{cn} F$, $(1 - k^2 \sin^2 \theta)^{1/2} = \operatorname{dn} F$ are expanded as power series in F we observe that in the series for $(\operatorname{sn} F)/F$ the coefficients of F^{2n} are reciprocal polynomials of degree n in k^2, so that $(\operatorname{sn} F)/F$ is unchanged by the substitution $(k, F) \to (1/k, kF)$. However, this substitution interchanges the functions $\operatorname{cn} F$ and $\operatorname{dn} F$. Hence we try a modification of (3) in the form

$$1 - \frac{\operatorname{sn} F}{F} \sim \frac{1 - \operatorname{cn} F}{3 - aF^2/10} + \frac{1 - \operatorname{dn} F}{3 - bF^2/10} \tag{13}$$

and find that

$$a + k^2 b = 1 - k^2 + k^4, \qquad b(k) = k^2 a(1/k). \tag{14}$$

The necessary conditions (14) are met by taking $a = 1 - k^2/2$, $b = k^2 - 1/2$, or $a = b = (1 - k^2 + k^4)/(1 + k^2)$. We find that the function

$$\frac{\operatorname{sn} F}{F} + \frac{1 - \operatorname{cn} F}{3 - F^2(2 - k^2)/20} + \frac{1 - \operatorname{dn} F}{3 - F^2(2k^2 - 1)/20} - 1 \tag{15}$$

behaves like $\theta^6(-1 + 12k^2 + 12k^4 - k^6)/15(7!)$ for θ near 0, and is near zero for all values of F between 0 and π. (It is not near 0 for large F, however.)

Nearly as good is the approximation obtained from (13) taking $a = b$, which gives a formula similar to (4)

$$
(16)\qquad F = \int_0^\theta (1 - k^2 \sin^2 \phi)^{-1/2} d\phi = \frac{\sin \theta(3 - a\theta^2/10)}{1 + (1 - k^2 \sin^2 \theta)^{1/2} + \cos \theta - a\theta^2/10}, \qquad a \sim \frac{1 - k^2 + k^4}{1 + k^2}.
$$

A similar result obtained for elliptic integrals of the second kind is

$$
(17)\qquad E = \int_0^\theta (1 - k^2 \sin^2 \phi)^{1/2} d\phi = \frac{\sin \theta(3 - q\theta^2/10)}{3 - (1 - k^2 \sin^2 \theta)^{1/2} + \cos \theta - q\theta^2/10}, \qquad q \sim 2 + k^2 - (1 - k^2 \sin^2 \theta).
$$

Formula (17) reduces to an identity for $k^2 = 1$, whereas formula (16) may be refined to the inequality

$$
(18)\qquad \frac{\sin \theta(3 - \theta^2/20)}{1 + 2\cos \theta - \theta^2/20} < \int_0^\theta \sec \phi d\phi = \ln(\sec \theta + \tan \theta) < \frac{\sin \theta(3 - \theta^2/10)}{1 + 2 \cos \theta - \theta^2/10}, \qquad 0 < \theta < \frac{5\pi}{12}.
$$

Thus the meridian distance on a Mercator chart from the equator to a latitude $\theta < 57°$ may be approximated by $3 \sin \theta/(1+2 \cos \theta)$ with about the same relative error ($<1\%$) that θ is approximated by $3 \sin \theta/(2+\cos \theta)$.

References

1. J. M. Bruce, Approximation to a circular arc, this MONTHLY, vol. 49, 1942, pp. 184–185.
2. ———, Approximation to a central angle, this MONTHLY, vol. 50, 1943, pp. 42–45.
3. N. Cusanus, De Mathematica Perfectione, Opera, Paris, 1514.
4. J. S. Frame, Solving a right triangle without tables, this MONTHLY, vol. 50, 1943, pp. 622–623. Also, Historical note, this MONTHLY, vol. 51, 1944, p. 210.
5. ———, A trigonometric approximation, this MONTHLY, vol. 53, 1946, p. 454.
6. ———, An approximation for the dip of a catenary, Pi Mu Epsilon Jour., vol. 1, April 1950, pp. 44–46.
7. W. A. Hurwitz, Editorial comment, this MONTHLY, vol. 27, 1920, pp. 365–366.
8. R. A. Johnson, Determination of an angle of a right triangle without tables, this MONTHLY, vol. 27, 1920, p. 368.

AN INTEGRAL INEQUALITY

KY FAN, University of Notre Dame, and G. G. LORENTZ, University of Toronto

1. In a recent note in this Monthly [4] one of the authors gave necessary and sufficient conditions for a continuous function $\Phi(t, u_1, \cdots, u_n)$, $0 \leqq t \leqq 1$, $u_i \geqq 0$, in order that

$$
(1)\qquad \int_0^1 \Phi(t, f_1, \cdots, f_n) dt \leqq \int_0^1 \Phi(t, f_1^*, \cdots, f_n^*) dt
$$

for each set of positive bounded measurable functions $f_i(t)$, namely (5) and (6) below. By $f_i^*(t)$ we denote the decreasing rearrangement of $f_i(t)$. Two particular cases of (1) were given before by Ruderman [5]. In this note we discuss similar inequalities which govern the relation $f \prec g$. Theorem 1 gives necessary and sufficient conditions for a continuous function $\Phi(t, u_1, \cdots, u_n)$, defined for $0 \leqq t \leqq 1$ and all values of the u_i, in order that

$$\int_0^1 \Phi(t, f_1, \cdots, f_n)dt \leqq \int_0^1 \Phi(t, g_1, \cdots, g_n)dt \tag{2}$$

for each system of decreasing bounded functions f_i, g_i where $f_i \prec g_i$, $i = 1, \cdots, n$. The relation $f \prec g$ (compare [3, p. 45]) means that

$$\int_0^x f(t)dt \leqq \int_0^x g(t)dt, \qquad 0 \leqq x \leqq 1, \tag{3}$$

and

$$\int_0^1 f(t)dt = \int_0^1 g(t)dt. \tag{4}$$

(Sometimes condition (4) is omitted in this definition, see §3.) If f_i is decreasing, then $f_i = f_i^*$, so that (2) is *not* a generalization of (1). Nevertheless the conditions on Φ in (2) are exactly the same as in (1), except that the u_i now may be negative and Φ must be convex in each of the u_i. If Φ satisfies all these conditions, we have as a combination of (1) and Theorem 1, that relation (2) holds if $f_i^* \prec g_i$, $i = 1, \cdots, n$ and the g_i are decreasing. We indicate several special cases of (2) in §3.

2. We shall often omit those arguments of Φ which are the same in a certain formula, and, for a given i, $1 \leqq i \leqq n$, shall use the notation U_I for the set of all u_j with $j \neq i$. In [4] it has been shown that necessary and sufficient conditions for Φ in (1) are

$$\Phi(u_i + h, u_j + h) - \Phi(u_i + h, u_j) - \Phi(u_i, u_j + h) + \Phi(u_i, u_j) \geqq 0, \tag{5}$$

$$\int_0^\delta \{\Phi(a + \delta + t, u_i) - \Phi(a + \delta + t, u_i + h) + \Phi(a + t, u_i + h) - \Phi(a + t, u_i)\}dt \geqq 0, \tag{6}$$

for all $0 \leqq a \leqq 1 - 2\delta$, $\delta > 0$, $u_k \geqq 0$, $k = 1, \cdots, n$, $h \geqq 0$ and $i \neq j$. We shall also use the condition of convexity

$$\Phi(u_i + h) - 2\Phi(u_i) + \Phi(u_i - h) \geqq 0 \tag{7}$$

for $i = 1, \cdots, n$.

THEOREM 1. *Conditions* (5), (6) *and* (7) (*with the u_k now of arbitrary sign*) *are necessary and sufficient in order that Φ satisfies* (2).

PROOF. (a) *Necessity*. Let $0 \leqq a < a+2\delta \leqq 1$, $i \neq j$, $h_i \geqq 0$, $h_j \geqq 0$, u_k, $k=1, \cdots, n$ arbitrary. We put

$$f_i(t) = \begin{cases} u_i + h_i, & 0 \leqq t \leqq a \\ u_i, & a < t \leqq a + 2\delta, \\ u_i - h_i, & a + 2\delta < t \leqq 1, \end{cases} \qquad g_i(t) = \begin{cases} u_i + h_i, & 0 \leqq t \leqq a + \delta \\ u_i - h_i, & a + \delta < t \leqq 1, \end{cases}$$

$$f_j(t) = g_j(t) = \begin{cases} u_j + h_j, & 0 \leqq t \leqq a + \delta \\ u_j, & a + \delta < t \leqq 1, \end{cases}$$

$$f_k(t) = g_k(t) = u_k, \quad 0 \leqq t \leqq 1, \ k \neq i, \ k \neq j.$$

Then $f_k \prec g_k$ for $k=1, \cdots, n$. The inequality (2) reduces in this case to

$$\begin{aligned} (8) \quad \int_0^\delta \{ &\Phi(a + t, u_i + h_i, u_j + h_j) - \Phi(a + t, u_i, u_j + h_j) \\ &+ \Phi(a + \delta + t, u_i - h_i, u_j) - \Phi(a + \delta + t, u_i, u_j) \} dt \geqq 0. \end{aligned}$$

Putting here $h_j = 0$, dividing by δ and making $\delta \to 0$, we obtain (7).

To deduce (6), we recall that a convex continuous function is an integral of an increasing function. Let

$$\Phi(t, u_1, \cdots, u_n) = \int_0^{u_i} \phi(t, v, U_J) dv. \tag{9}$$

Inequality (8) for $h_i = h$, $h_j = 0$ may be written in the form

$$\int_0^\delta \left\{ \int_{u_i}^{u_i+h} \phi(a + t, v) dv - \int_{u_i-h}^{u_i} \phi(a + \delta + t, v) dv \right\} dt \geqq 0,$$

or also

$$\frac{1}{h} \int_{u_i}^{u_i+h} dv \int_0^\delta \phi(a + t, v) dt - \frac{1}{h} \int_{u_i-h}^{u_i} dv \int_0^\delta \phi(a + \delta + t, v) dt \geqq 0. \tag{10}$$

For fixed a and δ, both inner integrals are increasing functions of v. For all those v for which they both are continuous, hence a.e., we get from (10) for $h \to 0$:

$$\int_0^\delta \phi(a + t, v) dt - \int_0^\delta \phi(a + \delta + t, v) dt \geqq 0.$$

Integrating this with respect to v from u_i to $u_i + h$ and using the relation (9), we obtain (6).

Finally, dividing (8) by δ and making $\delta \to 0$, we get

$$\Phi(u_i + h_i, u_j + h_j) - \Phi(u_i, u_j + h_j) + \Phi(u_i - h_i, u_j) - \Phi(u_i, u_j) \geqq 0,$$

or, using (9),

$$\frac{1}{h_i}\int_{u_i}^{u_i+h_i}\phi(v,\,u_j+h_j)dv-\frac{1}{h_i}\int_{u_i-h_i}^{u_i}\phi(v,\,u_j)dv\geqq 0.$$

For $h_i\to 0$ we obtain that $\phi(v,\,u_j+h_j)-\phi(v,\,u_j)\geqq 0$ a.e. Integrating this from u_i to u_i+h_i with respect to v we obtain

$$\Phi(u_i+h_i,u_j+h_j)-\Phi(u_i,u_j+h_j)-\Phi(u_i+h_i,u_j)+\Phi(u_i,u_j)\geqq 0, \tag{11}$$

which reduces to (5) for $h_i=h_j=h$.

(b) *Sufficiency*. Clearly it is sufficient to prove (2) in the case when $f_k=g_k$ except for $k=i$, for the general result follows by successive application of the special one. We may further assume that all f_k and g_k are step-functions, constant on each of the intervals $(s/p,\,(s+1)/p)$, $s=0,\cdots,p-1$, for arbitrary f_k, g_k may be approximated by step-functions of this kind. In this case, by [3, p. 47, Lemma 2], f_i may be transformed into g_i by successive application of a finite number of transformations T. A transformation T changes the value u_i of f_i on an interval $(q/p,\,(q+1)/p)$ into u_i+h_i, the value u_i' of f_i on $(r/p,\,(r+1)/p)$ into $u_i'-h_i$, where $u_i'\leqq u_i$, $h_i\geqq 0$, $q<r$, and does not change the values of f_i on other intervals.† The increase of the integral

$$\int_0^1\Phi(t,f_1,f_2,\cdots,f_n)dt$$

corresponding to a transformation T is equal to

$$\begin{aligned}\Delta=\int_0^\delta\{&\Phi(a+l\delta+t,\,u_i'-h_i,\,U_J)-\Phi(a+l\delta+t,\,u_i',\,U_J)\\&+\Phi(a+t,\,u_i+h_i,\,U_J+H_J)-\Phi(a+t,\,u_i,\,U_J+H_J)\}dt,\end{aligned}$$

where $\delta=1/p$, $a=q/p$, $l=r-q$ and u_j+h_j and u_j, $j\in J$, are the values of $f_j(t)$ on $(q/p,\,(q+1)/p)$ and $(r/p,\,(r+1)/p)$, respectively.

Now from (5) and (6) it follows that [4, formula (10)]

$$\begin{aligned}\int_0^\delta\{&\Phi(a+\delta+t,\,u_i,\,U_J)-\Phi(a+\delta+t,\,u_i+h_i,\,U_J)\\&+\Phi(a+t,\,u_i+h_i,\,U_J+H_J)-\Phi(a+t,\,u_i,\,U_J+H_J)\}dt\geqq 0.\end{aligned} \tag{12}$$

We add to (12) the $(l-1)$ inequalities obtained from (12) by replacing a by $a+\delta,\cdots,a+(l-1)\delta$ and H_J each time by zero. This gives

$$\begin{aligned}\int_0^\delta\{&\Phi(a+l\delta+t,\,u_i,\,U_J)-\Phi(a+l\delta+t,\,u_i+h_i,\,U_J)\\&+\Phi(a+t,\,u_i+h_i,\,U_J+H_J)-\Phi(a+t,\,u_i,\,U_J+H_J)\}dt\geqq 0.\end{aligned} \tag{13}$$

† In addition, we may assume $h_i\leqq u_i'$, if f_i, g_i are positive.

From the convexity of Φ with respect to u_i it follows (for instance by means of (9)) that

$$\Phi(u_i + h_i) - \Phi(u_i) - \Phi(u_i') + \Phi(u_i' - h_i) \geqq 0$$

if $u_i' \leqq u_i$, $h_i \geqq 0$. Replacing here t by $a + l\delta + t$ and adding to the integrand of (13), we obtain $\Delta \geqq 0$, which completes the proof.

In particular we obtain (compare [4]):

COROLLARY. *If Φ has continuous second derivatives*, (2) *holds if and only if*

$$\frac{\partial^2 \Phi}{\partial u_i \partial u_j} \geqq 0, \qquad \frac{\partial^2 \Phi}{\partial t \partial u_i} \leqq 0, \qquad i, j = 1, \cdots, n. \tag{14}$$

3. We consider some examples. Let $\Phi = \phi(t)\Psi(u_1, \cdots, u_n)$. Then condition (6) reduces to

$$[\Psi(u_i + h_i) - \Psi(u_i)] \int_0^\delta \{\phi(a + \delta + t) - \phi(a + t)\} dt \leqq 0$$

or to

$$[\Psi(u_i + h_i) - \Psi(u_i)][\phi_1(a + 2\delta) - 2\phi_1(a + \delta) + \phi_1(a)] \leqq 0,$$

where $\phi_1(t)$ is an integral of $\phi(t)$. Hence (6) holds if and only if either $\phi(t)$ is a constant, or $\phi(t)$ and $\Psi(u_i)$ for each i are monotone in the opposite sense.

Further examples are $\Phi = F(u_1)$ with a convex F; this gives an inequality of Hardy, Littlewood and Pólya ([3], [2]), and the following two inequalities which appear to be new:

$$\int_0^1 F[f_1(t) + \cdots + f_n(t)]dt \leqq \int_0^1 F[g_1(t) + \cdots + g_n(t)]dt, \tag{15}$$

$$\int_0^1 F[f_1(t) + \cdots + f_n(t)]dt \leqq \int_0^1 F[g_1(t) + \cdots + g_n(t)]dt. \tag{16}$$

In (15), $F(u)$ is a convex function defined for all values of u; in (16), $F(u)$ is convex and increasing for $u \geqq 0$. The inequalities follow at once from Theorem 1, since $\Phi = F(u_1 + \cdots + u_n)$ and $\Phi = F(u_1 \cdots u_n)$ satisfy all conditions (5), (6), and (7).

If we define the relation $f \prec g$ by (3), omitting (4), we obtain similar results. If $f = u$, $g = v$ are constants with $u \leqq v$, then $f \prec g$. Hence Φ in (2) must be increasing in each variable u_i. Together with (5), (6), and (7) this is also sufficient for (2). This is seen from the proof of Theorem 1 if we supplement the transformations T used there by a transformation T' which increases (in the wide sense) each of the values of a step-function f_i. As $f \prec g$ on (0, 1) now implies $f \prec g$ on (0, x), $0 \leqq x \leqq 1$, we may replace the limits of integration 0, 1 in (2) by 0, x. (A similar remark applies to (1) if Φ is increasing.)

We could have required that (2) holds for all $f_i \prec g_i$ (this relation defined by (3) and (4)), dropping the assumption that f_i, g_i are decreasing. This gives, as in Theorem 1, (a), the relation (11) with the h_j now of arbitrary sign. It follows that $\Phi(u_i+h_i)-\Phi(u_i)$ and therefore also $\partial\Phi/\partial u_i$ does not depend on the u_j with $j\neq i$. Integrating we get $\Phi=\phi(t, u_i)+\psi(t, U_J)$. In this way we obtain

THEOREM 2. *A function Φ satisfies (2) with f_i, g_i not necessarily decreasing if and only if Φ is of the form*

$$\Phi(t, u_1, \cdots, u_n) = \sum_{i=1}^{n} \Phi_i(t, u_i),$$

where each Φ_i is a function of two variables which satisfies (6) and (7).

References

1. L. Fuchs, A new proof of an inequality of Hardy, Littlewood and Pólya, Mat. Tidsskr., B, 1947, pp. 53–54.
2. G. H. Hardy, J. E. Littlewood and G. Pólya, Some simple inequalities satisfied by convex functions, Messenger Math., vol. 58, 1929, pp. 145–152.
3. ———, Inequalities, Cambridge, 1934.
4. G. G. Lorentz, An inequality for rearrangements, this MONTHLY, vol. 60, 1953, pp. 176–179.
5. H. D. Ruderman, Two new inequalities, this MONTHLY, vol. 59, 1952, pp. 29–32.

A PROOF OF THE IRRATIONALITY OF π

ROBERT BREUSCH, Amherst College

Assume $\pi=a/b$, a and b integers. Then, with $N=2a$, $\sin N=0$, $\cos N=1$, and $\cos(N/2)=\pm 1$.

If m is zero or a positive integer, then

$$A_m(x) \equiv \sum_{k=0}^{\infty} (-1)^k(2k+1)^m \frac{x^{2k+1}}{(2k+1)!} = P_m(x)\cos x + Q_m(x)\sin x$$

where $P_m(x)$ and $Q_m(x)$ are polynomials in x with integral coefficients. (Proof by induction on m: $A_{m+1}=x\,dA_m/dx$, and $A_0=\sin x$.)

Thus $A_m(N)$ is an integer for every positive integer m.

If t is any positive integer, then

$$\begin{aligned} B_t(N) &\equiv \sum_{k=0}^{\infty} (-1)^k \frac{(2k+1-t-1)(2k+1-t-2)\cdots(2k+1-2t)}{(2k+1)!} N^{2k+1} \\ &= \sum_{k=0}^{\infty} (-1)^k \frac{(2k+1)^t - b_1(2k+1)^{t-1} + \cdots \pm b_t}{(2k+1)!} N^{2k+1} \\ &= A_t(N) - b_1A_{t-1}(N) + \cdots \pm b_tA_0(N). \end{aligned}$$

Since all the b_i are integers, $B_t(N)$ must be an integer too.

Now

$$B_t(N) = \sum_{k=0}^{[(t-1)/2]} + \sum_{k=[(t+1)/2]}^{t-1} + \sum_{k=t}^{\infty}.$$

In the first sum, the numerator of each fraction is a product of t consecutive integers, therefore it is divisible by $t!$, and therefore by $(2k+1)!$ since $2k+1 \leqq t$. Thus each term of the first sum is an integer. Each term of the second sum is zero. Thus the third sum must be an integer, for every positive integer t.

This third sum is

$$\sum_{k=t}^{\infty} (-1)^k \frac{(2k-t)!}{(2k+1)!(2k-2t)!} N^{2k+1}$$
$$= (-1)^t \frac{t!}{(2t+1)!} N^{2t+1}\left(1 - \frac{(t+1)(t+2)}{(2t+2)(2t+3)} \frac{N^2}{2!} + \frac{(t+1)(t+2)(t+3)(t+4)}{(2t+2)(2t+3)(2t+4)(2t+5)} \frac{N^4}{4!} - \cdots\right).$$

Let $S(t)$ stand for the sum in the parenthesis. Certainly

$$|S(t)| < 1 + N + \frac{N^2}{2!} + \cdots = e^N.$$

Thus the whole expression is absolutely less than

$$\frac{t!}{(2t+1)!} N^{2t+1} e^N < \frac{N^{2t+1}}{t^{t+1}} e^N < (N^2/t)^{t+1} e^N;$$

for $t > t_0$, this is certainly less than 1.

Therefore necessarily $S(t) = 0$ for every integer $t > t_0$. But this is impossible, because

$$\lim_{t\to\infty} S(t) = 1 - \frac{1}{2^2}\cdot\frac{N^2}{2!} + \frac{1}{2^4}\cdot\frac{N^4}{4!} - \cdots = \cos(N/2) \neq 1.$$

It can be proved similarly that the natural logarithm of a rational number must be irrational: From $\log(a/b) = c/d$ would follow $e^c = a^d/b^d = A/B$. Then

$$B \cdot \sum_{k=0}^{\infty} \frac{(k-t-1)(k-t-2)\cdots(k-2t)}{k!} c^k$$

would have to be an integer for every positive integer t, and a contradiction results, as before.

CLASSROOM NOTES

EDITED BY G. B. THOMAS, Massachusetts Institute of Technology

All materials for this department should be sent to G. B. Thomas, Department of Mathematics, Massachusetts Institute of Technology, Cambridge 39, Mass.

A RULE FOR SOLVING A CERTAIN TYPE OF EXACT DIFFERENTIAL EQUATIONS

L. L. PENNISI, University of Illinois

In solving an exact differential equation

$$(1) \qquad M dx + N dy = 0 \qquad \left(\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}\right),$$

it often happens that M or N is a function of the type

$$(2) \qquad \sum_{i=1}^{n} \phi_i(x)\psi_i(y).$$

In that case, the following simple rule may be used to obtain the solution of (1).

Suppose, for definiteness, that N is of the form (2). *We write $N=N_1+N_2$, where N_2 is the sum of the terms of N for which $\phi_i(x)$ is a constant. The solution of equation* (1) *is then given by*

$$(3) \qquad \int M dx + \int N_2 dy = C,$$

where C is a constant.

Proof. Since equation (1) is exact, we have $dF = Mdx + Ndy$ and

$$F = \int M dx + \int \left\{N - \int \frac{\partial M}{\partial y} dx\right\} dy = \int M dx + \int \left\{N - \int \frac{\partial N}{\partial x} dx\right\} dy$$

$$= \int M dx + \int \{N - N_1\} dy = \int M dx + \int N_2 dy.$$

Similarly, if M is of the form (2) and $M = M_1 + M_2$ where M_2 is the sum of the terms of M for which $\psi_i(y)$ is a constant, then the solution of equation (1) is given by

$$\int N dy + \int M_2 dx = C.$$

Example: Solve the exact differential equation

$$(\tan y + \sec^2 x)dx + \sec y(\tan y + x \sec y)dy = 0.$$

Observe that $N_2 = \sec y \tan y$. Hence $\int Mdx + \int N_2 dy = C$, becomes $x \tan y + \tan x + \sec y = C$.

REMARKS ON YATES' NOTE "DIFFERENTIATING THE LOGARITHM"

ALBERT WILANSKY, Lehigh University

In the note mentioned (this MONTHLY, February 1954, p. 120) it was shown that the Δ-process works more naturally with $x \ln x$ than with $\ln x$ in that, at a certain stage in the differentiation of $\ln x$ "it is necessary to introduce a carefully selected coefficient." Thence, by differentiating $x \ln x$ as a product the derivative of $\ln x$ is obtained *if it has a derivative*, an assumption omitted in the note. The assumption can, of course, be dispensed with if $x \ln x/x$ be differentiated as a quotient. However if we are willing to assume that $\ln x$ is differentiable the following remarks may be even more natural. If f is a differentiable function defined for $x>0$ such that $f(ax)=f(a)+f(x)$, differentiating gives $af'(ax)=f'(x)$, hence $f'(ax)=f'(x)/a$, thus $f'(a)=c/a$ where $c=f'(1)$. The derivative of $\ln x$ at $x=1$ is $\lim \ln (1+h)^{1/h}$, and no "carefully selected coefficient" is necessary.

TOSSING A COIN

KARL MENGER, Illinois Institute of Technology

A basic fact concerning random experiments treated in practically all books on probability and statistics is stated in one of them* in the following words: "If an ordinary coin is rapidly spun several times, and if we take care to keep the conditions of the experiment as uniform as possible in all respects, we shall find that we are unable to predict whether, in a particular instance, the coin will fall 'heads' or 'tails'." The page devoted to a general discussion of this experiment ends with the following remark. "A moment's reflection will show that even extremely small changes in the initial state of the motion must be expected to have a dominating influence on the result. In practice, the initial state will never be exactly known, but only to a certain approximation."

Since a quantitative description of the problem can be achieved in terms of the most elementary mathematics and in about as many words as are required by a general discussion, there is no reason why every beginner should not learn the quantitative details of the case. If the following remarks† should be contained somewhere in the literature, they do not seem to have found their way into the standard books on probability.

We choose a coin of radius 1, lift it from a table, bring it into a vertical position, spin it about its horizontal diameter, and drop it. All that is relevant to the problem occurs in the vertical plane which passes through the center of the lifted coin and is perpendicular to the axis of rotation. At any instant, the intersection of this plane with the coin is a flat rectangle of length 2. Of its two long sides, one may be assumed to be marked H(ead), and the other T(ail).

---

* Cramér, Mathematical Methods of Statistics, Princeton University Press, 1946, pp. 138 seq.

† They elaborate on an exercise in Menger, Calculus, A Modern Approach, 2nd edition, Illinois Institute of Technology Bookstore, 1953, p. 174. (There, the initial position of the coin is horizontal.) The author is indebted to Prof. Eli Sternberg for suggestions concerning this elaboration.

We thus consider a vertical halfplane, bounded by a horizontal line. A rod of length 2 is lifted and brought into a vertical position (say, with the mark H on right side, and T on the left). Then, within the halfplane, the rod is spun about its center, and dropped with the initial velocity 0. If the rod is assumed to be homogeneous and ideally slender, and the resistance of the air is neglected, the laws of mechanics imply:

A. The distance travelled by the center of the rod is proportional to the square of the time elapsed since the release.

B. The initial angular velocity of the rod, due to the spin, remains constant until the rod touches the bottom line.

Moreover, we make the following (more convenient than realistic) assumptions which neglect the effects of the spin after the first contact between the rod and the bottom.

C. If, when first touching the bottom line, the rod is in a vertical position, then it will remain in this unstable equilibrium and we say that the outcome of the tossing is V(ertical).

D. If case C is not present, then, unless the rod reaches the bottom line in a horizontal position, it will pivot about the first point of contact and come to a definite horizontal rest after traversing the smaller of the two angles that it forms with the bottom line on first contact.

D implies that the mark (H or T) which is on the upper side of the rod at the instant of its first contact will remain on the upper side, and thus determine the outcome of the tossing.

By virtue of A and B, if proper units of time and distance are chosen, t units of time after the release of the rod, the altitude of its lowest point is

$$h(t) = a - t^2 - |\cos ct| \tag{1}$$

units of length, if a denotes the initial altitude of the center of the rod and c the angular velocity. Clearly, $a \geqq 1$.

The rod is in a vertical position at $t = k\pi/c$ for $k = 0, 1, \cdots$. If $c > 0$, that is, if the spin is counterclockwise, then, since at the initial instant the mark H is assumed to be on the right side, H is on top during the intervals

$$H^+:\quad 2m\pi/c < t < (2m+1)\pi/c, \qquad m = 0, 1, \cdots,$$

and T is on top during the intervals

$$T^+:\quad (2m-1)\pi/c < t < 2m\pi/c, \qquad m = 1, 2, \cdots.$$

If $c < 0$, the situation is reversed, that is, $H^- = T^+$ and $T^- = H^+$.

Let k be that integer for which

$$k^2\pi^2/c^2 + 1 \leqq a < (k+1)^2\pi^2/c^2 + 1. \tag{2}$$

If $t < k\pi/c$, then, by (1),

$$h(t) = a - t^2 - |\cos ct| > a - k^2\pi^2/c^2 - 1$$

and thus $h(t) > 0$, by (2). Therefore, since h is a continuous function of t, the instant t_1 of the first contact of the bar with the bottom (*i.e.*, the smallest solution of the equation $h(t) = 0$) satisfies the condition

$$k\pi/c \leqq t_1 < (k + 1)\pi/c. \tag{3}$$

Conversely, (3) implies (2) since, if the altitude lay in an interval different from (2), t_1 would lie in an interval different from (3).

By assumption C, the outcome is V if in (3), and therefore in (2), the equality sign holds. In case of inequalities in (2) and (3), by assumption D, the outcome, provided $c > 0$, is H or T depending on whether k in (2) is even or odd and t_1 accordingly belongs to H^+ or T^+. The situation is reversed if $c < 0$.

The initial state of the rod, and therefore the outcome of the tossing, is determined by the numbers a and c. Each possible initial state can be represented by a point in an (a, c) halfplane $(a \geqq 1)$. The points on the curves

$$C_k: \quad a = k^2\pi^2/c^2 + 1 \qquad\qquad (k = 1, 2, \cdots)$$

and on the lines

$$C_0: \quad c = 0 \text{ and } a = 1$$

represent initial states resulting in the outcome V. The states in the boomerang shaped regions between C_{2m} and C_{2m+1} for $c > 0$ and between C_{2m-1} and C_{2m} for $c < 0$ result in H. The states in the other regions result in T.

Suppose now that the initial state is not accurately known and all that can be ascertained is that the initial altitude lies between a and $a+\alpha$, and the angular velocity between c and $c+\gamma$. If a and c satisfy (2), the number $(a-1)c^2$ is confined to the interval between $k^2\pi^2$ and $(k+1)^2\pi^2$. If

$$(a + \alpha - 1)(c + \gamma)^2 > (k + 1)^2\pi^2, \tag{4}$$

then initial conditions which are indistinguishable from (a, c) lead to a different outcome, *and the outcome of the tossing is unpredictable.**

For instance, this is the case if $\alpha = 0$ (that is to say, if the altitude can be accurately determined), $c > 0$, and

$$\gamma > \frac{(k + 1)\pi}{\sqrt{a - 1}} - c > 0.$$

Another example is the case where $\gamma = 0$ (that is to say, the angular velocity can be accurately determined) and

$$\alpha > \frac{(k + 1)^2\pi^2}{c^2} - a + 1 > 0.$$

* Besides the inaccuracies in determining a and c, the facts that the initial verticality of the coin and the horizontality of its axis of rotation can be only approximately materialized add to the unpredictability of the outcome of the tossing.

ON THE ITERATION OF CONTINUOUS NON-INCREASING FUNCTIONS

D. S. GREENSTEIN, University of Pennsylvania

Let $f(x)$ be continuous and non-increasing, and let x_0 be an arbitrary real number. We consider sequences obtained by iteration of $f(x)$, *i.e.*, sequences defined by $x_n = f(x_{n-1})$ $(n=1, 2, \cdots)$. Theorems are presented on the behavior of such sequences. Included among these theorems are a useful sufficient condition for convergence and theorems about the set of all x_0 for which x_n converges. In all the theorems x_n and $f(x)$ are as defined above.

THEOREM 1. *If $x_n \to x$ then $x = f(x)$.*

Proof. Since $f(x)$ is continuous, $x = \lim f(x_n) = f(\lim x_n) = f(x)$.

THEOREM 2. *For all x_0 for which x_n converges, $\lim x_n$ has the same value.*

Proof. Since $f(x)$ is non-increasing, there can be at most one solution of the equation $x = f(x)$.

THEOREM 3. *Each sequence satisfies one of the following sets of relations:*

a. (1) $x_0 \leqq x_2 \leqq \cdots \leqq x_{2n} \leqq \cdots \leqq x_{2n+1} \leqq \cdots \leqq x_3 \leqq x_1$

(2) $x_1 \leqq x_3 \leqq \cdots \leqq x_{2n+1} \leqq \cdots \leqq x_{2n} \leqq \cdots \leqq x_2 \leqq x_0$.

b. (1) $\cdots \leqq x_{2n} \leqq \cdots \leqq x_2 \leqq x_0 \leqq x_1 \leqq x_3 \leqq \cdots \leqq x_{2n+1} \leqq \cdots$

(2) $\cdots \leqq x_{2n+1} \leqq \cdots \leqq x_3 \leqq x_1 \leqq x_0 \leqq x_2 \leqq \cdots \leqq x_{2n} \leqq \cdots$.

Proof. This is an immediate consequence of the non-increasing behavior of $f(x)$. Thus it is seen that the subsequences x_{2n} and x_{2n+1} are always monotonic.

THEOREM 4. *Let x_n be bounded, $\underline{x} = \underline{\lim}\, x_n$, and $\bar{x} = \overline{\lim}\, x_n$. Then $\bar{x} = f(\underline{x})$ and $\underline{x} = f(\bar{x})$.*

Proof. From Theorem 3 it as follows that either

$$\underline{x} = \lim x_{2n} \quad \text{and} \quad \bar{x} = \lim x_{2n+1}$$

or

$$\underline{x} = \lim x_{2n+1} \quad \text{and} \quad \bar{x} = \lim x_{2n}.$$

Taking limits in $x_{2n} = f(x_{2n-1})$ and $x_{2n+1} = f(x_{2n})$ gives the desired results.

THEOREM 5. *A sufficient condition for the convergence of x_n is that it be bounded and that the only solutions of the equations $y = f(x)$ and $x = f(y)$ which lie within these bounds have $y = x$.*

Proof. By Theorem 4, $\underline{\lim}\, x_n = \overline{\lim}\, x_n$.

The condition on the solutions of $y = f(x)$ and $x = f(y)$ can be interpreted geometrically. It says that the curve $y = f(x)$ and its mirror image through the

line $y=x$ intersect only on the line $y=x$ (at least within the bounds of x_n). Sometimes (*e.g.*, if $f(x)=e^{-x}$) it is readily seen, either by drawing the curves or by simple analysis, that the curves have exactly one point of intersection, which must lie on $y=x$, since (y_0, x_0) is a point of intersection if (x_0, y_0) is.

THEOREM 6. *Let the curves $y=f(x)$ and $x=f(y)$ have exactly one point of intersection. Furthermore let $f(x)$ be either bounded from below or bounded from above. Then x_n converges for all x_0.*

Proof. Suppose $-\infty<a\leqq f(x)$ $(-\infty<x<+\infty)$.

Then $a\leqq x_n\leqq f(a)$ $(n\geqq 2)$. Hence by Theorem 5 x_n converges. A similar argument holds if $f(x)\leqq b<\infty$ $(-\infty<x<+\infty)$. Thus since $e^{-x}\geqq 0, f(x)=e^{-x}$ yields a convergent sequence for all x_0.

THEOREM 7. *Let x_n converge when $x_0=a$. Then for all x_0 such that min $(a, f(a))$ $\leqq x_0\leqq$ max $(a, f(a))$, x_n converges.*

Proof. Iteration gives the desired result.

THEOREM 8. *Let C be the set of all x_0 for which x_n converges. Then C is nonvacuous and includes all x_0 satisfying inf $C<x_0<$ sup C.*

Proof. Choose any x_0. Either $x_0\leqq f(x_0)$ and $x_1\geqq f(x_1)$ or $x_0\geqq f(x_0)$ and $x_1 \leqq f(x_1)$. Thus there exists a unique p such that min $(x_0, x_1)\leqq p\leqq$ max (x_0, x_1) and $p=f(p)$. Clearly p belongs to C. Now suppose that inf $C<x_0<$sup C and $x_0\neq p$. If $x_0<p$, choose a belonging to C such that $a<x_0$; if $x_0>p$, choose a belonging to C such that $a>x_0$. In either case, min $(a, f(a))\leqq x_0\leqq$ max $(a, f(a))$. Hence by Theorem 7 x_0 belongs to C.

THEOREM 9. *Let $\alpha=$ inf C and $\beta=$ sup C. Then either $\alpha=-\infty$ and $\beta=+\infty$ or α and β are both finite with $\alpha=f(\beta)$ and $\beta=f(\alpha)$.*

Proof. Suppose α is finite. Clearly $\alpha\leqq p=f(p)$. Hence $f(\alpha)\geqq\alpha$. Now consider the points $f(\alpha-\epsilon)$ $(\epsilon>0)$. These points do not belong to C; otherwise $\alpha-\epsilon$ would. Since $f(\alpha)=f(\alpha-0)\leqq f(\alpha-\epsilon)$, we must by Theorem 8 have $\beta\leqq f(\alpha)$. On the other hand since there are points arbitrarily close to α which belong to C, there must be points arbitrarily close to $f(\alpha)$ which belong to C. Therefore $\beta\geqq f(\alpha)$. Thus $\beta=f(\alpha)$. Similarly if β is finite, then $\alpha=f(\beta)$.

THEOREM 10. *Let the curves $y=f(x)$ and $x=f(y)$ have exactly one point of intersection. Then C is either all the reals or just $p=f(p)$.*

Proof. This is an immediate consequence of Theorem 9. That p can be the only point of C can be seen by taking $f(x)=-2x$. By taking $f(x)=-x/2$, it is seen that C can be all the reals without $f(x)$ being bounded.

So far we have assumed that $f(x)$ is non-increasing for all real x. Obviously some of our results are valid if $f(x)$ is non-increasing over an interval containing x_n. For example, consider $f(x)=\cos x$. Irrespective of x_0, $0\leqq x_n\leqq 1$ $(n\geqq 2)$.

Since the cosine is non-increasing over this range, we may write the equations:

$$\bar{x} = \cos \underline{x}$$

$$\underline{x} = \cos \bar{x}$$

$$\bar{x} - \underline{x} = 2 \sin \frac{\bar{x} + \underline{x}}{2} \sin \frac{\bar{x} - \underline{x}}{2}.$$

Assume $\bar{x} \neq \underline{x}$. Then we may write

$$\frac{\sin \dfrac{\bar{x} - \underline{x}}{2}}{\left(\dfrac{\bar{x} - \underline{x}}{2}\right)} = \frac{1}{\sin \dfrac{\bar{x} + \underline{x}}{2}}. \tag{1}$$

Since $0 < [(\bar{x} + \underline{x})/2] < 1 < (\pi/2)$, the right hand side of (1) is greater than one. But the left hand side is less than one. Therefore x_n converges for all x_0.

ON AN APPLICATION OF THE VANDERMONDE DETERMINANT

W. V. PARKER, Alabama Polytechnic Institute

In this MONTHLY for December 1953, Harley Flanders gave a solution of the following problem. Let $x_1, x_2, \cdots, x_n$ be n numbers such that $\sum x_i = \sum x_i^2 = \cdots = \sum x_i^n = 0$; to prove that $x_1 = x_2 = \cdots = x_n = 0$. He used the Vandermonde determinant and an induction process.

The following solution, which uses the Vandermonde determinant but avoids the induction process, may be of interest. Let the solution be such that n_i of the x's are equal to $a_i \neq 0$, $i = 1, 2, \cdots, k$, and $a_i \neq a_j$, $i \neq j$, and the remaining x's are zero. Then we have $\sum n_i a_i = \sum n_i a_i^2 = \cdots = \sum n_i a_i^n = 0$. The first k of these equations in $n_1, n_2, \cdots, n_k$ constitute a homogeneous system whose determinant is $a_1 \cdot a_2 \cdot \cdots \cdot a_k \cdot V(a_1, a_2, \cdots, a_k)$ where $V(a_1, a_2, \cdots, a_k)$ is a nonvanishing Vandermonde determinant. Hence, $n_i = 0$, $i = 1, 2, \cdots, k$, and $x_1 = x_2 = \cdots = x_n = 0$ is the only solution.

REMARK ON THE GENERAL POLYNOMIAL OF THE SECOND DEGREE

LEONARD GILLMAN, Purdue University

In a previous note (this MONTHLY, vol. 61, 1954, p. 191), F. W. Perkins points out that in establishing the invariance under rotation of $B^2 - 4AC$ (*etc.*) in the general polynomial of the second degree, the labor can be considerably reduced by employing the *derivatives* of the new coefficients $a, b, \cdots$ (with respect to the rotation angle θ). The method presented avoids the calculation of b^2 and of $4ac$, but it does require the long computation of the coefficients $a, b, \cdots$ themselves.

Here is an alternate procedure that avoids both. Let x, y denote the old coordinates, and u, v the new. Then

$$(1)\qquad Ax^2 + Bxy + Cy^2 + Dx + Ey + F = au^2 + buv + cv^2 + du + ev + f,$$

where

$$(2)\qquad u = x\cos\theta + y\sin\theta, \qquad v = -x\sin\theta + y\cos\theta.$$

Using primes to denote derivatives with respect to θ, we have, from (2),

$$(3)\qquad u' = v, \qquad v' = -u.$$

We compute the derivative of the right side of (1). Since (1) is an identity, this derivative is zero. With the substitutions (3), we get (after collecting terms),

$$(4)\quad (a' - b)u^2 + (b' + 2a - 2c)uv + (c' + b)v^2 + (d' - e)u + (e' + d)v + f' = 0.$$

Since (4), too, is an identity, all the coefficients are zero. Thus $a' = b$, *etc.*, and, as before, $(b^2 - 4ac)' = 0$.

If the only invariants of concern are those involving the first three coefficients, one may prefer to proceed by first reducing (1) to

$$(5)\qquad Ax^2 + Bxy + Cy^2 = au^2 + buv + cv^2$$

(which is an immediate consequence of (1) and (2)).

A disadvantage in all this is that some additional work must be done in order to derive the formula $\cot 2\theta = (A - C)/B$ for removing the cross-term. One way is to set $b = 0$ in (5) and then substitute various combinations of 1's and 0's for x and y (or for u and v). Being derivative-happy at the moment, however, we cannot resist pointing out the following. From the formulas $b' = 2(c - a)$, $(c-a)' = -2b$, we obtain $b'' = -4b$. The student does not know how to solve this differential equation, but he can verify that

$$(6)\qquad b = k_1 \sin 2\theta + k_2 \cos 2\theta$$

is a solution. Differentiation in (6) yields

$$(7)\qquad c - a = \tfrac{1}{2}b' = k_1 \cos 2\theta - k_2 \sin 2\theta.$$

Since $a = A$, $b = B$ and $c = C$ when $\theta = 0$, we see that $k_2 = B$ and $k_1 = C - A$. Thus $b = (C - A)\sin 2\theta + B\cos 2\theta$, as desired.

COMPLEX ROOTS OF AN INTEGRAL RATIONAL EQUATION

D. TRIFAN, University of Arizona

The following note concerns a method of proving the theorem that an integral rational equation with real coefficients has complex roots occurring in conjugate pairs.

If $x = a + bi$ is a root of the integral rational equation $f(x) = 0$, then as a result of the factor theorem

$$f(x) = [x - (a + bi)]Q(x).$$

Separating $Q(x)$ into its real and imaginary parts we have

$$f(x) = [x - (a + bi)][F(x) + iG(x)].$$

Since $f(x)$ has real coefficients the imaginary part must be zero *i.e.*

$$xG(x) - aG(x) - bF(x) = 0$$

or

$$F(x) = \frac{(x - a)G(x)}{b}.$$

Therefore

$$f(x) = [x - (a + bi)]\left(\frac{(x - a)G(x)}{b} + iG(x)\right)$$

$$= [x - (a + bi)]\left(\frac{x - (a - bi)}{b}\right)G(x).$$

Thus

$$f(a - bi) = 0.$$

TWO THEOREMS ON GROUPS

R. V. ANDREE and G. M. PETERSEN, University of Oklahoma

THEOREM 1. *If G is a group having more than two elements, then there exist two distinct elements a, b, neither of which is the identity, such that $ab = ba$.*

If there exists an element $x \in G$ which is not self-inverse, $x \neq x^{-1}$, then let $a = x$, $b = x^{-1}$.

If for every $x \in G$, $x = x^{-1}$, then for all $a, b \in G$, $(ab) \cdot (ba) = a(bb)a = aa = e$. Thus, $(ab)^{-1} = ba$. Since $x = x^{-1}$ by assumption, $(ab)^{-1} = ab$. Thus, $ab = ba$ for all $a, b \in G$.

The latter paragraph suffices to prove theorem 2.

THEOREM 2. *A group in which each element is self-inverse is necessarily commutative.*

Theorem 1 has proved useful for classroom examination purposes. It is simple enough to be proved by an average student and can be "half proved" by the less alert, who may fail to consider the case covered by Theorem 2.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1136. *Proposed by W. A. Bowers, University of North Carolina*

An elementary physics text, wishing to avoid using calculus, and also hoping (unsuccessfully!) to avoid the appearance of pulling the answer out of a hat, gives the following recipe for calculating the work done in moving a positive unit charge from r_1 to r_2 against the attraction of a negative unit charge at the origin. "Since the force varies with distance, we cannot simply multiply 'the force' by the distance; instead we must take the geometric mean of the initial and final values of the force, and multiply it by the distance." Noting with amazement that this does indeed work, we ask: what is the most general function for which this is true, that is, for which the average value over an arbitrary interval equals the geometric mean of the values at the end points?

E 1137. *Proposed by Simon Green, Philander Smith College, Arkansas*

Let $N=1+1/2+1/3+\cdots+1/n$. Prove that $e^N>n+1$.

E 1138. *Proposed by J. P. Ballantine, University of Washington*

For any triangle prove that: (1) if $B=2A$, then $b^2=a^2+ac$, (2) if $B=3A$, then $b^3-ab^2-a^2b-ac^2+a^3=0$.

E 1139. *Proposed by N. A. Court, University of Oklahoma*

Three collinear points P, Q, R are marked on the sides BC, CA, AB of a triangle ABC. Starting with an arbitrary point X of the line BC, the following points are constructed successively:

$$Y=(XR, CA), \qquad Z=(YP, AB), \qquad X'=(ZQ, BC);$$
$$Y'=(X'R, CA), \qquad Z'=(Y'P, AB), \qquad X''=(Z'Q, BC).$$

Show that points X and X'' coincide.

E 1140. *Proposed by Albert Wilansky, Lehigh University*

Let sequences $\{x_n\}$, $\{y_n\}$ be called related if $\sum_{k=1}^{n}(x_k-y_k)$ is a bounded function of n. For example, $\{(-1)^n\}$ and $\{0\}$ are related. Is there a bounded sequence not related to any convergent sequence?

SOLUTIONS

The Intersection of Two Secants

E 1106 [1954, 194]. *Proposed by C. I. Lubin, University of Cincinnati*

Two non-parallel, non-coincident lines which cut the circle $|z| = r$ in the points a, b and c, d respectively, where a, b, c, d are complex numbers not necessarily all different, intersect in the point z given by

$$z = (a^{-1} + b^{-1} - c^{-1} - d^{-1})/(a^{-1}b^{-1} - c^{-1}d^{-1}).$$

Solution by J. R. Hatcher, B. F. Peery, and Ineatha Walker, Fisk University. Since $a\bar{a} = b\bar{b} = c\bar{c} = d\bar{d} = r^2$, the equations of the lines are $r^2z + ab\bar{z} = r^2(a+b)$ and $r^2z + cd\bar{z} = r^2(c+d)$. Hence the intersection point z is given by

$$z = [cd(a + b) - ab(c + d)]/(cd - ab),$$

so that dividing numerator and denominator by $abcd$ gives the desired result.

Also solved by Hüseyin Demir, F. A. Ficken, Norman Greenspan, Louisa Grinstein, J. R. Jackson, M. W. Oliphant, J. H. Simester, F. Underwood, and the proposer.

A Pencil of Planes Associated with a Tetrahedron

E 1107 [1954, 194]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

On the edges AB, AC, AD of a tetrahedron $ABCD$ are marked points M, N, P such that $AB = nAM$, $AC = (n+1)AN$, $AD = (n+2)AP$. Show that the plane MNP contains a fixed line as n varies.

I. *Solution by Hüseyin Demir, Zonguldak, Turkey.* From the relations it is evident that the ranges of points $[M]$ and $[P]$ are projective. But since A is a self-corresponding element, the projectivity is a perspectivity. Hence MP is on a fixed point P'. Similarly MN is on a fixed point N'. Hence the plane MNP is on the fixed line $P'N'$.

II. *Solution by C. P. Pinzka, Princeton, N. J.,* Let X be the fourth vertex of the parallelogram determined by AB and BC. Since MX divides AC in the same ratio as N, MN must pass through X. Similarly, NP passes through Y, the fourth vertex of the parallelogram determined by AC and CD. These conclusions hold for any n. It follows that plane MNP must pass through the fixed line XY.

Also solved by Paul Berry, J. R. Jackson, Josef Langr, D. C. B. Marsh, D. B. Mumford, C. S. Ogilvy, J. A. Tierney, F. Underwood, Chih-yi Wang, and the proposer.

Sum of Cubes as Difference of Two Squares

E 1108 [1954, 194]. *Proposed by René Bloch, Humanistisches Gymnasium, Basle, Switzerland*

Let n, k, $a+1$ be three positive integers which are not all odd. Express $\sum_{i=0}^{k} (n+ai)^3$ as a difference of two integer squares.

Solution by the Proposer. We have

$$\sum_{i=0}^{k} (n+ai)^3 = (k+1)n^3 + 3n^2a\binom{k+1}{2}$$
$$+ na^2k(2k+1)(k+1)/2 + a^3\binom{k+1}{2}^2$$
$$= \left[n(k+1) + \binom{k+1}{2}a\right]\left[\binom{n+ak+1}{2} + \binom{n}{2} + k\binom{a}{2}\right].$$

Denoting the first factor by $p-q$ and the second by $p+q$ we find

$$p = \binom{n+1}{2} + nk(a+1)/2 + \binom{a+1}{2}\binom{k+1}{2},$$
$$q = \binom{n}{2} + nk(a-1)/2 + \binom{a}{2}\binom{k+1}{2}.$$

Since n, k, $a+1$ are not all odd it follows that p and q are integers, and

$$\sum_{i=0}^{k} (n+ai)^3 = p^2 - q^2.$$

Also solved by A. R. Hyde, M. J. Pascual, and Walter Penney.

Reversed Squares

E 1109 [1954, 194]. *Proposed by Erich Michalup, Caracas, Venezuela*

Find numbers so that their squares, when reversed, are the squares of the reversed numbers.

Contributions by W. V. Gamzon, Vern Hoggatt, Edgar Karst, D. C. B. Marsh, and A. P. Rhodes.

It can be shown that suitable numbers can contain no digits other than 0, 1, 2, 3, that they must not begin or end with 0, that if one end is 3 the next place cannot be 2, that when such a number is squared there must be no carry-overs in the addition part of the multiplication algorithm. With these restrictions in mind it is easy to construct numbers of the desired sort.

An Ideal Cryptarithm

E 1110 [1954, 194]. *Proposed by Edgar Karst, Independence, Missouri*

An *ideal* cryptarithm has only one solution and involves all ten digits. Solve the following ideal addition cryptarithm based upon the name of the daily newspaper, Hannoversche Presse:

$$\begin{array}{cccccc} & H & A & N & N & O & V \\ & E & R & S & C & H & E \\ \hline & P & R & E & S & S & E \end{array}.$$

Solution by M. A. Kirchberg, Eastern Illinois State College. Obviously $V=0$, $A=9$, $H+E+1=P$, whence we must have $NNO+SCH=1ESS$. Now $N+C \geqq S-1$, $10+E \geqq N+S$, $C+E+11 \geqq 2S$, $S=7$ or 8; $14 \geqq O+H \neq S+10$, $14 \geqq N+C \neq S+10$, $10+E=N+S$, $3 \leqq N \leqq 7$. Upon checking the ten remaining combinations of S and N it is readily seen that the unique solution is $293360+178521=471881$.

Also solved by P. M. Berry, R. L. Caskey, J. E. Darraugh, Hüseyim Demir, William Douglas, Jean Gamzon, A. St. C. G. Grant, Louisa Grinstein, J. D. Haggard, W. P. Hennessy, Vern Hoggatt, A. R. Hyde, Virginia Johnson, Diane Kocher, Sidney Kravitz, D. C. B. Marsh, Don Mattis, J. H. Means, D. B. Mumford, David Muskat, J. B. Muskat, Walter Penney, Elizabeth de Picciotto, G. G. Roberts, Azriel Rosenfeld, David Rothman, C. W. Thomson, C. W. Trigg, W. D. Ward, Monica Wyzalek, the proposer, and a solver with an illegible signature. Late solution by W. T. Grant.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4608. *Proposed by José L. Massera, Instituto de Matematica y Estadistica, Montevideo, Uruguay*

Let a_i, b_i ($i=0, 1, \cdots, 2n$) be real numbers satisfying the following assumptions:

(a) $$a_i + a_{i+1} \geqq 0 \qquad \text{for } i = 0, 1, \cdots, 2n-1;$$

(b) $$a_{2i+1} \leqq 0 \qquad \text{for } i = 0, 1, \cdots, n-1;$$

(c) $$\sum_{h=2p}^{2q} b_h > 0 \qquad \text{for } 0 \leqq p \leqq q \leqq n.$$

Prove that

$$\sum_{i=0}^{2n} (-1)^i a_i b_i \geqq 0$$

and that the equality sign holds only if $a_i = 0$ for all i.

4609. *Proposed by Vern Hoggatt, San Jose State College, California*

Let $q_k(F_n)$ be the kth digit (from the right) of F_n, the nth Fibonacci number $(F_0=0,\ F_1=1,\ F_{n+2}=F_{n+1}+F_n)$. Let s be defined such that $q_k(F_n)=0$ for $n=0, 1, 2, \cdots, s$ but $q_k(F_{s+1}) \neq 0$. (1) Show that the sequence $q_k(F_n)$ has at least another run of $s+1$ consecutive zeros. (2) Prove or disprove: the sequence $q_k(F_n)$ has never more than $s+1$ consecutive zeros.

4610. *Proposed by Joseph Lehner, Los Alamos Scientific Laboratory*

Show that for almost all real, irrational θ, the series

$$\sum_{n=1}^{\infty} \log \left| \sin n\pi\theta \right| \cdot x^n$$

converges in the unit circle.

4611. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Given five spheres, if one of them is orthogonal to the four others, then the centers of the four are the vertices of an orthocentric tetrahedron whose orthocenter coincides with the center of the fifth sphere.

4612. *Proposed by D. J. Newman, Republic Aviation Corporation, Farmingdale, N. Y.*

Let a_0, a_1 be arbitrary and $a_n = a_{n-1} + a_{n-2}/n(n-1)$ for $n > 1$. Find

$$\lim_{n\to\infty} a_n.$$

SOLUTIONS

Systems of Numeration Having Squares of Special Form

4543 [1953, 423]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Determine systems of numeration in which there are one or more four-digit squares of the form *aabb* and such that $a+b$ equals the sum of the digits of the square root.

Partial Solution by W. J. Blundon, Memorial University of Newfoundland. The assumption is that, if x is the base of the system of numeration, then integers p, q exist such that

$$(ax^2 + b)(x + 1) = (px + q)^2,$$

where a, b, p, $q < x$ and $a+b = p+q$. Two particular cases give interesting sets of solutions:

Case I. Let $a+b=p+q=x+1$.
Elimination of b, q from the given equation gives

$$(1) \qquad (ax - a + 1)(x + 1)^2 = (px + x - p + 1)^2.$$

Therefore $x+1$ divides $px+x-p+1$, so that $x+1$ divides $2p$. Since $p<x$, it follows that $x+1=2p$. Substituting in (1) we have $p=2a-1$, $q=2a-1$, $x=4a-3$, $b=3a-2$.

Case II. Let $a+b=p+q=x-1$. Then

$$(2) \qquad (ax + a + 1)(x + 1) = (x - 1)(p + 1)^2.$$

Therefore $4a+2=m(x-1)$, where $m=1, 2, 3, 4$ (since $a \leqq x-1$). (2) may now be written

$$(4a + 2m + 1)^2 - m(2p + 2)^2 = 1.$$

If m is 1 or 4, there is no solution in integers. If m is 2 or 3, we have Pell's equation. If m is 3, solutions never yield integral values of x, but if m is 2 the equation becomes

$$(4a + 5)^2 - 2(2p + 2)^2 = 1$$

the smallest solution of which is $a=3$, $p=5$, successive values being given by the recurrence relations

$$a' = 17a + 12p + 32, \qquad p' = 24a + 17p + 46.$$

That the above two cases do not give the complete solution is shown by the examples: $x=97$, $a=34$, $b=94$, $p=57$, $q=71$; $x=161$, $a=148$, $b=142$, $p=154$, $q=136$.

Also partially solved by R. Venkatachalam Iyer, S. Parameswaran, and the Proposer.

Probability of an Odd Determinant

4544 [1953, 423]. *Proposed by D. J. Newman, Republic Aviation Corporation, Farmingdale, N. Y.*

The elements of a determinant are arbitrary integers. Determine the probability that the value of the determinant is odd.

Solution by E. H. Cutler, Lehigh University. Let P_n be the required probability where n, the order of the determinant, is fixed. Then P_n is also the probability that a determinant of elements 0 and 1 shall have its value $\equiv 1 \pmod 2$, and we now consider only values (mod 2) of n-rowed determinants with elements 0 or 1.

The probability that the first row shall be $(0, 0, \cdots, 0)$ is 2^{-n} and the probability that such a determinant have value 1 is 0. If the first row is $(1, 0, 0, \cdots, 0)$, the probability of the value 1 is P_{n-1}.

If the first row is $(1, x_2, \cdots, x_n)$, there is an elementary transformation re-

ducing each such determinant to one with first row $(1, 0, \cdots, 0)$, and this transformation is its own inverse (mod 2). Thus there exists a 1-1 value-preserving correspondence between the set of all determinants with the one first row and the set of all those with the other first row. Since the order of columns is immaterial, the probability that a determinant with given first row have the value 1 is P_{n-1} unless the first row is $(0, 0, \cdots, 0)$. Hence

$$P_n = (1 - 2^{-n})P_{n-1} = (1 - 2^{-1})(1 - 2^{-2}) \cdots (1 - 2^{-n})$$

since P_1 is $1/2$.

Solved also by W. J. Blundon, Leonard Carlitz, N. J. Fine, Harley Flanders, D. S. Greenstein, D. L. Johnson, D. N. Mesner, G. E. Raynor, E. H. Umberger, and the Proposer.

Editorial Note. As noted by several solvers, the present problem is merely the case $m=2$ of a result obtained by N. J. Fine and Ivan Niven. See their paper, The probability that a determinant be congruent to a (mod m), *Bulletin of the American Mathematical Society*, vol. 50, 1944, pp. 89–93.

A Divergent Series with a Special Property

4545 [1953, 423]. *Proposed by G. G. Lorentz, Wayne University, Detroit*

Let us say that a series, $\sum v_k$, $v_k \geqq 0$, is dominated by the series $\sum u_n$, $u_n \geqq 0$ if $v_k \geqq u_{n_k}$, $k=1, 2, \cdots$ for a sequence $n_1 < n_2 < \cdots$ of indices n. Construct a divergent series $\sum u_n$ such that each dominated series with decreasing terms is convergent.

Solution by L. A. Ringenberg, Eastern Illinois State College. For $m=0, 1, 2, \cdots$, let n_m be the smallest integer such that $\sum_{i=1}^{n_m} 1/i \geqq 2^m$. Let $\sum u_k$ be the series which results when the parentheses are removed from the following series:

$$\begin{aligned} &1 + \frac{1}{2}\left(\frac{1}{4} + \frac{1}{3} + \frac{1}{2} + 1\right) + \frac{1}{4}\left(\frac{1}{n_2} + \frac{1}{n_2 - 1} + \cdots + 1\right) + \cdots \\ &\quad + \frac{1}{2^m}\left(\frac{1}{n_m} + \frac{1}{n_m - 1} + \cdots + 1\right) + \cdots . \end{aligned} \tag{1}$$

Note that (1) dominates the series $1+1+1+\cdots$; hence (1) and also $\sum u_k$ are divergent. Note also that the terms in $\sum u_k$ may be grouped into blocks of consecutive terms (as suggested by the form of (1)) such that the terms within each block increase with the subscript. Let $\sum v_k$ be any series of nonnegative terms, terms nonincreasing as k increases, dominated by $\sum u_k$. Consider the terms in $\sum v_k$ which are dominated by terms in the block

$$\frac{1}{2^m}\cdot\frac{1}{n_m} + \frac{1}{2^m}\cdot\frac{1}{n_m - 1} + \cdots + \frac{1}{2^m}\cdot$$

There may be n_m terms, each one not exceeding $1/2^m n_m$; or there may be $n_m - 1$

terms, each one not exceeding $1/2^m(n_m-1)$; etc. Hence the sum of these terms of the v_k series does not exceed $1/2^m$ and therefore $\sum v_k \leqq 2$.

Also solved by George Piranian and the Proposer.

An Improper Integral

4546 [1953, 423]. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn, New York*

Evaluate

$$I = \int_0^\infty \frac{\sin \frac{mx}{2} \sin \frac{nx}{2} \sin \frac{m+1}{2} x \sin \frac{n+1}{2} x}{x^2 \sin^2 \frac{x}{2}} dx$$

where m and n are integers and $m \geqq n$.

Solution by J. V. Whittaker, University of California, Los Angeles. This integral is merely

$$\int_0^\infty \frac{1}{x^2} \sum_{r=1}^{m} \sum_{s=1}^{n} \sin rx \sin sxdx.$$

Integrating by parts and applying the sum and difference formulas, we obtain a known form:

$$\begin{aligned}\int_0^\infty \frac{\sin rx \sin sxdx}{x^2} &= \int_0^\infty \frac{r \cos rx \sin sx + s \sin rx \cos sx}{x} dx \\ &= \pi s/2 \qquad\qquad (r \geqq s).\end{aligned}$$

Summing over r and s, we find the value of I to be $(\pi/12)n(n+1)(3m-n+1)$.

Also solved by A. D. Anderson, B. A. Fleishman, Emil Grosswald, L. A. Ringenberg, O. E. Stanaitis, Chih-yi Wang, and the Proposer.

A Special Case of Familiar Inequalities

4547 [1953, 424]. *Proposed by H. S. Shapiro, New York University*

Let $0 \leqq a_i < 1$, $i = 1, \cdots, n$ and put $A = \sum a_i$. Prove

$$\sum_{i=1}^{n} \frac{a_i}{1-a_i} \geqq \frac{nA}{n-A},$$

equality occurring only if all the a_i are equal.

I. *Solution by L. A. Ringenberg, Eastern Illinois State College.*

Let $b_i = 1 - a_i$, $B = \sum b_i$, $B_{ij} = b_i/b_j + b_j/b_i$. Then

$$B_{ij} = \frac{b_i^2 + b_j^2}{b_i b_j} \geqq 2,$$

$$\sum_{i=1}^{n} \frac{B}{b_i} = n + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} B_{ij} \geqq n + 2\sum_{i=1}^{n-1} (n - i) = n^2,$$

whence, after dividing by B,

$$\sum_{i=1}^{n} \left(\frac{1}{b_i} - 1\right) \geqq \frac{n(n - B)}{B},$$

which is the required inequality for $a_i < 1$. Equality holds only if every $B_{ij} = 2$, only if all the b_i are equal, only if all the a_i are equal. If one or more of the $a_i = 1$, the inequality may be justified in the sense that, in the extended real number system, if the right member is $+\infty$ then so is the left member.

II. *Solution by Hermann von Schelling, Groton, Connecticut.* For $0 < b_i$, by the well known relation between the harmonic and arithmetic mean we have

$$\frac{1}{n} \sum_{i=1}^{n} \frac{1}{b_i} \geqq \frac{1}{\sum b_i/n}.$$

Putting $b_i = 1 - a_i$, and taking $a_i < 1$, this becomes

$$\frac{1}{n} \sum_{i=1}^{n} \frac{1}{1 - a_i} \geqq \frac{n}{n - A},$$

which reduces easily to the required inequality.

III. *Solution by Robert Frucht, Technical University Santa Maria, Valparaiso, Chile.* The inequality that has to be proved is only the special case: $m_1 = m_2 = \cdots = m_n = 1, f(x) = x/(1-x), a = 0, b = 1$, of the following more general theorem:* If, for $a \leqq x < b, f(x) \geqq 0$ and $f''(x) > 0$, then

$$\frac{\sum m_i f(a_i)}{\sum m_i} \geqq f\left(\frac{\sum m_i a_i}{\sum m_i}\right)$$

if $a \leqq a_i < b$ and $m_i > 0$ for $i = 1, 2, \cdots, n$; equality occurs only if all the a_i are equal.

Also solved by T. M. Apostol, W. J. Blundon, D. G. Duncan, W. B. Fulks and P. G. Kirmser, Harry Furstenberg, H. E. Goheen, D. S. Greenstein, J. W. Hardy, Jr., Vern Hoggatt, A. R. Hyde, J. B. Kelly, Joseph Lehner, Stanislaw Leja, D. C. Lewis, Viktors Linis, A. E. Livingston, D. C. B. Marsh, D. J. Newman, S. Parameswaran, George Piranian, H. W. E. Schwerdtfeger, Michael Skalskyj, O. E. Stanaitis, G. R. Trimble, Jr., Chih-yi Wang, Morgan Ward, J. V. Whittaker, and the Proposer.

* See Polya–Szegö, Aufgaben und Lehrsätze aus der Analysis, vol. 1, p. 53, Problem 74.

Topological Space in Which Every Permutation Is Continuous

4548 [1953, 482]. *Proposed by Joel Pitcairn and Robert Ellis, University of Pennsylvania*

Let X be a topological space with the property that every one-to-one transformation of X onto itself is continuous. Characterize the open subsets of X.

Solution by H. E. Vaughan, University of Illinois. It is slightly more convenient to characterize the closed subsets. Using "condensed" to describe a space which has no closed, non-empty, proper subsets and "discrete" to describe one each of whose subsets is closed, one has the following

THEOREM. *A space X has the property that every one-to-one transformation of X onto itself is continuous if and only if X is either condensed or discrete or there exists a transfinite cardinal $n \leqq \overline{\overline{X}}$ such that a proper subset F of X is closed if and only if $\overline{\overline{F}} < n$. ($\overline{\overline{F}}$ is* the cardinal number of the set F.)

Proof: The sufficiency is obvious. Suppose, then, that every one-to-one transformation of X onto itself is continuous (and hence a homeomorphism) and that X is neither condensed nor discrete. Let F be a closed non-empty proper subset of X, p a point of F, and q a point of $X-F$. Then, clearly, $(F-\{p\})+\{q\}$ is also closed. Hence every subset of F is, as an intersection of F and such sets, also closed. In particular X is a T_1-space and, since it is not discrete, is infinite. Furthermore, X is not the join of two closed proper subsets: If it were, then the two sets could, by the preceding result, be chosen disjoint. Now if p is any point of X and F is that one of the two sets to which p belongs, while q is a point of $X-F$, then $((F-\{p\})+\{q\})+(X-F)=X-\{p\}$ is closed. But this contradicts the assumption that X is not discrete.

It now follows that no closed proper subset of X has the same cardinal number as X. For if F were such a set then one would have $\overline{\overline{X-F}} \leqq \overline{\overline{X}} = \overline{\overline{F}}$ and there would be a one-to-one transformation of X on itself mapping $X-F$ in F. (If $\overline{\overline{X-F}} = \overline{\overline{F}}$, merely interchange the two sets. If $\overline{\overline{X-F}} < \overline{\overline{F}}$, then map $X-F$ on a subset F_1 of F. Since $\overline{\overline{F_1}} < \overline{\overline{X}}$ and X is infinite, $\overline{\overline{X-F_1}} = \overline{\overline{X}} = \overline{\overline{F}}$ and the transformation can be completed by mapping F on $X-F_1$.) Since every subset of F is closed and the transformation is continuous, $X-F$ is closed. But X is not the join of two closed proper subsets. Furthermore, if F is a closed proper subset of X then so is every subset of X which has the same cardinal number. For if F_1 is such a set, then, since $\overline{\overline{F}} < \overline{\overline{X}}$, $\overline{\overline{X-F}} = \overline{\overline{X}} = \overline{\overline{X-F_1}}$ and there exists a one-to-one transformation mapping F_1 on F and $X-F_1$ on $X-F$. Since this transformation is continuous, F_1 is closed.

It is now clearly sufficient to take n to be the least cardinal number greater than that of each closed proper subset of X.

It would be of interest to find a solution which does not, like the above, depend on the axiom of choice.

Also solved by G. U. Brauer, Helen F. Cullen, David Ellis and Robert Bagley, Azriel Rosenfeld, J. J. Schäffer, W. R. Scott, G. H. M. Thomas, and the Proposers.

The Four Feuerbach Tangents

4549 [1953, 482]. *Proposed by Richard Oblâth, Budapest, Hungary*

The Gauss-Newton line of the complete quadrilateral formed by the four Feuerbach tangents of a triangle is the Euler line of the triangle.

Solution by R. Goormaghtigh, Bruges, Belgium. The property has been given before by W. Gallatly, *Mathesis*, 1908, p. 33. It may be derived immediately from the fact that the Feuerbach tangents are the reciprocal lines to those joining the centroid G to the in- and excenters; hence, the considered tangents are common to the nine-point circle and the inner Steiner ellipse, and the Newton line of the quadrilateral formed by these tangents, being the locus of the centers of the conics tangent to them, is the Euler line which passes through the nine-point center and through G.

Generalization. The present writer has proved (*Mathesis*, 1922, p. 164) that if P is one of the points where a circumdiameter meets the *seventeen-point cubic*,* the orthopole of that diameter is the projection of P on the reciprocal line to PG. Hence:

If P_1, P_2, P_3, P_4 are any four points on the seventeen-point cubic, the perpendiculars erected at the orthopoles of the circumdiameters passing through P_1, P_2, P_3, P_4 on the lines joining these orthopoles to P_1, P_2, P_3, P_4, respectively, form a quadrilateral, the Newton line of which passes through the centroid.

When the points P_1, P_2, P_3, P_4 are the in- and excenters, the considered perpendiculars are tangent to the nine-point circle, and the Newton line passes then also through the center of that circle.

Consecutive Cubes Whose Difference Is a Triangular Number

4550 [1953, 482]. *Proposed by P. L. Chessin, Cooper Union, New York City*

I. The smallest consecutive cubes whose difference is a triangular number are 6^3 and 5^3. What are the next two pairs?

II. Under the assumption in I, show that this difference is equal to the difference of two squares of the form $(9a+1)^2-9a^2$.

To Victor Thébault this problem is humbly dedicated.

Solution by R. Venkatachalam Iyer, Trivandrum, India. The hypothesis gives

$$(N+1)^3 - N^3 = x(x+1)/2$$

which reduces to the Pellian

$$6q^2 + 3 = p^2$$

with $p=2x+1$ and $q=2N+1$. Positive integral solutions are given from $p_i+q_i\sqrt{6}=(3+\sqrt{6})(5+2\sqrt{6})^i$, and both p and q satisfy the recursion formula $u_i=10u_{i-1}-u_{i-2}$. The consecutive cubes, N^3 and $(N+1)^3$ are now easily found

* Locus of the points such that the joins of these points to their counter-points pass through the centroid.

to be 5^3 and 6^3, 54^3 and 55^3, 539^3 and 540^3, $\cdots$.

From $p_1=3$ and $p_2=27\equiv 3 \pmod{24}$ and the recursion formula, it follows that every $p_i\equiv 3 \pmod{24}$. Then $p=24a+3$ implies $x=12a+1$, whence

$$(N+1)^3-N^3=(12a+1)(12a+2)/2=(9a+1)^2-9a^2.$$

Also solved by Norman Anning, Leon Bankoff, W. J. Blundon, W. B. Carver, M. A. Kirchberg, D. C. B. Marsh, Eric Michalup, S. Parameswaran, M. J. Pascual, L. A. Ringenberg, R. E. Shafer, G. W. Walker, Chih-yi Wang and the Proposer, many of whom referred to the analogous problems E 702 [1946, 464] and 4299 [1950, 189]. and also a note by V. Thébault [1949, 174].

A Limit Involving the Gamma Function

4551 [1953, 482]. *Proposed by K.-F. Moppert, Basle, Switzerland*

For $k=0, 1, 2, \cdots$, prove

$$\lim_{n\to\infty} n^{\alpha-k}\sum_{\nu=0}^{n-1}(-1)^\nu\gamma^k\binom{\alpha}{\gamma}=\frac{1}{(k-\alpha)\Gamma(-\alpha)}\cdot$$

Remark: for $\alpha=-1$ this gives a well known asymptotic value for the sum of the kth powers.

Solution by Joseph Lehner, Los Alamos Scientific Laboratory. We first prove the formula for $k=0$, $\alpha\neq 0, 1, 2, \cdots$. Since

$$\binom{\alpha}{\nu}=\binom{\alpha-1}{\nu}+\binom{\alpha-1}{\nu-1}, \qquad \nu\geqq 1,$$

and $\binom{\alpha}{0}=\binom{\alpha-1}{0}=1$, we get

$$\begin{aligned}\sum_{\nu=0}^{n-1}(-1)^\nu\binom{\alpha}{\nu}&=\sum_{\nu=0}^{n-1}(-1)^\nu\binom{\alpha-1}{\nu}+\sum_{\nu=1}^{n-1}(-1)^\nu\binom{\alpha-1}{\nu-1}\\&=(-1)^n\binom{\alpha-1}{n-1}=\binom{-\alpha+n-1}{n-1}\\&\sim\frac{n^{-\alpha}}{\Gamma(-\alpha+1)}, \qquad n\to\infty,\end{aligned}$$

by Stirling's formula.

Now suppose the proposed formula proved for $0\leqq k<j$ and $\alpha\neq 0, 1, 2, \cdots$. We use

$$\binom{\alpha-1}{\nu}=\binom{\alpha}{\nu}\left[1-\frac{\nu}{\alpha}\right], \qquad \nu=0, 1, 2, \cdots,$$

and obtain

$$\sum_{\nu=0}^{n-1}(-1)^\nu\nu^{j-1}\binom{\alpha-1}{\nu}=\sum_{\nu=0}^{n-1}(-1)^\nu\nu^{j-1}\binom{\alpha}{\nu}-\frac{1}{\alpha}\sum_{\nu=0}^{n-1}(-1)^\nu\nu^j\binom{\alpha}{\nu}.$$

When α is not equal to a non-negative integer, this is true also of $\alpha-1$. We apply the induction hypothesis, getting

$$\lim_{n\to\infty} n^{\alpha-j}\sum_{\nu=0}^{n-1}(-1)^\nu \nu^j\binom{\alpha}{\nu} = \alpha \lim_{n\to\infty} n^{\alpha-j}\sum_{\nu=0}^{n-1}(-1)^\nu \nu^{j-1}\binom{\alpha}{\nu}$$

$$- \alpha \lim_{n\to\infty} n^{\alpha-1-(j-1)}\sum_{\nu=0}^{n-1}(-1)^\nu \nu^{j-1}\binom{\alpha-1}{\nu}$$

$$= 0 - \frac{\alpha}{(j-\alpha)\Gamma(-\alpha+1)} = \frac{1}{(j-\alpha)\Gamma(-\alpha)},$$

as required.

Both members of the formula are continuous functions of α for all α and so it still holds for $\alpha=0, 1, 2, \cdots$. The right member then has the value

$$\frac{d}{dx}\left(\frac{1}{\Gamma(x)}\right)_{x=-k}$$

for $\alpha=k$, and 0 for $\alpha\neq k$.

Also solved by H. W. Gould, O. E. Stanaitis, and Chih-yi Wang.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to E. P. Vance, Oberlin College, Oberlin, Ohio.

First Course in Abstract Algebra. By R. E. Johnson. New York, Prentice-Hall, Inc., 1953. 8+257 pages. $5.50.

This book is an introductory text in abstract algebra, intended as a year course for upper division students. It is not a survey of modern algebra, but is prepared as a working text on basic material, in which the hand and mind of the student are carefully guided, by easy stages and frequent illustrative examples, from concrete to abstract.

Starting with an introduction to mappings and relations, the elementary number systems, through the reals, are developed from the Peano axioms. Various categories of rings—integral domains, fields, polynomial rings—are studied in some detail, while the concept ring *per se* and the concept of ideal are only very lightly touched on at the conclusion of the book. Abstract groups and the symmetric group are treated in some twenty-five pages, ending with the mapping theorem. Linear theory—matrices, linear equations and determinants—are presented in modern form (about sixty pages). The book concludes with a very

short discussion of Boolean algebras.

In the opinion of the reviewer this is a significant contribution to the literature of this intermediate level. The arguments are clear and the student should be eased into the spirit of modern algebra without being frightened by too much generality or pathology. An adequate supply of exercises is offered, most of which are well within the range of the average student.

While a selection of material is of course required, it nevertheless seems unfortunate that the author thought it desirable to completely omit such conventional material as characteristic value theory and quadratic forms. Also a somewhat expanded treatment of groups, with a few applications, would have been welcome.

A. L. Foster
University of California

Principles of Numerical Analysis. By A. S. Householder. New York, McGraw-Hill, Inc., 1953. x+274 pages. $6.00.

With the recent increase in the application of science to practical problems and the attendant requirement for numerical answers has come a demand for computational procedures which are more efficient and suited to modern computing machinery. This has led to a resurgence of interest in numerical analysis and the development of new computational techniques with the result that many methods applicable to a wide range of special cases are now known. Many are new, others are modifications and refinements of older methods, some of which had long been forgotten. The results of these researches of the past ten years are scattered through the literature, both formal and informal, much of it not readily available nor widely known.

In *Principles of Numerical Analysis* A. S. Householder has assembled many of these results in a form conveniently available to the computer. This is a real service which will be welcomed by all who are interested in the art of computation. In spite of the author's statement in the preface that this text can be read by those who have had a course in calculus, this would seem to be a book for readers with some mathematical maturity and particularly with some knowledge of the theory of matrices and vector spaces. The treatment of these topics in the text could be improved and, instead of trying to make the book self-contained, it might have been better to have omitted the theory given there and have referred the reader to one of the standard works on the subject. The author's approach throughout the book is theoretical which should be welcomed by those mathematicians who have tried to learn numerical analysis by following detailed computations. Theories are given in their most general form, often containing as special cases older standard methods of computation, as the author points out. The author has limited himself for the most part to algebraic problems and has not treated at all the large field of differential and integral equations. This subject, however, is better known and has already been covered in several excellent books.

This reviewer was glad to see that the subject of errors has not been neglected. The book opens with a general discussion of the nature and source of errors arising in computation and later when treating special methods error estimates are frequently given. It would have been desirable to have had more of this, for an estimate of the error involved is an important part of any calculation

A large part of the book is devoted to computational problems involving matrices. Direct and iterative methods for solving systems of linear equations and finding the characteristic roots and vectors of matrices are treated. For many of the methods operational counts are given. These are the approximate number of multiplications, divisions and recordings required in the computation which, when using digital computers, is useful in estimating time requirements. The remainder of the book is devoted to non-linear equations, interpolation, orthogonal functions and numerical differentiation and integration. The book closes with a brief description of the Monte Carlo method. It is too bad there is not more on this important new computational technique. There is an extensive bibliography.

Unfortunately, the author has allowed an occasional inaccurate statement to mar the text. He frequently uses "complex" when he means "non-real." On page 122 occurs the statement " . . . if the equation is real, then the complex roots occur in conjugate pairs, . . . ," which is not true as it stands. The author often uses algebraic equation when he means polynomial equation. The expressions on page 138 for $u(x, y)$ and $v(x, y)$ are true if $f(z)$ is a polynomial with real coefficients. These are minor points which should cause the experienced reader no trouble, but they do detract from the over-all appearance of the book.

This should be a useful book, both as a reference for the experienced computer and as a text for one who wishes to learn the subject.

W. H. DURFEE
Operations Research Office

1. *Trigonometry*. By J. F. Randolph. New York, The Macmillan Company, 1953. 10+186+34 pages. $3.00.

2. *Plane Trigonometry*. By P. R. Rider. New York, The Macmillan Company, 1953. 8+152+28 pages. $3.00.

3. *Plane Trigonometry*, Second Edition. By A. W. Weeks and H. G. Funkhouser. New York, D. Van Nostrand and Company, Inc., 1953. 8+193+42 pages. $2.68 (without tables), $2.88 (with tables).

A successful editor once said that a good text should have at least one innovation. These three texts all qualify in this respect.

In their careful attention to form and presentation and teaching devices the authors are doing something about the difficult problem of teaching trigonometry. Each of these three texts meets the needs of the course for which it was designed.

If the present tendency to use smaller and still smaller type continues the student will soon need a high power magnifier. Since the war is terminated surely paper is not so scarce or expensive.

This reviewer is not commenting on the tables except to state that if the tables in each text are not considered adequate to meet the needs of a particular institution, separate tables are available.

Randolph seems to have introduced more innovations than the others. The distance formula is made basic in the subject yet his approach may cause the reader to overlook its importance. The definition of the co-function on page 34 is excellent. The use of projections in writing coordinates is very good. But has the author overlooked many of the advantages of the use of the projection theorems?

An important teaching aid is the use of the identity sign $\equiv$ instead of the equal sign $=$ when working with identities. The introduction of the scale factor a in the multiple angle formulas gives generality in a sense but may cause confusion in the mind of the student. On page 82 the introduction of the idea that if the product of two numbers is zero then at least one of the numbers is zero emphasizes an important concept that is frequently overlooked.

It will be interesting to note the reaction of educators to the introduction of complex numbers as the algebra of ordered pairs. From the advanced viewpoint it has much to recommend it. From the teaching approach is there already so much material (new to the student) in the course, that additional ideas serve only to confuse? This is a text which the experienced teacher will need to study carefully before each class exercise. It has a wonderful new approach and it seems especially good for the gifted student.

The Rider text is well written. The approach to logarithms and approximate numbers is excellent. The stress on the importance of an orderly arrangement of the work in numerical trigonometry is almost nil, although he uses a good form in the examples. The derivation of the law of tangents by geometric methods is good. The approach to Heron's formula is interesting. The treatment of vectors is brief. In view of its applications, has it been given sufficient attention? The material is condensed and the student should grasp the ideas readily.

The text by Weeks and Funkhouser is an excellent standard text with a large number of good problems. The check formula on page 55 by the use of projections is different and interesting. The development of sin $(A+B)$ by areas is an easy way for the student to grasp the relationship. The chapter on applications is good but some instructors may object to the emphasis given to problems connected with the military.

The authors include a little spherical trigonometry in the applications to navigation.

These three texts are all excellent and each one contains excellent teaching aids.

FRED ROBERTSON
Iowa State College

NEW BOOKS RECEIVED

Differential and Integral Calculus. By C. E. Love and E. D. Rainville. New York, The Macmillan Company, 1954. xiv+526 pages.

Fundamentals of College Mathematics. By J. C. Brixey and R. V. Andree. New York, Henry Holt and Company, 1954. xiv+609 pages. $5.90.

The Method of Trigonometrical Sums In the Theory of Numbers. By I. M. Vinogradov. New York, Interscience Publishers, Inc., 1954. x+180 pages. $5.00.

Gas Dynamics of Thin Bodies. By F. I. Frankl and E. A. Karpovich (Translated by M. D. Friedman). New York, Interscience Publishers, Inc., 1954. viii+175 pages. $5.75.

Eingefrangenes Unendilich Bekenntnis Zur Geschichte Der Mathematik. By Franz Von Krbek. Leipzig, Geest and Portig K. G., 1954. iv+332 pages. $5.25.

Analog Methods in Computation and Simulation. By W. W. Soroka. New York, McGraw-Hill Book Company, Inc., 1954. xii+390 pages. $7.50.

The Kinematics of Vorticity. By C. Truesdell. Bloomington, Indiana, Indiana University Press, 1954. xvii+232 pages. $6.00.

Wave Motion and Vibration Theory. Volume V. Proceedings of the Fifth Symposium in Applied Mathematics of the American Mathematical Society. Editor: Albert E. Heins. New York, McGraw-Hill Book Company, Inc., 1954. v+169 pages. $7.00.

Mathematics for the Secondary School. By W. D. Reeve. New York, Henry Holt and Company, 1954. xii+547 pages. $5.95.

Theoretical Physics. By F. W. Constant. Cambridge, Addison-Wesley Publishing Company, Inc., 1954. xiv+281 pages. $6.50.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

NUMERICAL ANALYSIS RESEARCH, UCLA

Numerical Analysis Research, University of California at Los Angeles, is continuing much of the research program of the Institute for Numerical Analysis (INA) of the National Bureau of Standards; INA was disbanded on June 30, 1954.

The new organization, administered by the UCLA Mathematics Department, will receive support from the Office of Naval Research and the Office of Ordnance Research. Its primary mission includes pertinent fundamental research in mathematics and science and research in the application of computers to problems occurring in science and other applied fields. It will be aided in at-

taining these objectives by the use of the National Bureau of Standards Western Automatic Computer (SWAC) and other equipment which has been loaned to UCLA. In addition, the new organization has the use of INA's library. Most of the research staff of INA has joined the new organization.

The organization offers a training program in the efficient application of high-speed digital computers. A number of graduate assistantships offered in co-operation with various university departments are available each year. Seminars and formal and informal courses in numerical analysis are conducted.

Correspondence should be addressed to: Numerical Analysis Research, University of California, 405 Hilgard Avenue, Los Angeles 24, California.

PRELIMINARY ACTUARIAL EXAMINATIONS PRIZE AWARDS

The winners of the prize awards offered by the Society of Actuaries to the nine undergraduates ranking highest on the score of Part 2 of the 1954 Preliminary Actuarial Examination are as follows:

First Prize of $200

Monsky, Paul	Swarthmore College

Additional Prizes of $100

Bowers, N. L.	Yale University
Croteau, Robert	University of Montreal
Driscoll, F. T.	Yale University
Fike, C. T.	University of the South
Freeman, D. N.	Yale University
Huff, R. W.	College of Wooster
Reinken, D. L.	Princeton University
Shapland, Robert	Drake University
Strang, W. G.	Massachusetts Institute of Technology

The Society of Actuaries has authorized a similar set of nine prizes for the 1955 examinations on Part 2.

The Preliminary Actuarial Examination consists of the following three parts:

Part 1. *Language Aptitude Test.*
(Reading comprehension, meaning of words and word relationships, antonyms, and verbal reasoning.)

Part 2. *General Mathematics Examination.*
(Algebra, trigonometry, coordinate geometry, differential and integral calculus.)

Part 3. *Special Mathematics Examination.*
(Finite differences, probability and statistics.)

The 1955 Preliminary Actuarial Examinations will be prepared by the Educational Testing Service and will be administered by the Society of Actuaries at centers throughout the United States and Canada on May 11, 1955 (tentative

date). The closing date for applications is March 15, 1955.

Detailed information concerning the Examinations can be obtained from: The Society of Actuaries, 208 South LaSalle Street, Chicago 4, Illinois.

SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS

The Society for Industrial and Applied Mathematics will hold its first national meeting in conjunction with the annual meetings of the American Mathematical Society, the Mathematical Association of America, and the Association for Symbolic Logic at the University of Pittsburgh on December 27–29. The following addresses will be presented to an evening meeting: "The History of a Problem," by Dr. Brockway McMillan, Bell Telephone Laboratories; "The Control of Industrial Operations," by Prof. Herbert A. Simon, Carnegie Institute of Technology; "Probability Theory in Liability and Property Insurance," by Mr. C. W. Crouse, Actuary, Preslan and Company. Further information can be obtained from H. W. Kuhn, Dalton Hall, Bryn Mawr College, Bryn Mawr, Pa.

PERSONAL ITEMS

Professor H. S. Pollard of Miami University was appointed to represent the Association at the inauguration of President H. B. Young, Western College for Women, on October 9, 1954.

Professor T. J. Higgins of the University of Wisconsin has been presented with the George Westinghouse award of $1,000 for outstanding teaching by the American Society for Engineering Education.

Dr. W. A. Shewhart has been given the first Honorary Professorship in Statistical Quality Control by Rutgers University.

Assistant Professor J. V. Talacko of Marquette University has received a Ford Foundation Fellowship and plans to spend most of the year at the Statistical Laboratory, University of California.

The Educational Testing Service announces the following awards: Mr. R. F. Boldt, graduate student at Princeton University, and Mr. D. P. Estavan, graduate student at Stanford University, have received fellowship awards for graduate study in psychology at Princeton University; Mr. J. A. Keats, member of the staff of the Australian Council for Educational Research in Melbourne, Australia, has been awarded the Visiting Psychometric Fellowship for the second year; Mr. Norman Cliff, Mr. Bertram Karon, Mr. Anton Morton, and Mr. Robert Sadacca have been re-appointed as Psychometric Fellows.

Mr. M. L. Anthony, former senior research engineer at the Armour Research Foundation, Chicago, Illinois, is a research engineer at American Machine and Foundry Company, Chicago.

Dr. R. W. Bagley of the University of Florida has accepted a position as an assistant professor at the University of Kentucky.

Mr. E. H. Batho, previously a teaching assistant at the University of Wisconsin, has been appointed to an instructorship at the University of Rochester.

Mr. R. E. Bayles, a graduate student of Harvard University, is an actuarial assistant for the John Hancock Mutual Life Insurance Company, Boston, Massachusetts.

Assistant Professor Helen P. Beard of Newcomb College has returned from her leave of absence at the Statistical Laboratory, University of California.

Miss Barbara J. Beechler, formerly an instructor at Smith College, is now at the State University of Iowa as a part-time instructor.

Professor L. M. Blumenthal of the University of Missouri will be on leave during 1954–55 to serve as Fulbright Lecturer at the University of Leiden, The Netherlands.

Dr. R. N. Bradt, previously a graduate assistant at Stanford University, has been appointed to an assistant professorship at the University of Kansas.

Mr. A. R. Brown, Jr., recently a mathematician at Aberdeen Proving Ground, Maryland, has been appointed Associate Professor at Drury College.

Associate Professor F. J. H. Burkett of Union College has been promoted to a professorship.

Assistant Professor V. B. Caris of Ohio State University has retired.

Mr. R. C. Courter, formerly a student at Columbia University, is now a mathematician at the Ballistics Research Laboratory, Aberdeen Proving Ground, Maryland.

Mr. G. L. Crumley, recently a graduate student at the University of Kansas, is now an instructor at Coffeyville College.

Mr. R. A. Dean, previously a lieutenant in the U. S. Navy, is a fellow at the California Institute of Technology.

Assistant Professor R. D. Depew of Florence State Teachers College, Alabama, has been promoted to an associate professorship.

Associate Professor R. J. Dunholter of the University of Cincinnati has been promoted to a professorship.

Reverend A. J. Eiardi, formerly chairman of the Mathematics Department of Boston College, has accepted a position as an associate professor at Fairfield University.

Mrs. Cecile S. Feder, instructor at Hunter College, has been appointed Registrar and Assistant Professor of Mathematics at Stern College for Women, Yeshiva University.

Mr. G. F. Feeman, graduate assistant at Lehigh University, has been appointed Instructor in Mathematics and Physics at Muhlenberg College.

Mr. H. H. Fox, previously a research physicist at Mound Laboratories, Miamisburg, Ohio, has accepted a position as a mathematician at the Applied Physics Laboratory, Johns Hopkins University.

Mr. R. A. Gambill, a former member of the Federal Public Housing Administration, Indianapolis, Indiana, is engaged as a mathematician at the U. S. Naval Ordnance Laboratory, Indianapolis.

Assistant Professor S. T. Gormsen of the University of Florida has been appointed to a professorship at Rollins College.

Mr. J. C. Gould, recently a student at Wabash College, is with the United States Army as a supply records specialist.

Professor Simon Green of Philander Smith College has accepted a position as an assistant professor at the University of Tulsa.

Mr. R. L. Gulley, formerly a mathematician with the Bureau of Ships, Navy Department, Washington, D. C., is a mathematician with the Naval Electronics Laboratories, Point Loma, California.

Dr. Henry Helson of Yale University has been promoted to an assistant professorship.

Associate Professor Carl Holtom of the U. S. Air Force Institute of Technology, Wright-Patterson Air Force Base, Dayton, Ohio, has resigned to become a staff member of the University of California's Los Alamos Scientific Laboratory.

Mrs. Betty W. Holz, who was an engineer with Melpar Incorporated, Alexandria, Virginia, has accepted a position as an operations research analyst with Johns Hopkins University.

Professor S. B. Jackson of the University of Maryland has succeeded Professor M. H. Martin as Head of the Mathematics Department.

Professor E. F. W. Jones of Rollins College retired in June, 1954.

Professor Chosaburo Kato, formerly chairman of the Mathematics Department of Denison University, has been promoted to the newly created senior professorship.

Mr. M. A. Kirchberg, former student at Eastern Illinois State College, has accepted a position as teacher at Hopkins Township School, Hopkins, Michigan.

Mr. Charles Kurland, a project engineer at the Chevrolet Aviation Engine Company, Buffalo, New York, is now a test engineer for the Chevrolet Engineering Laboratory, Detroit, Michigan.

Mr. Milton Lees, a former student at the University of California, has been promoted to a research assistantship.

Professor S. L. Levy of Brown University has been appointed Manager of the Applied Physics Division of the Midwest Research Institute.

Professor M. H. Martin of the University of Maryland has been appointed Director of the University's Institute for Fluid Dynamics and Applied Mathematics.

Mr. K. A. McGown, former instructor at Lafayette College, is teaching at Boonton High School, Boonton, New Jersey.

Mr. Michael Montalbano, previously employed as Chief of the Programming Section of the Computer Control Company, Port Mugu, California, is now a mathematician at the Kaiser Steel Corporation, Fontana, California.

Reverend T. F. Mulcrone has been transferred from Spring Hill College to St. Charles College.

Mr. R. C. Nickerson, recently a student at Brown University, has a position as engineer at the General Electric Company, Physics Training Program, Utica, New York.

Mr. A. H. Payne, formerly a mathematician at the Ballistics Research Laboratories, Aberdeen Proving Ground, Maryland, is an analyst for the Raytheon Manufacturing Company, Waltham, Massachusetts.

Mr. W. J. Pervin, previously a teaching assistant at Carnegie Institute of Technology, is a senior scientist at Westinghouse Atomic Power Division, Pittsburgh, Pennsylvania.

Dr. S. R. Peterson of Union College has been promoted to the position of Assistant Professor of Philosophy.

Assistant Professor T. J. Pignani of Loyola University is at the University of North Carolina for the year 1954–55.

Mr. E. J. Polak, senior project engineer at the Curtiss-Wright Corporation, Carlstadt, New Jersey, has been appointed to an assistant professorship at Bucknell University.

Associate Professor S. E. Rauch has been promoted to a professorship at the University of California, Santa Barbara.

Mr. Saul Rosen, associate research engineer at Burroughs Adding Machine Company, Philadelphia, Pennsylvania, has been appointed to an associate professorship at Wayne University and is a member of the staff of the Computation Laboratory.

Mr. A. I. Rosenfeld, a graduate student at Columbia University, has a position as a junior physicist at the Freed Electronics and Controls Corporation, New York City.

Professor W. E. Roth of the University of Tulsa has resigned from this position and retired from teaching.

Miss Miriam J. Russell, previously an instructor at the University of Arizona, is now a teacher at Wakefield High School, Arlington, Virginia.

Associate Professor W. A. Rutledge of Alabama Polytechnic Institute has accepted a position as an assistant professor at the University of Tulsa.

Mr. W. B. Stovall, Jr., formerly Chief of the Vital Statistics Section of the State Board of Health, Jacksonville, Florida, has accepted a position as instructor at the University of Florida.

Associate Professor W. B. Temple of Louisiana Polytechnic Institute has been promoted to a professorship.

Dr. L. E. Ward, Jr., of the University of Nevada has been appointed to a professorship at the University of Utah.

Mr. W. G. Younkin, formerly a student at the University of California, Los Angeles, is a physicist at the U. S. Naval Ordnance Test Station, China Lake, California.

Emeritus Professor E. W. Pehrson of the University of Utah died on October 11, 1953. He was a charter member of the Association.

Emeritus Dean R. P. Stephens of the University of Georgia died on June 1, 1954. He was a charter member of the Association.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE THIRTY-FIFTH SUMMER MEETING OF THE ASSOCIATION

The thirty-fifth summer meeting of the Mathematical Association of America was held at the University of Wyoming, Laramie, Wyoming, on Monday and Tuesday, August 30 and 31, 1954, in conjunction with the summer meetings of the American Mathematical Society and the Canadian Mathematical Congress. About four hundred and fifty adults were registered including the following two hundred and thirty-five members of the Association:

L. V. Ahlfors, C. B. Allendoerfer, J. M. Anderson, R. D. Anderson, R. V. Andree, H. A. Antosiewicz, T. M. Apostol, Nachman Aronszajn, W. F. Atchison, D. F. Atkins, R. W. Ball, C. F. Barr, M. A. Basoco, S. Louise Beasley, E. M. Beesley, E. G. Begle, B. C. Bellamy, J. S. Bendat, A. I. Benson, Arthur Bernhart, R. H. Bing, R. G. Blake, G. U. Brauer, W. E. Briggs, R. W. Brink, Leonard Bristow, J. R. Britton, Myrtle C. Brown, R. G. Buschman, R. K. Butz, J. E. Byrne, R. L. Calvert, F. M. Carpenter, R. E. Carr, Harold Chatland, R. V. Churchill, S. D. Conte, T. F. Cope, K. L. Cooke, W. R. Cowell, H. M. Cox, R. B. Crouch, A. B. Cunningham, J. H. Curtiss, P. H. Daus, W. M. Davis, Jim Douglas, Jr., T. C. Doyle, D. M. Dribin, W. L. Duren, Jr., J. C. Eaves, G. M. Ewing, H. J. Fletcher, J. S. Frame, W. B. Fulks, M. G. Gaba, J. W. Gaddum, R. A. Gambill, A. E. Gault, H. M. Gehman, Michael Goldberg, Michael Golomb, S. H. Gould, H. T. Guard, R. R. Gutzman, W. T. Guy, Jr., Marian S. Gysland, Edwin Halfar, P. C. Hammer, J. R. Hanna, W. L. Hart, C. L. Hassell, Jr., F. C. Hatfield, Nola L. A. Haynes, Leota C. Hayward, T. J. Head, I. L. Hebel, E. R. Heineman, Melvin Henriksen, Anna S. Henriques, J. G. Herriot, I. N. Herstein, Fritz Herzog, J. F. Heyda, J. J. L. Hinrichsen, S. T. Hu, Ralph Hull, J. A. Hummel, M. Gweneth Humphreys, N. C. Hunsaker, J. W. Hurst, W. R. Hutcherson, C. A. Hutchinson, Jane C. Ingersoll, L. K. Jackson, R. D. James, C. A. Johnson, L. W. Johnson, Ernest Johnston, L. S. Johnston, D. H. Jones, F. B. Jones, P. S. Jones, Dora E. Kearney, M. W. Keller, J. B. Kelly, D. E. Kibbey, E. C. Kiefer, V. L. Klee, Jr., J. C. Knipp, F. W. Kokomoor, R. B. Kriegh, Ruth G. Lane, C. E. Langenhop, Leo Lapidus, E. H. Larguier, W. I. Layton, J. R. Lee, D. H. Lehmer, R. B. Leipnik, W. J. LeVeque, Gene Levy, F. A. Lewis, C. H. Lindahl, W. D. Lindstrom, Charles Loewner, A. J. Lohwater, R. G. Long, A. T. Lonseth, L. H. Loomis, Lee Lorch, M. L. Madison, Morris Marden, Margaret E. Mauch, B. H. McCandless, Garner McCrossen, W. C. McDaniel, A. W. McGaughey, S. W. McInnis, J. H. McKay, E. B. McLeod, Jr., D. F. Mela, A. S. Merrill, B. C. Meyer, R. J. Michel, R. J. Mihalek, E. B. Miller, C. B. Morrey, Jr., E. D. Mouzon, Jr., M. E. Mullings, C. H. Murphy, Jr., W. H. Myers, Greta Neubauer, T. A. Newton, C. O. Oakley, Morris Ostrofsky, T. G. Ostrom, L. J. Paige, B. J. Pettis, C. G. Phipps, George Piranian, Everett Pitcher, G. B. Price, M. H. Protter, W. T. Puckett, Jr., Henry Rainbow, Gordon Raisbeck, J. F. Randolph, O. M. Rasmussen, G. E. Raynor, O. H. Rechard, O. W. Rechard, A. W. Recht, P. V. Reichelderfer, Haim Reingold, H. B. Ribeiro, D. E. Richmond, F. A. Rickey, J. D. Riley, E. K. Ritter, Calvin A. Rogers, G. F. Rose, Mary E. Rudin, R. G. Sanger, Robert Schatten, O. F. G. Schilling, Nathan Schwid, W. R. Scott, George Seifert, A. L. Shields, Sister Mary Corona, Sister Mary Felice, Abe Sklar, M. F. Smiley, A. J. Smith, G. W. Smith, S. R. Smith, W. N. Smith, L. C. Snively, Andrew Sobczyk, J. McD. Staley, R. D. Stalley, D. W. Starr, P. O. Steen, B. M. Stewart, Ruth W. Stokes, E. B. Stouffer, D. R. Sudborough, J. G. Sutton, P. M. Swingle, J. M. Thomas, C. J. Thorne, R. M. Thrall, W. J. Thron, Wilmont Toalson, C. B. Tompkins, Leonard Tornheim, C. W. Trigg, A. W. Tucker, W. R. Utz, Jr., V. J. Varineau, R. W. Veatch, G. L. Walker, J. A. Ward, L. A. Ware, M. S. Webster, J. G. Wendel, A. L. Whiteman, L. R. Wilcox, W. L. Williams, G. M. Wing, R. M. Winger, Oswald Wyler, C. R. Wylie, Jr., Max Wyman.

Sessions of the Association were held on Monday morning and afternoon and on Tuesday morning in the Education Auditorium of the University of Wyoming, with Professors C. F. Barr, R. D. James, and J. S. Frame presiding. The third series of Earle Raymond Hedrick Lectures was delivered by Professor L. H. Loomis of Harvard University. The Program Committee for the meeting consisted of Marshall Hall, Jr., Chairman; Harold Chatland, B. E. Meserve, and G. deB. Robinson.

FIRST SESSION OF THE ASSOCIATION

The Earle Raymond Hedrick Lectures: "Convex Sets," Lecture I, by Professor L. H. Loomis, Harvard University.

Symposium on Educational Aspects of Computers

"The Educational Impact of the Illiac," by Professor W. F. Atchison, University of Illinois.

"Effects of Large Digital Computers on Numerical Analysis Curricula," by Professor C. B. Tompkins, University of California at Los Angeles.

"A Machine's-eye View of Numerical Analysis," by Professor D. H. Lehmer, University of California.

SECOND SESSION OF THE ASSOCIATION

The Earle Raymond Hedrick Lectures: "Convex Sets," Lecture II, by Professor L. H. Loomis, Harvard University.

Joint Session with the Canadian Mathematical Congress

"Some Geometrical Applications of Taylor's Formula," by Professor Peter Scherk, University of Saskatchewan.

"Asymptotic Expansions," by Professor Max Wyman, University of Alberta.

"Systems of Congruences," by Professor B. M. Stewart, Michigan State College.

THIRD SESSION OF THE ASSOCIATION

The Earle Raymond Hedrick Lectures: "Convex Sets," Lecture III, by Professor L. H. Loomis, Harvard University.

"The Preparation of College Mathematics Teachers," by Professor P. V. Reichelderfer, Ohio State University.

"Integral Transformations and Differential Equations," by Professor R. V. Churchill, University of Michigan.

"Metrics and Matrices," by Professor A. T. Lonseth, Oregon State College.

MEETING OF THE BOARD OF GOVERNORS

The Board of Governors of the Association met on Monday evening in the Education Lounge of the University of Wyoming. Twenty-one of the forty

members of the Board were present. Among the more important items of business transacted were the following:

Professor E. N. Oberg of the State University of Iowa was elected to serve as Governor from the Iowa Section for a term expiring in June 1956 to succeed the late Professor D. L. Holl.

Professor Mark Kac of Cornell University was invited to deliver the fourth series of Hedrick Lectures at the 1955 Summer Meeting.

Announcement was made of the following grants received by the Association during the current year: a grant of $15,000 from the National Science Foundation for the support of the Program of Visiting Lecturers; a grant of $2,500 from the Social Science Research Council for the work of the Committee on the Undergraduate Program; a grant of $1,000 from the National Research Council to assist with the publication of Slaught Paper Number 3 which contains the "Proceedings of the Symposium on Special Topics in Applied Mathematics." In each case, the Association will supplement from its own funds the amounts granted by these other organizations.

It was announced that the membership of the Association was 5,615 on August 20.

MEETING OF SECTION OFFICERS

A meeting of Officers of the Sections of the Association was held on Tuesday evening in the Education Lounge. Thirty-four persons were present representing twenty-three of the twenty-five Sections of the Association. The following topics were discussed: publication of accounts of Section Meetings in the MONTHLY, the Program of Visiting Lecturers, the Committee on Employment Opportunities, programs of Section Meetings, finances of the Sections, Section activities, such as contests and lecturers, and reports of special committees of certain Sections. A report was distributed from the Illinois Section Committee on Strengthening the Teaching of Mathematics in Secondary Schools.

MEETINGS OF OTHER ORGANIZATIONS

Sessions of the American Mathematical Society began on Tuesday afternoon and continued through Friday morning. Invited addresses were given by Professors R. D. James, Charles Loewner, R. S. Phillips, J. G. Wendel, and R. E. Bellman.

The Canadian Mathematical Congress met jointly with the Mathematical Association of America on Monday afternoon.

ARRANGEMENTS, ENTERTAINMENT AND RECREATION

The Committee on Arrangements for the meeting consisted of: C. F. Barr, Chairman; H. M. Gehman, J. W. Green, Greta Neubauer, W. T. Puckett, O. H. Rechard, Nathan Schwid, S. R. Smith, W. N. Smith, P. O. Steen, V. J. Varineau.

Registration headquarters was in the reception room of Wyoming Hall of the University of Wyoming. Dormitory accommodations were available in Wyoming Hall and in Knight Hall from Sunday, August 29 until Saturday, Sep-

tember 4. Meals were served cafeteria style in the Knight Hall Cafeteria. A tea for the women attending the meeting was held on Monday afternoon in Knight Hall Banquet Room. On Monday evening movies showing scenes in Wyoming were shown in the Liberal Arts College Auditorium. On Tuesday afternoon a trip was conducted to Vedauwoo, a place of great natural beauty in the mountains east of Laramie.

On Wednesday afternoon there was a trip by bus and car to Snowy Range, lying forty miles west of Laramie, which was followed by a wild-game (deer, antelope, elk) steak fry at the University Recreation Camp, at the foot of Snowy Range.

A square dance was held on Thursday evening. A local square dance group gave several exhibition numbers during the evening.

At the steak fry on Wednesday afternoon, President George D. Humphrey of the University of Wyoming expressed the greetings of the University to the assembled mathematicians. Professor W. L. Duren responded for the mathematical organizations and presented a resolution of thanks which was adopted unanimously expressing appreciation to the officers of the University and to the members of the Mathematics Department for their efforts in making this meeting so successful and enjoyable.

HARRY M. GEHMAN, *Secretary-Treasurer*

THE MAY MEETING OF THE MINNESOTA SECTION

The May meeting of the Minnesota Section of the Mathematical Association of America was held at Hamline University in St. Paul, Minnesota, on May 8, 1954. Sessions were held in the forenoon, at luncheon and in the afternoon. Professors K. H. Bracewell, G. K. Kalisch and A. G. Hill, Chairman of the Section, presided at the respective sessions.

Seventy-one persons attended the meeting including the following fifty-eight members of the Association:

N. E. Albrecht, H. M. Anderson, Souren Babikian, E. J. Berger, H. D. Block, C. R. Bonnell, K. H. Bracewell, G. U. Brauer, R. W. Brink, W. H. Bussey, Anne W. Calloway, J. M. Calloway, E. J. Camp, C. S. Carlson, Elizabeth Carlson, H. D. Colson, W. L. Duren, Jr., Helen Engebretson, I. C. Fischer, Walter Fleming, Abraham Franck, Gladys Gibbens, Ruby M. Grimes, K. L. Hankerson, Charles Hatfield, F. C. Hatfield, A. G. Hill, Hildegarde H. Howden, R. T. John, D. A. Johnson, G. K. Kalisch, W. S. Loud, W. J. Lyche, H. B. MacDougal, K. O. May, W. H. McBride, W. R. McEwen, E. O. Nelson, J. M. H. Olmsted, J. C. Peterson, Ruth Scholten, Sister M. Bibiana, Sister M. Joanne, Sister Mary Leontius, F. C. Smith, Marion V. Smith, O. E. Stanaitis, A. G. Swanson, F. J. Taylor, Takashi Terami, Matilda B. Thompson, H. L. Turrittin, Frances E. Walsh, K. W. Wegner, Irene L. Wente, R. P. Winter, F. L. Wolf, L. G. Woodby.

At the business meeting, the following officers were elected for the coming year: Chairman, Sister Mary Leontius, College of St. Teresa; Secretary, Professor F. C. Smith, College of St. Thomas; Executive Committee, Professor Walter Fleming, Mankato State Teachers College, Professor A. G. Hill, North

Dakota Agricultural College, Professor Fulton Koehler, University of Minnesota.

By invitation of the Executive Committee, Professor W. L. Duren, Jr., of Tulane University delivered an address at the morning session entitled "Boundary Conditions for the Undergraduate Program in Mathematics."

The following short papers were presented:

1. *The game tree for a simplified form of chess*, by Professor F. L. Wolf, Carleton College.

A simplified form of chess which uses a board of only nine squares and has each player starting with just a king and a queen was considered. A game tree was constructed for the game and optimal strategies were immediately found from the tree. Even without an effective stop rule to limit the number of moves, it is possible to show all essentials of the tree by making use of the fact that the physical position of the board is repeated in any cyclic sequence of moves. The optimal strategies for chess itself can be found (theoretically) in a similar manner.

2. *The tangential approach to area integration*, by Mr. C. R. Bonnell, Cedar Engineering Corporation.

The area bounded by a curve $y=f(x)$ and the x-axis in the interval $a \leqq x \leqq b$ can be computed by the following formula:

$$\int_a^b f(x)dx = \frac{1}{2}\frac{[f(b)]^2}{f'(b)} - \frac{1}{2}\frac{[f(a)]^2}{f'(a)} + \frac{1}{2}\int_a^b \frac{[f(x)]^2 f''(x)}{[f'(x)]^2}dx.$$

The development of the area equation indicates that another rule for computing areas using tangents and subtangents can be added to the list with Simpson's rule.

A special case of interest is $f(x)=ax^2$ because the two right triangles at the end points have a difference in area of 3/4 that of the area bounded.

3. *Calculating machines and mathematicians as they seemed to Oliver Wendell Holmes*, by Professor W. H. Bussey, University of Minnesota.

Professor Bussey read the following quotation from *The Autocrat of the Breakfast Table* and made some remarks about it: "Given certain factors, a sound brain should always evolve the same fixed product with the certainty of Babbage's calculating machine. What a satire, by the way, is that machine on the mere mathematician! A Frankenstein monster, a thing without brains and without heart, too stupid to make a blunder; that turns out results like a corn sheller, and never grows any wiser or better though it grind a thousand bushels of them!

"I have an immense respect for a man of talents *plus* 'the mathematics.' But the calculating power alone should seem to be the least human of qualities and to have the smallest amount of reason in it; since a machine can be made to do the work of three or four calculators and better than any of them. Sometimes I have been troubled that I had not a deeper intuitive apprehension of the relations of numbers. But the triumph of the ciphering hand-organ has consoled me. I always fancy I can hear the wheels clicking in a calculator's brain."

4. *A theorem on transcendentality*, by Mr. Richard Juberg, University of Minnesota, introduced by Professor S. E. Warschawski.

Define $p^{(n)}=p(p-1)\cdots(p-n+1)$; $p^{(0)}=1$. Let a be algebraic and non-rational; let $b=ac+d$ where c and d are rational.

Theorem: The analytic function $f_\lambda(z)$, generated by the power series $\sum_{n=0}^{\infty}(an+b)^{(n+\lambda)}z^n/n!$, $(\lambda=0, 1, 2, \cdots)$, has the property that for all algebraic z except $z=0$, the corresponding functional values are all transcendental. An outline of the proof was given under the assumptions that

c and d were integers and that z was confined to the circle of convergence of the generating functional element.

5. *On convolutions in number theory*, by Professor Jan Popken, University of Utrecht and the University of Minnesota, introduced by Professor S. E. Warschawski.

Let $f(n)$ and $g(n)$ be two arithmetical functions. The new arithmetical function of n defined by $\sum_{d\delta=n} f(d)g(\delta)$ is called the convolution of $f(n)$ and $g(n)$. It is denoted by $f(n) * g(n)$. The commutative and associative laws hold.

Some proofs in the theory of numbers can be simplified considerably by the introduction of these convolutions. This was shown first for a simple example and then for the elementary proof of the prime number theorem.

6. *On the minimality of the variational principles of classical particle mechanics*, by Professor H. D. Block, University of Minnesota.

An elementary device, apparently overlooked in the standard techniques of the calculus of variations, is particularly effective in dealing with the variational integrals of mechanics and enables us to settle the following question.

Which of the variational principles are minimal principles (under suitable conditions), which of them are not minimal principles under any conditions and what are the "suitable conditions"?

F. C. SMITH, *Secretary*

THE MAY MEETING OF THE UPPER NEW YORK STATE SECTION

The tenth annual meeting of the Upper New York State Section of the Mathematical Association of America was held at New York State College for Teachers at Albany, New York, on May 1, 1954. The Chairman of the Section, Professor Harriet F. Montague of the University of Buffalo, presided at the morning session, and the Vice-Chairman, Professor J. R. F. Kent of Harpur College, presided at the afternoon session.

Seventy-three persons attended the meeting, including the following fifty-three members of the Association:

E. B. Allen, M. R. Bates, R. A. Beaver, Col. W. W. Bessell, Harry Birchenough, E. A. Butler, Ethel B. Callahan, J. D. Campbell, A. J. Coleman, E. J. Downie, O. J. Farrell, A. H. Fox, N. S. Free, J. E. Freund, H. M. Gehman, Lillian Gough, N. G. Gunderson, A. S. Hendler, H. K. Holt, H. F. Hunter, J. R. F. Kent, D. E. Kibbey, Violet H. Larney, R. D. Larsson, Roger Lessard, Caroline A. Lester, J. V. Limpert, R. C. Luippold, Ingo Maddaus, Jr., Dis Maly, J. N. Mangnall, June M. McArtney, Myles McConnon, Norman Miller, Harriet F. Montague, Mabel D. Montgomery, D. S. Morse, C. W. Munshower, C. V. Newsom, F. D. Parker, M. J. Pascual, Valdemars Punga, Rev. Timothy Reardon, D. A. Robinson, H. M. Rosenbaum, Edith R. Schneckenburger, Sister Noel Marie, Ruth W. Stokes, W. C. Stone, R. L. Swain, Nura D. Turner, F. C. Warner, W. G. Warnock.

At the business meeting the following officers were elected: Chairman, Professor J. R. F. Kent, Harpur College; Vice-Chairman, Professor C. E. Rhodes, Alfred University; Secretary, Professor N. G. Gunderson, University of Rochester. A motion was passed which created an Executive Committee consisting of the officers of the Section and the Sectional Governor. The motion directed

the Executive Committee to prepare a revision of the By-Laws of the Section and to present the revision at the 1955 meeting for the consideration of the members.

The following papers were presented:

1. *A program for the preparation of secondary school teachers of mathematics*, by Professor R. A. Beaver, New York State College for Teachers at Albany.

The program for training junior and senior high school teachers of mathematics at the New York State College for Teachers at Albany was described. In terms of semester credit hours, the usual program consists of distributional requirements other than mathematics, 38 hours; professional courses, 18 hours; mathematics, 24 hours; a minor field, 24 hours; free electives, 20 hours.

2. *Potential of a charged cylinder between two parallel grounded planes*, by Dr. Hillel Poritsky, General Electric Company; read by Mr. R. A. Powell, General Electric Company.

A solution is obtained for the harmonic function $V(x, y)$ in the region outside a circle $r=a$, $a<\pi/2$, and between the lines $x=\pm\pi/2$, subject to the boundary conditions $V=1$ on $r=a$, $V=0$ on $x=\pm\pi/2$.

The conditions $V=0$ on $x=\pm\pi/2$ are satisfied by putting $V=\text{Re}\,[f(z)]$, $z=x+iy$, and letting $f(z)=\sum C_n F^{(n)}(z)$,where $F=-\ln\tan(z/2)$, C_n are real, and n is even. By expanding $F+\ln(z/2)$ in powers of z, differentiating to obtain similar expansions for $F^{(n)}$, and applying the condition $V=1$ on $r=a$, one obtains an infinite set of linear equations in C_n. These are solved by an iteration method based on the observation that the main diagonal coefficients are large compared to the remaining coefficients.

3. *Statistics and engineering*, by Professor Roger Lessard, École Polytechnique, Montreal.

The speaker reviewed the developments of the engineering applications of statistics. By several examples, he showed why numerous societies have recommended the introduction of statistical courses in all engineering curricula, and gave the different ways used by universities to solve this problem. He also recommended that the statistical concept be introduced as soon as possible in high schools.

4. *The 1953 Summer Conference in Collegiate Mathematics*, by Professor Ethel B. Callahan, Hartwick College.

The speaker described the objectives and organization of the Colorado Conference, and reported on the courses and other activities offered.

5. *Representations of compact Lie groups*, by Professor A. J. Coleman, University of Toronto.

The classification of simple compact Lie groups and their representations, originally due to Cartan, is presented using the in-the-large methods of H. Weyl and H. Hopf. The Pontrjagin duality theory permits the association with the group G of an n-dimensional lattice, characterizing G, if G has rank n. All possible lattices arising in this way have been classified by Coxeter [*Can. J. of Math.* vol. 3, 1951, pp. 391–441] who associates with each lattice a connected graph. The use of the Coxeter graph in studying the properties of a representation is illustrated by a simple solution of the problem of determining the number of homologues of any weight in a representation. The following inclusion relations amongst the exceptional Lie groups is pointed out: $G_2\subset F_4\subset E_6 \subset E_7\subset E_8$.

6. *Mathematics as taught to engineers by Dr. Steinmetz*, by Mr. P. L. Alger, General Electric Company, introduced by Miss Nura D. Turner.

Dr. Charles P. Steinmetz was a great teacher, who did far more than any one else to educate electrical engineers in the then new science of alternating currents. His 300-page book *Engineering Mathematics* (1911) begins with arithmetic and goes through trigonometry, differential equations, and the theory of functions. He made mathematics appear as a simple well-knit whole, and all its processes appear to be merely extensions of arithmetic. By using general numbers, $a+jb$, throughout, instead of only real numbers, he telescoped all kinds of mathematics into simple and usable forms, especially useful to engineers.

7. *A non-linear problem from nuclear reactor theory*, by Professor A. H. Fox, Union College.

Consideration of the relation between power and temperature in a nuclear reactor with a negative temperature coefficient of reactivity leads to a non-linear differential equation for the transient temperature rise from steady state operation. A study of this equation determines the stability of the system for typical values of the parameters involved.

8. *Modern differential geometry and its application to relativity*, by Professor Valdemars Punga, Rensselaer Polytechnic Institute.

The speaker discussed the growth of the concepts of differential geometry from Riemannian space to Finsler space and to the affinely connected manifolds, and the associated development of relativity. He included a discussion of the principle of parallel displacement of tensors defined by affine connection $\Gamma^{\alpha}_{\beta\gamma}$ and its possible generalizations.

N. G. GUNDERSON, *Secretary*

THE JUNE MEETING OF THE PACIFIC NORTHWEST SECTION

The eighth annual meeting of the Pacific Northwest Section of the Mathematical Association of America was held at Reed College, Portland, Oregon, on June 18, 1954, in conjunction with the five hundred fourth meeting of the American Mathematical Society. Professors R. D. James and Ivan Niven were invited by Professor Harold Chatland, Chairman of the Section, to share the duties of presiding at the afternoon session.

Seventy-three persons were in attendance, including the following fifty-five members of the Association:

C. B. Allendoerfer, J. J. Andrews, T. M. Apostol, M. G. Arsove, R. W. Ball, R. A. Beaumont, R. F. Bell, J. L. Botsford, J. L. Brenner, L. G. Butler, F. M. Carpenter, Harold Chatland, P. A. Clement, K. L. Cooke, C. M. Cramlet, D. B. Dekker, K. S. Ghent, W. M. Gilbert, S. G. Hacker, Mary E. Haller, Burrowes Hunt, H. H. Irwin, R. D. James, J. M. Kingston, M. S. Knebelman, D. H. Lehmer, R. B. Leipnik, A. E. Livingston, R. G. Long, A. T. Lonseth, C. F. Luther, J. H. McKay, A. F. Moursund, B. N. Moyls, D. C. Murdoch, W. M. Myers, Jr., J. A. Nickel, Ivan Niven, C. O. Oakley, Gloria Olive, T. G. Ostrom, T. S. Peterson, Ruth E. Porter, C. A. Pursel, L. B. Rall, J. B. Roberts, E. M. Scheuer, A. J. Smith, W. M. Stone, D. B. Tillotson, J. R. Vatnsdal, Sylvia Vopni, L. B. Williams, R. M. Winger, F. H. Young.

Following a joint dinner with the American Mathematical Society, a business meeting was held in the evening at which the following officers were elected: Chairman, Professor Ivan Niven, University of Oregon; Vice-Chairman, Professor D. C. Murdoch, University of British Columbia; Secretary-Treasurer, Professor J. M. Kingston, University of Washington. The by-laws of the Section were amended to provide for an Executive Committee consisting of the officers and the Governor of the Section. Reports of contests held in British Columbia, Washington and Oregon were received. A Program Committee for the 1955 meeting was appointed, consisting of Professor D. C. Murdoch, Chairman, and Professors J. L. Botsford, and W. M. Myers, Jr.

The afternoon session consisted of the following invited hour address, two fifteen-minute papers and a symposium by a panel of four speakers:

1. Invited address: *Modern mathematics for freshmen*, by Professor C. O. Oakley, Haverford College.

The standard undergraduate curriculum in mathematics, especially that of the freshman year consisting of trigonometry, algebra and analytic geometry, is obsolete. In order to keep the youth of the nation abreast of the current developments in mathematics, in the physical sciences and in the social sciences, it is essential that modern mathematical ideas be introduced at a very early date—preferably in the high school. A detailed curriculum for college freshmen was suggested including logic, sets, groups, fields, functions (ordered pairs), limits, polynomial calculus, probability and statistics *and enough* trigonometry, algebra and analytic geometry for a following and substantial course in calculus.

2. *The most desirable undergraduate preparation for advanced study in mathematics*, by Professor D. B. Tillotson, Northwest Nazarene College.

A survey was made of the opinions of the chairmen of the mathematics departments of leading graduate schools concerning a program of fifteen semester hours of advanced undergraduate mathematics which would best prepare a student in a liberal arts college to pursue graduate work. The replies seem to indicate that about six hours of advanced algebra and at least six hours of advanced calculus including differential equations are very desirable. The remaining three hours were generally given either to geometry or to more analysis.

3. *On simplification of certain probability density functions*, by Professor W. M. Stone, Boeing Airplane Company and Oregon State College.

A number of interesting problems in communication theory involve the distribution of a constant amplitude sinusoidal signal in the presence of normal background noise. It seems that much greater simplicity is obtained if a more realistic randomly modulated signal amplitude is postulated. For example, the distribution of current in the presence of normal noise,

$$f(x) = \pi^{-1}\int_0^{\pi} \phi(x - y\cos\theta)d\theta$$

where $\phi(x) = (2\pi)^{-1/2}\exp(-x^2/2)$ and y is constant, reduces to a simple normal law if y is assumed to be Rayleigh distributed.

4. The Symposium: *The content of undergraduate mathematics courses*, moderated by Professor Ivan Niven, University of Oregon.

Panel:

(1) *College mathematics for engineers and scientists*, by Professor D. R. Crosby, University of Alberta.

The speaker expressed his opposition to the introduction into universities of the course recently proposed by Professors Allendoerfer and Oakley.* His opposition was based on the low entrance requirement for the course. In the preface it is stated that "A course based on (the first five chapters) is now given at the University of Washington with only two years of (United States) high school mathematics as a prerequisite. . . . After Chapters VI to X (which require intermediate algebra as a prerequisite) a student is prepared to *BEGIN* a standard course in calculus." For engineers and scientists, beginning calculus in second year University is far too late. The speaker did not believe that the "democratic" United States high schools would improve or even maintain their already depleted mathematics courses for the benefit of students of science, if the United States universities established new lows in entrance requirements for the large mass of students. Although sympathetic to introducing modern mathematics earlier, the speaker thought that the Allendoerfer-Oakley course would do more harm than good.

(2) *The content of calculus courses*, by Professor R. D. James, University of British Columbia.

Professor Menger, in the preface to his book, *Calculus, A Modern Approach*, speaks of " a project to rescue calculus from one of the greatest dangers that may befall a discipline—the danger of petrifaction." This paper suggests ways and means of avoiding the danger.

(3) *Suggested revisions in college mathematics*, by Professor R. B. Leipnik, University of Washington.

The traditional undergraduate curriculum, stemming from Euler's classic text of 1748, is admirably shaped for application to Newtonian mechanics. However, notions of logic, set algebra, probability, and vector space are now needed by students of electrical engineering, physics, and quantitative social and biological science. Experience with students at the University of Washington indicates that some of these topics can be successfully introduced much earlier than is usually believed possible.

(4) *Some practical aspects of the teaching of modern mathematics to undergraduates* by Professor M. G. Arsove, University of Washington.

It is both desirable and inevitable that "conventional" undergraduate mathematics courses give way to courses containing more modern abstract material. Practical aspects of this transition require the consideration of problems in course content, pedagogy, and administration. Course content will depend partly on circumstances at individual institutions. However, it is important to discuss mathematical logic and the structure of a mathematical system, so that the rules are known before the game begins. From the pedagogical viewpoint it is desirable that the abstract systems considered be limited in number and well delineated. The administrative problems arising from curriculum revision are perhaps the most troublesome of the problems encountered.

J. MAURICE KINGSTON, *Secretary*

* Principles of Mathematics, by Carl B. Allendoerfer and Cletus O. Oakley, published in mimeographed form by McGraw-Hill Book Co., 1953.

EMPLOYMENT OPPORTUNITIES

The MONTHLY is devoting this space to paid announcements of employment opportunities for mathematicians. The text of such announcements should be in want-ad form and must be in the hands of the editor (C. B. Allendoerfer, Mathematics Department, University of Washington, Seattle 5, Wash.) before the first day of the month preceding the issue in which the notice is to appear Advertisers will be billed at the rate of $1.50 per line. Rates for display advertising may be obtained from the Advertising Manager.

CALENDAR OF FUTURE MEETINGS

Thirty-eighth Annual Meeting, University of Pittsburgh, Pittsburgh, Pennsylvania, December 30, 1954.

Thirty-sixth Summer Meeting, University of Michigan, Ann Arbor, Michigan, August 29–30, 1955.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Pittsburgh, Pennsylvania, May, 1955.

ILLINOIS, Monmouth College, Monmouth, May 13–14, 1955.

INDIANA, Butler University, Indianapolis, May, 1955.

IOWA, St. Ambrose College, Davenport, April 15–16, 1955.

KANSAS

KENTUCKY

LOUISIANA-MISSISSIPPI, Buena Vista Hotel, Biloxi, Mississippi, February 18–19, 1955.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Georgetown University, Washington, D. C., December 4, 1954.

METROPOLITAN NEW YORK, Queens College, Flushing, New York, April 30, 1955.

MICHIGAN, Michigan State College, East Lansing, Spring, 1955.

MINNESOTA, College of St. Teresa, Winona, Minnesota, May, 1955.

MISSOURI, University of Kansas City, Spring, 1955.

NEBRASKA

NORTHERN CALIFORNIA, University of California, Berkeley, January 15, 1955.

OHIO

OKLAHOMA

PACIFIC NORTHWEST, University of British Columbia, Vancouver, June 17, 1955.

PHILADELPHIA, Princeton University, Princeton, New Jersey, November 27, 1954.

ROCKY MOUNTAIN, University of Wyoming, Laramie, Spring, 1955.

SOUTHEASTERN, Tennessee Polytechnic Institute, Cookeville, March 11–12, 1955.

SOUTHERN CALIFORNIA, Santa Monica City College, March 12, 1955.

SOUTHWESTERN, University of New Mexico, Albuquerque, Spring, 1955.

TEXAS, Abilene Christian College, Abilene, April, 1955.

UPPER NEW YORK STATE, University of Buffalo, May 14, 1955.

WISCONSIN, Cardinal Stritch College, Milwaukee, May, 1955.

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

INDEX TO VOLUME 61, 1954

THE AMERICAN MATHEMATICAL MONTHLY

GENERAL MATHEMATICAL PAPERS

ALGEBRA

ANALYSIS

APPLIED MATHEMATICS

EDUCATION

GEOMETRY, TOPOLOGY

HISTORY

PROBABILITY

MATHEMATICAL NOTES

Edited by F. A. FICKEN, University of Tennessee

CLASSROOM NOTES

Edited by G. B. THOMAS, Massachusetts Institute of Technology

PROBLEMS AND SOLUTIONS

Edited by HOWARD EVES, University of Maine
and E. P. STARKE, Rutgers University

AUTHORS

Numbers refer to pages, boldface type indicating a problem solved and solution published; italics, a problem solved but the solution not published; ordinary type, a problem proposed.

SOLUTIONS

Numbers in boldface type refer to problems, those in lightface to pages.

RECENT PUBLICATIONS

Edited by E. P. VANCE, Oberlin College

NEW BOOKS RECEIVED

REVIEWS

Names of authors are in ordinary type, those of reviewers, in capitals.

NEWS AND NOTICES

Edited by EDITH R. SCHNECKENBURGER, University of Buffalo

GENERAL INFORMATION

NECROLOGY

REPORTS AND ANNOUNCEMENTS OF THE ASSOCIATION AND ITS SECTIONS

MEETINGS AND ANNOUNCEMENTS OF THE ASSOCIATION

MEETINGS OF ITS SECTIONS

PERSONAL INFORMATION

Newly elected members of the Association, 72–74, 145–146, 288–290, 441–442, 499–501, 585–587.
The following persons presented papers at meetings of the Association and its Sections.

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 61 NUMBER 10

CONTENTS

DECEMBER 1954

The AMERICAN MATHEMATICAL MONTHLY

(Founded in 1894 by Benjamin F. Finkel)

EDITORIAL CORRESPONDENCE should be addressed to the Editor, C. B. Allendoerfer, Department of Mathematics, University of Washington, Seattle 5, Washington.

ADVERTISING CORRESPONDENCE should be addressed to J. F. Randolph, Morey Hall, University of Rochester, Rochester 3, N. Y.

NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the Monthly should be sent to the Secretary-Treasurer, H. M. Gehman, University of Buffalo, Buffalo 14, N. Y.

THIS IS THE OFFICIAL JOURNAL OF THE

MATHEMATICAL ASSOCIATION OF AMERICA, INC.

(Devoted to the Interests of Collegiate Mathematics)

Entered as second class matter at the post office at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

Subscription Price: To Members, $4 a Year, To Others, $5 a Year.

Published by the Association at Menasha, Wisconsin, and Seattle, Wash. during the months of January, February, March, April, May, June–July, August–September, October, November, December.

PRINTED IN THE UNITED STATES OF AMERICA

CHECKER BOARDS AND POLYOMINOES

S. W. GOLOMB, Harvard University

Our starting point is the well-known problem: Given a checker board with a pair of opposite corners deleted (Fig. 1), and given a box of dominoes, where each domino covers exactly two squares of the checker board, is it possible to cover this checker board exactly with dominoes? The answer is "no"; for suppose that the checker board is colored in the usual manner (Fig. 1). Then each domino covers one light square and one dark square. Thus n dominoes would cover n light squares and n dark squares, that is, an equal number of each. But the checker board of Fig. 1 has more dark squares than light squares, and so it can not be covered with dominoes.

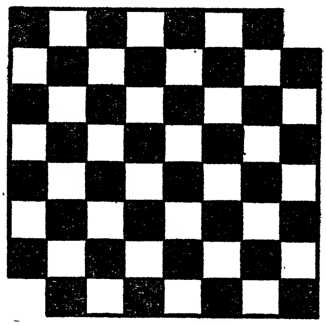


FIG. 1

We shall retain the 8×8 checker board as our "canonical domain," but we shall generalize the "domino" to the "polyomino," and our theorems will involve all the simpler polyominoes, shown in Figure 2. More precisely, we *define* an n-omino as a simply-connected set of n squares of the checker board which are "rook-wise connected"; that is, a rook placed at any square of the n-omino must be able to get to any other square, in a finite number of moves.

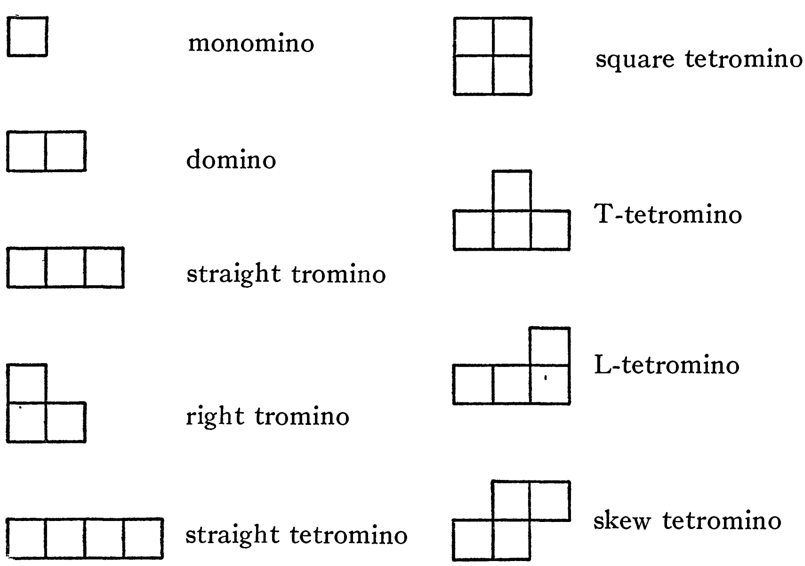


FIG. 2

First we consider trominoes. Clearly it is impossible to cover the 8×8

checker board with trominoes, because 64 is not divisible by 3. Instead, we will try to cover the checker board with 21 trominoes and one monomino.

It is impossible to cover the checker board with 21 *straight* trominoes, and a monomino in the upper left-hand corner of the board. To prove this, we color the checker board in three colors, a, b, and c, as shown in Figure 3. There are 21 a-colored squares, 22 b-colored squares, and 21 c-colored squares. Every straight tromino covers one a-colored square, one b-colored square, and one c-colored square, so that the straight trominoes will always cover an equal number of a-squares, b-squares, and c-squares. But the upper left-hand corner is an a-square, so that covering it with a monomino leaves only 20 a-squares, but 22 b-squares and 21 c-squares. These numbers are unequal; hence the covering with straight trominoes is impossible.

a	b	c	a	b	c	a	b
b	c	a	b	c	a	b	c
c	a	b	c	a	b	c	a
a	b	c	a	b	c	a	b
b	c	a	b	c	a	b	c
c	a	b	c	a	b	c	a
a	b	c	a	b	c	a	b
b	c	a	b	c	a	b	c

X

FIG. 3

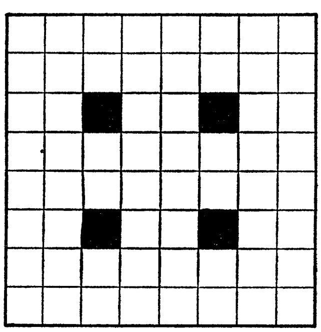


FIG. 4

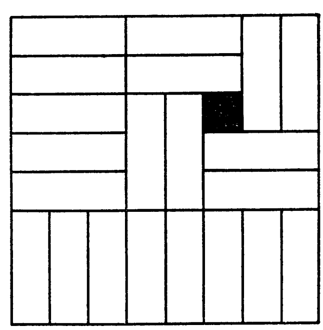


FIG. 5

The same argument shows that if the monomino is placed on any a-square, or on any c-square, the rest of the board cannot be covered with straight trominoes. Finally, suppose the monomino is placed on a b-square. For example, the lower left-hand corner. But by symmetry in the x-axis, this is really the same problem as when the monomino was in the *upper* left-hand corner, and

that covering was impossible. In fact, only b-squares which are symmetric to other b-squares are possible locations for the monomino. The only such squares are the four dark squares of Figure 4. And from Figure 5, we see that if the monomino is placed on one of these squares, the rest of the board really *can* be covered with straight trominoes.

Thus we have proved: *A necessary and sufficient condition for the checker board to be coverable with* 21 *straight trominoes and one monomino is that the monomino be placed on one of the dark squares of Figure 4.*

So far we have worked mainly with "non-existence" proofs. Our next result is in the opposite direction:

No matter where on the checker board a monomino is placed, the remaining squares can always be covered with 21 *right trominoes.*

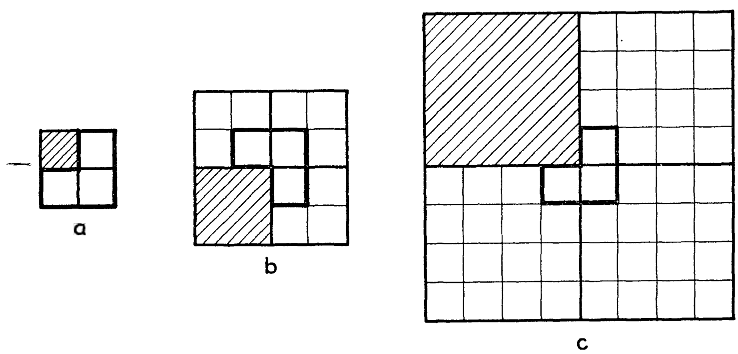


FIG. 6a–c

The proof proceeds by induction in $2^n \times 2^n$ checker boards. For a 2×2 checker board (Fig. 6-a), it is clear that wherever a monomino is placed, the rest can be covered by a right tromino. Given a 4×4 checker board, we divide it into quadrants (Fig. 6-b). Let a monomino be in one of these quadrants, the third, say. Each of the quadrants is a 2×2 checker board; so we can put a monomino in each quadrant, and cover the rest of the board with right trominoes. In the third quadrant the monomino is already assigned. In the other three quadrants, we place our monominoes in the center (Fig. 6-b); and then we can replace these three monominoes by a single tromino. Thus we have covered the 4×4 checker board with one monomino in a preassigned position, and 5 right trominoes. The 8×8 case, and indeed the general case, is treated in the same way (Fig. 6-c). We divide into quadrants; the preassigned monomino lies in one of these quadrants, and we can finish covering that quadrant with right trominoes by the previous case. In each of the other three quadrants, we introduce monominoes in the central squares. Again, by the previous case, the rest of these quadrants can be covered by right trominoes. And the monominoes in the three central squares can be replaced by a single right tromino.

This proof is similar to the Bolzano-Weierstrass Theorem in two dimensions. We keep subdividing into quadrants until we locate a monomino, our discrete analogue of a cluster point.

Now we come to the tetrominoes. It is easy to cover the checker board entirely with straight tetrominoes, or with square tetrominoes, or with T-tetrominoes, or with L-tetrominoes. This should be clear from Figure 7. On the other hand, it is impossible to cover the board entirely with skew tetrominoes. In particular, it is impossible to place skew tetrominoes on the board in such a way that even a single edge is completely covered. Here I leave the details as an exercise.

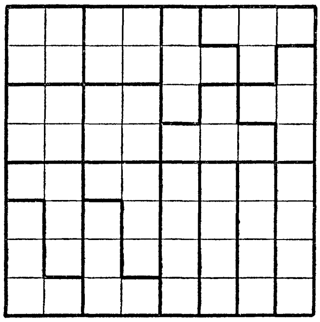


FIG. 7

Interesting tetromino problems are obtained in the following way: Suppose we try to cover the checker board with 15 T-tetrominoes and one square tetromino. But this is impossible. For consider the ordinary checker board coloring (*cf.* Fig. 1). Here a square tetromino covers two dark squares and two light squares—an even number of each. But a T-tetromino will cover three light and one dark square, or else three dark and one light square, in any case an odd number of each. There are 15 T-tetrominoes, an odd number, each covering an odd number of light and an odd number of dark squares. By a result of elementary arithmetic, an odd number of odd numbers is odd; by another result, an even number plus an odd number is odd. Thus 15 T-tetrominoes and one square tetromino cover an odd number of dark and an odd number of light squares. But with the ordinary checker board coloring, there are an even number of squares of each type.

A similar result holds for L-tetrominoes: *The checker board cannot be covered with 15 L-tetrominoes and one square tetromino.* To prove this, we introduce the coloring of Figure 8. Here, again, a square tetromino covers two light and two dark squares, while each L-tetromino covers three of one and one of the other, an odd number of each. Thus again 15 L-tetrominoes and a square tetromino will cover an odd number of dark and an odd number of light squares; but

clearly there are an even number of light and of dark squares in Figure 8.

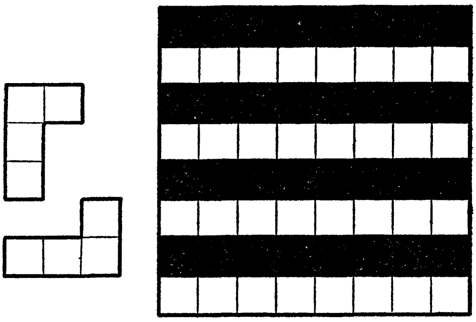


FIG. 8

We prove the corresponding result for both straight tetrominoes and skew tetrominoes in a single theorem: *The checker board cannot be covered with a square tetromino and* 15 *other tetrominoes of which some* (*or all*) *are straight and the rest are skew.*

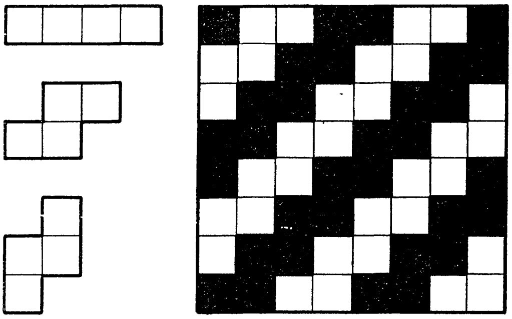


FIG. 9

This time we introduce the coloring of Figure 9. Here it is the *square* tetromino which covers an odd number of dark and an odd number of light squares, while straight tetrominoes always cover two of each, and skew tetrominoes cover either two of each, or four of one and none of the other. In any case, straight as well as skew tetrominoes cover an even number of dark and an even number of light squares. Adding on the *odd* number of squares of each color covered by the square tetromino, we find that the 16 tetrominoes in question always cover an odd number of dark and an odd number of light squares. But in each row of Figure 9 there are four light and four dark squares; hence the entire board consists of an *even* number of each, and cannot be covered with the given tetrominoes.

There are several directions in which we may now proceed.

1. *Problems on the torus.* If we bend the checker board around until the right and left sides meet, we get a cylinder. If we take this cylinder and bend it around until the top and bottom meet, we get a doughnut, or *torus.* Any covering with polyominoes which was possible before, on the ordinary board, is still possible on the torus. The impossibility theorems, however, must be re-examined.

We know that we can cover the checker board with a monomino and 21 straight trominoes (Fig. 5), provided the monomino is placed on a suitable square. On the torus, every square is just like every other, so that the covering is possible *wherever* the monomino is placed. (Note that on the torus, a straight tromino will *not* always cover one a-square, one b-square, and one c-square, in the coloring of Figure 3.)

The tetromino theorems are the same on the torus, because the various tetrominoes still cover the same number of light and dark squares as on the flat checker board, at least for the ordinary checker board coloring and the colorings of Figures 8 and 9. Moreover, we can use this fact to conclude something about the original checker board: Since the four corner squares of the checker board form a square tetromino on the torus, it is impossible to remove the four corner squares from a checker board and cover the rest with 15 T-tetrominoes, or with 15 L-tetrominoes, or with a combination of 15 straight and skew tetrominoes.

2. *Algebraic significance of the colorings.* Considering the squares of the checker board as lattice points in the plane, the light and dark squares of the ordinary coloring are the two residue classes of the Gaussian integers $a+bi$ modulo $1+i$. The coloring of Figure 3 shows the three residue classes of $a+b\omega$ modulo $1+\omega$, where ω is a primitive cube root of unity.

3. *Pseudo-polyominoes.* If we admit "queen-wise" as well as "rook-wise" connected polyominoes, we get the *pseudo-polyominoes*, the simplest of which appear in Figure 10.

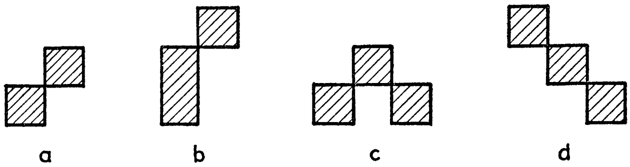


FIG. 10

I recommend the following problems:

(i) Show that the checker board cannot be covered with 21 pseudo-trominoes of types 10-c and 10-d, and one monomino.

(ii) Show that the checker board *can* be covered with 21 pseudo-trominoes of type 10-b and one monomino. Where must the monomino be placed?

(iii) Find all the pseudo-tetrominoes.

We can generalize even further, to the *quasi-polyomino*, which need not be connected at all. Figure 11 shows first a certain quasi-tromino; then it shows how two of these may be combined to form a certain hexomino; and using 10 of these hexominoes, with one of the original quasi-trominoes, and one monomino, how to cover the checker board. Thus it is possible to cover the checker board with 21 of these quasi-trominoes, and one monomino.

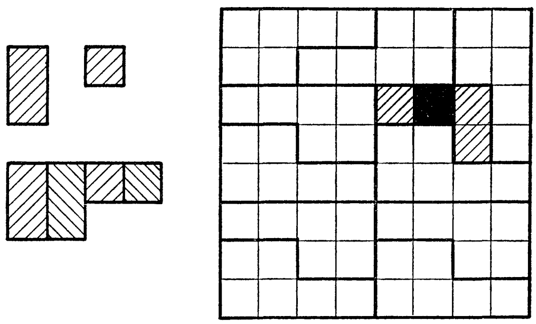


FIG. 11

4. *Pentominoes.* There are twelve distinct pentominoes. They all appear in Figure 12, which moreover solves the problem: Is it possible to place all 12 pentominoes on the checker board at the same time? The solution pictured here is the "best," in the sense that the four squares left over not merely form a tetromino, but a square tetromino, in the center of the board.

Other interesting pentomino problems arise when one tries to cover the checker board with 12 pentominoes of a single type, and one square tetromino.

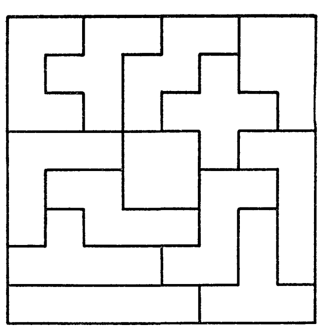


FIG. 12

5. *The board.* The shape of the checker board can be altered at will to obtain new problems. For example, is it possible to fit all twelve distinct pentominoes

on a 3×20 board? Also, polyomino problems in 3 or even more dimensions can be considered. However, I stated earlier that I would confine myself at present to the 8×8 checker board.

Another possible modification, of a rather fundamental sort, would be the use of hexagonal rather than square tiles for the board and for the objects covering it.

ZERO-FREE REGIONS OF POLYNOMIALS*

V. F. COWLING, University of Kentucky and W. J. THRON, Washington University

1. Introduction. In this paper we are concerned with the derivation of certain results about the location of the zeros of a polynomial. We use a method which was developed by Leighton and Thron [4] to study the convergence of continued fractions, was then adapted by Cowling [1] to obtain results on the distribution of values of the partial sums of a Taylor series, and has now been further modified by us to make it as suitable as possible for our present purpose. This method is developed in Section 2. The next two sections bring a number of applications of this method. The last section is devoted to a discussion of the relation of our results to other results of a similar nature.

2. Notation and basic considerations. Let V be a region in the complex plane; then $1/V$ and $V+a$ are, respectively, the regions consisting of all z such that $1/z\in V$ and all w such that $w-a\in V$. Further, throughout this paper indices k, m, p and q will be used to indicate the following ranges:

$$k = 1, \cdots, n; \quad m = 1, \cdots, n-1; \quad p = 0, \cdots, n-1; \quad q=0, \cdots, n.$$

Now let

$$P(z) = a_0 + a_1 z^{\lambda_1} + \cdots + a_n z^{\lambda_n}, \tag{2.1}$$

where all a_q are assumed to be different from zero, and where the λ_k are positive integers satisfying the relation $\lambda_1 < \lambda_2 < \cdots < \lambda_n$. It is convenient also to introduce $\lambda_0 = 0$. The integers α_k are to be defined as follows:

$$\alpha_k = \lambda_k - \lambda_{k-1}.$$

* This research was supported in part by the United States Air Force under contract No. AF 18(600)-419 monitored by the office of Scientific Research, Air Research and Development Command.

It is clear that

$$P(z) = a_0\left(1 + \frac{a_1}{a_0} z^{\alpha_1}\left(1 + \frac{a_2}{a_1} z^{\alpha_2}\left(1 + \cdots\left(1 + \frac{a_n}{a_{n-1}} z^{\alpha_n}\right)\cdots\right)\right)\right), \tag{2.2}$$

and the result below follows.

LEMMA 2.1. *Let two sets of regions $V_0, \cdots, V_{n-1}$ and $E_1, \cdots, E_n$ be given among which the following relations hold*:

$$1 + e_m v_m \in V_{m-1} \text{ for all } e_m \in E_m \quad \text{and} \quad v_m \in V_m, \quad 1 + E_n \subset V_{n-1}.$$

If

$$\frac{a_k}{a_{k-1}} z^{\alpha_k} \in E_k$$

for all k, then $P(z) \in a_0 V_0$.

In some cases, as for example in Section 4, the sets V_p can easily be computed from given sets E_k. Frequently, however, it is advisable to begin with a given set of V_p and construct the E_k from them. This construction is described below.

LEMMA 2.2. *Let regions $V_0, \cdots, V_{n-1}$ be given such that the regions*

$$E_m = \bigcap_{v_m \in V_m} \frac{1}{v_m}(V_{m-1} - 1),$$

$$E_n = V_{n-1} - 1,$$

are all non-empty. If

$$\frac{a_k}{a_{k-1}} z^{\alpha_k} \in E_k,$$

then $P(z) \in a_0 V_0$.

To prove this lemma it suffices to show that the E_k, constructed in the way described here, together with the V_p from which they are derived, satisfy the conditions of Lemma 2.1. Clearly If $E_n = V_{n-1} - 1$ then $1 + E_n \subset V_{n-1}$. For every $e_m \in E_m$ we have

$$e_m = (v'_{m-1} - 1)/v_m, \tag{2.3}$$

where v_m is an arbitrary element of V_m and $v'_{m-1} - 1$ is some element of V_{m-1}. Solving (2.3) for v'_{m-1} we obtain $v'_{m-1} = 1 + e_m v_m$ and hence $1 + e_m v_m \in V_{m-1}$ for all $e_m \in E_m$ and $v_m \in V_m$. We might observe at this point that indiscriminate choice of the V_p might lead to empty E_k and that to obtain regions E_k as large as possible, which is clearly our aim, requires a careful search for suitable regions V_p.

3. Circular value regions. As a first application of the methods outlined in

the preceding section we consider regions V_p defined by

$$|z| > r_p,$$

where the r_p are arbitrary positive numbers except for r_0 which we set equal to zero. E_m then are the regions defined by

$$|e| > (1 + r_{m-1})/r_m$$

while E_n is defined by $|e-1| > r_{n-1}$. However, for the sake of uniformity, we replace the last region by the slightly less inclusive one defined by $|e| > 1 + r_{n-1}$. If we now introduce $r_n = 1$, we have for every k, $e \in E_k$ if $|e| > (1+r_{k-1})/r_k$. It follows that the polynomial (2.1) does not assume the value zero for those z for which

$$\left| \frac{a_k}{a_{k-1}} z^{\alpha_k} \right| > \frac{1 + r_{k-1}}{r_k},$$

that is, for

$$|z| > \max \left[\left(\frac{1 + r_{k-1}}{r_k} \left| \frac{a_{k-1}}{a_k} \right|^{1/\alpha_k} \right) \right].$$

THEOREM 3.1. *The polynomial* (2.1) *assumes all its zeros for*

$$|z| > \max \left[\left(\frac{1 + r_{k-1}}{r_k} \left| \frac{a_{k-1}}{a_k} \right| \right)^{1/\alpha_k} \right].$$

where $r_0 = 0$, $r_n = 1$ *and the remaining* r_m *are arbitrary positive numbers.*

A well known result (see Dieudonné [3, p. 18]) is obtained as a corollary of this theorem if we set $r_k = 1$.

COROLLARY 3.1. *The polynomial* (2.1) *assumes all its zeros for*

$$|z| \leqq \max \left[\left| \frac{a_0}{a_1} \right|^{1/\alpha_1}, \left(2 \left| \frac{a_1}{a_2} \right| \right)^{1/\alpha_2}, \cdots, \left(2 \left| \frac{a_{n-1}}{a_n} \right| \right)^{1/\alpha_n} \right].$$

The choice of the free constants in Theorem 3.1 is facilitated by the introduction of $g_k = (1 + r_{k-1})/r_k$. Then we have

$$r_m = \left(\sum_{\nu=0}^{m-1} \prod_{\mu=1}^{\nu} g_\mu \right) \Big/ \prod_{\nu=1}^{m} g_\nu.$$

Here we use the convention that $\prod_{\mu=1}^{0} g_\mu = 1$. From these relations it follows that g_1 to g_{n-1} may be taken to be arbitrary positive numbers while g_n is determined, since

$$g_n = (1 + r_{n-1})/r_n = 1 + r_{n-1} = \left(\sum_{\nu=0}^{n-1} \prod_{\mu=1}^{\nu} g_\mu \right) \Big/ \prod_{\nu=1}^{n-1} g_\nu.$$

We can thus restate Theorem 3.1 in terms of g_m.

THEOREM 3.1′. *The polynomial* (2.1) *assumes all its zeros for*

$$|z| \leqq R_g = \max\left[\left(g_1\left|\frac{a_0}{a_1}\right|\right)^{1/\alpha_1}, \cdots, \left(g_{n-1}\left|\frac{a_{n-2}}{a_{n-1}}\right|\right)^{1/\alpha_{n-1}}, \left(\frac{\sum_{\nu=0}^{n-1}\prod_{\mu=1}^{\nu} g_\mu}{\prod_{\nu=1}^{n-1} g_\nu}\left|\frac{a_{n-1}}{a_n}\right|\right)^{1/\alpha_n}\right]$$

where $g_1, \cdots, g_{n-1}$ *are arbitrary positive numbers.*

In an application of this theorem the $|a_{k-1}/a_k|$ and the α_k would be given and it would be our task to select the g_m in such a way as to make R_g as small as possible. We would accordingly assign large values to those g_m whose corresponding $|a_{m-1}/a_m|$ are small. The obvious difficulty is that we have no free control over the coefficient of $|a_{n-1}/a_n|$. We illustrate the possible choices that can be made by a few examples. Taking $g_m = 1$ we obtain

$$R_g = \max\left[\left|\frac{a_0}{a_1}\right|^{1/\alpha_1}, \cdots, \left|\frac{a_{n-2}}{a_{n-1}}\right|^{1/\alpha_{n-1}}, \left(n\left|\frac{a_{n-1}}{a_n}\right|\right)^{1/\alpha_n}\right].$$

If we let $g_m = 1$, $m \neq r$, and $g_r = h$, we have

$$R_g = \max\left[\left|\frac{a_0}{a_1}\right|^{1/\alpha_1}, \cdots, \left(h\left|\frac{a_{r-1}}{a_r}\right|\right)^{1/\alpha_r}, \cdots, \left|\frac{a_{n-2}}{a_{n-1}}\right|^{1/\alpha_{n-1}}, \left(\frac{r + h(n-r)}{h}\left|\frac{a_{n-1}}{a_n}\right|\right)^{1/\alpha_n}\right].$$

If a_r differs markedly from the other coefficients a choice of $g_m = 1$, $m \neq r, r+1$, $g_r = h$, $g_{r+1} = 1/h$ might give a good value for R_g. It is

$$R_g = \max\left[\left|\frac{a_0}{a_1}\right|^{1/\alpha_1}, \cdots, \left(h\left|\frac{a_{r-1}}{a_r}\right|\right)^{1/\alpha_r}, \left(\frac{1}{h}\left|\frac{a_r}{a_{r+1}}\right|\right)^{1/\alpha_{r+1}}, \cdots \left|\frac{a_{n-2}}{a_{n-1}}\right|^{1/\alpha_{n-1}}, \left((n-1+h)\left|\frac{a_{n-1}}{a_n}\right|\right)^{1/\alpha_n}\right].$$

Finally the choice $g_m = |a_m/a_{m-1}|$ gives

$$R_g = \max\left[1, \left(\left(\sum_{\nu=0}^{n-1} |a_\nu|\right)\Big/ |a_n|\right)^{1/\alpha_n}\right].$$

For $\lambda_q = q$ this is a known result (see [3, p. 17]).

The fact that the roots of the polynomial

$$Q(z) = a_n + a_{n-1}z^{\lambda_n - \lambda_{n-1}} + \cdots + a_1 z^{\lambda_n - \lambda_1} + a_0 z$$

are the reciprocals of the roots of the polynomial (2.1) leads immediately to the result below.

THEOREM 3.2. *The polynomial* (2.1) *assumes all its zeros for*

$$|z| \geqq \min\left[\left(f_1\left(\left|\frac{a_{n-1}}{a_n}\right|\right)^{1/\alpha_n}, \cdots, \left(\left|\frac{a_0}{a_1}\right| \Big/ \left(\sum_{v=0}^{n-1}\prod_{u=1}^{v} f_u\right)\right)^{1/\alpha_1}\right]$$

where $f, \cdots, f_{n-1}$ *are arbitrary positive numbers.*

4. Angular openings as value regions. It follows immediately from formula (2.2) that requiring

$$|\arg((a_k/a_{k-1})z^{\alpha_k})| < \pi\alpha_k/\lambda_n$$

leads to V_p satisfying $0 \notin V_p$ and

$$|\arg z| < \pi(\lambda_n - \lambda_p)/\lambda_n.$$

The region V_0 thus does not contain that part of the real axis for which $R(z) \leqq 0$. It is then clear that under these conditions the polynomial $P(z)$ does not vanish. If we now restrict ourselves to $a_q > 0$ the condition imposed on $(a_k/a_{k-1})z^{\alpha_k}$ is satisfied provided $|\arg z| < \pi/\lambda_n$. We finally observe that $P(z)$ vanishes for $|\arg z| = \pi/\lambda_n$ if and only if $P(z) = a_0 + a_n z^{\lambda_n}$. The following theorem has now been proved.

THEOREM 4.1. *If in the polynomial* (2.1) *of degree* $\lambda_n \geqq 2$ *all coefficients are positive then* $P(z)$ *does not vanish for*

$$|\arg z| \leqq \pi/\lambda_n$$

unless it is of the form $a_0 + a_n z^{\lambda_n}$ *in which case it has two zeros satisfying* $|\arg z| = \pi/\lambda_n$.

This result is a special case of a more general criterion due to Obrechkoff [6] the proof of which is considerably more involved than the one presented here.

5. Comparison with other criteria. We have already pointed out that some of our results are equivalent to previously known ones. Another criterion essentially due to Fujiwara (see [**3**, p. 18] and [**5**, p. 105]) states that all roots of the polynomial (2.1) have an absolute value not greater than

$$R_\delta = \max\left[(\delta_p |a_p/a_n|)^{1/\lambda_n - \lambda_p}\right],$$

where the δ_p are positive numbers which satisfy the condition $\sum_{p=0}^{n-1} 1/\delta_p = 1$ but are otherwise arbitrary. This criterion may be shown to be equivalent to our Theorem 3.1′ if the best values for the δ_p and g_m can be determined. Since this however is likely to be difficult and since the main importance of these criteria seems to lie in the relative ease with which one can get fair approximations for the upper bound of the roots of a given polynomial, our results appear to be of value. Other zero-free regions, which are not contained in or equivalent to known criteria, have been obtained by the use of our method. The deriva-

tions, however, become sufficiently more complicated that it seemed advisable not to include them here.

Bibliography

1. V. F. Cowling, On the distribution of the values of the partial sums of a Taylor series, Proc. Amer. Math. Soc. vol. 2, 1951, pp. 732–738.
2. V. F. Cowling, W. Leighton and W. J. Thron, Twin convergence regions for continued fractions, Bull. Amer. Math. Soc., vol. 50, 1944, pp. 351–357.
3. J. Dieudonne, La theorie analytique des polynomes d'une variable. Memorial des Sciences Mathematiques, vol. 93, Paris, 1938.
4. W. Leighton and W. J. Thron, Continued fractions with complex elements, Duke Math. J., vol. 9, 1942, pp. 763–772.
5. M. Marden, The geometry of the zeros of a polynomial in a complex variable, Mathematical Surveys, number 3, New York, 1949
6. N. Obrechkoff, Sur un probleme de Laguerre, C. R. Acad. Sci. Paris, vol. 177, 1923, pp. 102–104.

SOME REMARKS ON STIRLING'S FORMULA

M. I. AISSEN, Radiation Laboratory, The Johns Hopkins University

Introduction. Stirling's Formula

$$n! \sim \sqrt{2\pi}n^{1/2}n^n e^{-n} \tag{1}$$

is often presented in various stages. Various methods are used to prove the weaker form of Stirling's Formula,

$$n! \sim Cn^{1/2}n^n e^{-n} \tag{2}$$

for some unspecified non-zero constant C. Then a standard elementary argument is used to show that $C=\sqrt{2\pi}$. [*Cf.* Courant: *Differential and Integral Calculus*, pp. 225, 363.]

In several of these methods, relatively crude arguments establish some tenuous connection between $n!$ and $n^n e^{-n}$. Then slightly refined arguments display the role of the factor $n^{1/2}$. It is the purpose of this note to present two lemmas, either of which can be used to supply the factor $n^{1/2}$. The two lemmas have little in common except for this possible application. Neither seems to be in the literature, although one (Lemma I) is practically a folk result. The second is perhaps new and was first noticed by A. Novikoff and the author in a different connection.

1. First method. It is perhaps natural to compare $n!$ and n^n since each is the

product of n simple factors. For $n>1$

$$1\cdot 2\cdot 3\cdots n<n\cdot n\cdot\cdots n$$

and we obtain the elementary inequality

$$n! < n^n. \tag{3}$$

To study the relationship between $n!$ and n^n more closely, we consider the ratio

$$u_n = n^n/n!. \tag{4}$$

The sequence $\{u_n\}$ clearly increases since

$$\frac{u_{n+1}}{u_n} = \left(1+\frac{1}{n}\right)^n. \tag{5}$$

We assume that the students have been exposed to this important sequence $\{(1+1/n)^n\}$ and know that it has a limit. (If not, this would seem to be a natural way to introduce it, rather than: "Consider the sequence $\{1+1/n\}^n$"). Thus we are led to study the sequence

$$V_n = \frac{u^n}{e^n} = \frac{n^n e^{-n}}{n!}. \tag{6}$$

Since the power series for e^x has positive coefficients, its sum is greater than any single term and we have $e^n>n^n/n!$ and thus $V_n<1$. More precisely, we learn that

$$\frac{V_{n+1}}{V_n} = \frac{\left(1+\frac{1}{n}\right)^n}{e}. \tag{7}$$

Before we exploit the connection between $n!$ and $n^n e^{-n}$ that we have established, we will state and prove the first of the Lemmas.

LEMMA I. *If*

$$y_{n+1} = y_n\left\{1+\frac{\alpha}{n}+O\left(\frac{1}{n^2}\right)\right\}$$

and $y_n\neq 0$ *for all* n, *then*

$$y_n \sim Cn^\alpha$$

for some non-zero constant C.

Proof. Let $W_n = n^{-\alpha} y_n$. Now

$$(n+1)^{-\alpha} = n^{-\alpha}\left(1+\frac{1}{n}\right)^{-\alpha} = n^{-\alpha}\left\{1-\frac{\alpha}{n}+O\left(\frac{1}{n^2}\right)\right\}.$$

Hence,

$$W_{n+1} = W_n\left(1+\frac{\alpha}{n}+O\left(\frac{1}{n^2}\right)\right)\left(1-\frac{\alpha}{n}+O\left(\frac{1}{n^2}\right)\right) = 1+O\left(\frac{1}{n^2}\right).$$

Therefore the infinite product

$$\frac{W_2}{W_1}\cdot\frac{W_3}{W_2}\cdot\frac{W_4}{W_3}\cdots = \prod_1^\infty\left(1+O\left(\frac{1}{n^2}\right)\right)$$

converges to some non-zero constant C. But if $\lim_{n\to\infty} W_n = C$, then

$$y_n \sim Cn^\alpha.$$

Now

$$\begin{aligned}(1+x)^{1/x} &= \exp.\left\{\frac{1}{x}\log(1+x)\right\}\\ &= e\left\{1-\frac{1}{2}x+O(x^2)\right\}.\end{aligned}$$

Hence:

$$\frac{\left(1+\frac{1}{n}\right)^n}{e} = 1-\frac{1}{2n}+O\left(\frac{1}{n^2}\right). \tag{8}$$

Combining (7), (8) and Lemma I, we obtain

$$V_n = \frac{n^n e^{-n}}{n!} \sim C' n^{-1/2}$$

for some non-zero constant C'. Hence, if $CC' = 1$, we get

$$n! \sim Cn^{1/2} n^n e^{-n}. \tag{2}$$

2. The second method. In the second method we consider log n! and represent it as a sum:

$$\log n! = \sum_{k=1}^n \log k. \tag{9}$$

It is quite natural to try to approximate the right hand side of (9) by $\int_1^n \log t\, dt$, and in fact it is true that

$$\log n! = \int_1^n \log t\, dt + O(1) = n \log n - n + O(1). \tag{10}$$

However, (10) is not equivalent to (2) although (2) does imply (10). A proposition equivalent to (2), involving $\log n!$ is

$$\log n! = (n + \tfrac{1}{2}) \log n - n + \log C + o(1). \tag{11}$$

From the monotonicity of $\log x$ it follows that

$$\int_1^n \log [t] dt \leqq \int_1^n \log t\, dt \leqq \int_1^n \log [t + 1] dt \tag{12}$$

where $[x]$ denotes the greatest integer in x. That is,

$$\log (n - 1)! \leqq \int_1^n \log t\, dt \leqq \log n!. \tag{13}$$

Thus monotonicity yields only

$$\log n! = \int_1^n \log t\, dt + O(\log n). \tag{14}$$

In order to improve (14) we use our second lemma which takes advantage of the concavity of $\log x$.

LEMMA II. *Let $f(x)$ be monotone increasing and concave for $x > a$. Then*

$$0 \leqq \sum_{k>a} \left\{ \int_k^{k+1} f(t)dt - \frac{f(k) + f(k+1)}{2} \right\} \leqq \frac{f(a+1) - f(a)}{2} \,.$$

Proof. The first inequality follows from the concavity of $f(x)$, since each term of the sum is non-negative. It expresses the fact that the graph of a concave function lies above the chord.

The second inequality states that the sum of the areas between the function and its chords is a convergent series if the end-points of the chords have abscissae at the integers.

To show this, we consider the point set I_n bounded by the graph of $f(x)$ and the chord joining the points $(n, f(n))$ and $(n+1, f(n+1))$ for $n > a$ and translate it (without rotation) so that the point $(n+1, f(n+1))$ becomes the point $(m+1, f(m+1))$ where $m = [a+1]$.

It follows from the concavity of $f(x)$, that after translation the interior of I_n will not intersect the interior of I_{n+1} and will lie completely below I_{n+1}. From the monotonicity of $f(x)$ it follows that for each n, I_n will lie between the horizontal lines $y = f(m)$, $y = f(m+1)$.

Hence, all these areas can fit in the triangle whose vertices are $(m, f(m))$, $(m, f(m+1))$, $(m+1, f(m+1))$ without overlapping (their boundaries may intersect), and hence if A_n denotes the area of I_n,

$$\sum_{n>a} A_n \leqq \frac{f(m+1)-f(m)}{2}.$$

But by the concavity of $f(x)$,

$$f(m+1)-f(m) \leqq f(a+1)-f(a).$$

Since

$$A_k = \int_k^{k+1} f(t)\, dt - \frac{f(k)+f(k+1)}{2},$$

we thus obtain Lemma II.

An alternative proof can be based on the inequality which we state here without proof:

$$A_n \leqq \tfrac{1}{2}\{f(n+1)-f(n)\} - \tfrac{1}{2}\{f(n+2)-f(n+1)\}.$$

We now apply Lemma II to our problem: Let $a=1, f(x)=\log x$. Then:

$$0 \leqq \sum_{k=2}^{\infty} \left\{\int_k^{k+1} \log t\, dt - \frac{\log k + \log (k+1)}{2}\right\} \leqq \tfrac{1}{2}\log 2.$$

Since the terms of the series are positive, we have

$$\lim_{N\to\infty} \sum_{k=2}^{N} \left\{\int_k^{k+1} \log t\, dt - \frac{\log k + \log (k+1)}{2}\right\} = D \leqq \tfrac{1}{2}\log 2,$$

or

$$\lim_{N\to\infty} \left\{\int_2^{N} \log t\, dt - \frac{\log (N-1)! + \log N!}{2}\right\} = D - \tfrac{1}{2}\log 2.$$

Since $\log N! = \log (N-1)! + \log N$, we get

$$\lim_{N\to\infty} \{N\log N - N - 2\log 2 + 2 - \log N! + \tfrac{1}{2}\log N\} = D - \tfrac{1}{2}\log 2.$$

$$\therefore\ \log N! = (N+\tfrac{1}{2})\log N - N - D_N,$$

where

$$\lim_{N\to\infty} D_N = D + \frac{3}{2}\log 2 - 2 = \log C.$$

Hence

$$\log N! = (N+\tfrac{1}{2})\log N - N + \log C + o(1),$$

and the proof is complete.

MATHEMATICAL NOTES

EDITED BY F. A. FICKEN, University of Tennessee

Material for this department should be sent to F. A. Ficken, University of Tennessee, Knoxville 16, Tenn.

NOTES ON THE SKEW QUADRILATERAL

O. BOTTEMA, Technological University, Delft, Netherlands

In the February 1953 issue of this journal Thébault* has established two interesting theorems on the skew quadrilateral. We give some supplementary remarks on both of them.

A. The first states: *a skew quadrilateral which has its opposite sides equal in pairs has its opposite angles equal in pairs and conversely.* The second part of the theorem is not so very simple; the geometrical proof given by the author is elegant, but requires some confidence in the possibility of a certain construction. A proof can also be given along algebraic lines. $ABCD$ be the skew quadrilateral, $AB=a$, $BC=b$, $CD=c$, $DA=d$, $(AD, AB)=\alpha$, $(BA, BC)=\beta$, $(CB, CD)=\gamma$, $(DC, DA)=\delta$. Given $\alpha=\gamma$, $\beta=\delta$. If $AC=p$, $BD=q$, we have in the triangles ABC and CDA respectively

$$p^2 = a^2 + b^2 - 2ab \cos \beta, \qquad p^2 = c^2 + d^2 - 2cd \cos \delta;$$

hence $\beta=\delta$ implies

$$\frac{a^2 + b^2 - p^2}{ab} = \frac{c^2 + d^2 - p^2}{cd}$$

or

$$(ab - cd)p^2 = -(ad - bc)(ac - bd). \tag{1}$$

Similarly

$$(ad - bc)q^2 = -(ab - cd)(ac - bd). \tag{2}$$

Now we have two cases: $ab-cd\neq 0$, and $ab-cd=0$.

I. If $ab-cd\neq 0$, then in view of (1) $ad-bc\neq 0$, $ac-bd\neq 0$. Thus

$$p^2 = -\frac{(ad - bc)(ac - bd)}{ab - cd}, \qquad q^2 = -\frac{(ab - cd)(ac - bd)}{ad - bc},$$

and therefore

$$p^2q^2 = (ac - bd)^2;$$

that means

$$ac = bd + pq \quad \text{or} \quad bd = ac + pq.$$

---

* V. Thébault, On the skew quadrilateral. This MONTHLY, vol. 60, 1953, pp. 102–105.

In both cases by a well-known theorem§ the four points A, B, C, and D are coplanar, so that the quadrilateral is not a skew one.

II. If $ab-cd=0$, then according to (2) $ad-bc=0$ and we have indeed $a=c$, $b=d$.

B. The second of Thébault's theorems says: *a skew quadrilateral has its opposite sides equal in pairs if and only if the perpendiculars at the vertices to the planes of the two sides meeting at these vertices constitute a hyperbolic group of lines.* He gives a short and interesting proof of both statements.

If $ABCD$ is the quadrilateral, $AB=a$, *etc.*, h_A the perpendicular at A to the plane DAB, *etc.*, then the theorem stating that $a=c$, $b=d$ implies that h_A, h_B, h_C, h_D, are on a hyperboloid was to the best of my knowledge first given by Bennett* in his study of the "isogram," which he considered above all from a kinematical standpoint. The converse theorem was given by me as a "problem" in a Dutch mathematical paper† in 1950; several solutions were sent in, and some of them seem as simple and straightforward as Thébault's. There would not be any reason to return to the matter, but for the fact that Bennett's proof of his theorem shows an additional property of the said hyperboloid. Therefore we reproduce it here in a few words.

For the quadrilateral, let $a+b=c+d$. Produce AB with $BP=b$ and AD with $DQ=c$, then $AP=AQ$. The outer bisecting plane W_B of the angle ABC intersects PC orthogonally in its midpoint; so does W_D the line QC. The inner bisecting plane V_A intersects PQ orthogonally in its midpoint. Hence V_A, W_B and W_D have a line r_1 in common and evidently the inner bisecting plane V_C passes through this line too. Now if $a=c$, $b=d$ we have not only $a+b=c+d$, but $a+d=c+b$ as well. That means: V_B, V_D, W_A and W_C pass through a line r_2. We fix a projectivity between the planes of the pencil (r_1) and those of the pencil (r_2) by associating with a plane of (r_1) that one of (r_2) which is perpendicular to it. Now it is a well-known theorem in projective geometry that the locus of the lines of intersection of corresponding planes in two projective pencils is a regulus. As V_A and W_A are perpendicular planes meeting in h_A, *etc.*, we conclude that h_A, h_B, h_C, h_D constitute a hyperbolic group of lines. But according to another well-known theorem of projective geometry, in view of the special manner in which the projectivity of the pencils is defined here, we have the additional property: the hyperboloid to which the four perpendiculars h_A, h_B, h_C and h_D belong is orthogonal.

---

§ A proof of Ptolemy's theorem in space can be given as follows: Use an inversion with its centre in A, transforming B, C and D into B_1, C_1, D_1. Then

$$B_1C_1 = \frac{k\cdot BC}{AB\cdot AC} = \frac{k\cdot b}{ap}, \; C_1D_1 = \frac{kc}{pd}, \; D_1B_1 = \frac{kq}{da}.$$

Hence $C_1D_1 = C_1B_1 + B_1D_1$ or $B_1C_1 = B_1D_1 + D_1C_1$; that means that B_1, C_1 and D_1 are collinear. Consequently A, B, C and D are coplanar (and of course on a circle).

* Proceedings of the London Math. Soc. vol. 13, 1914, p. 151.

† Wiskundige Opgaven met de oplossingen, dl. 19, 2, (1951) problem XLI, pp. 83–84.

THREE DIMENSIONAL HARMONIC FUNCTIONS GENERATED BY ANALYTIC FUNCTIONS OF A HYPERVARIABLE

E. P. MILES, Alabama Polytechnic Institute

1. Introduction. The study of harmonic functions in two dimensions is greatly simplified by the theorem that

$$V = f(x + iy) + g(x - iy) \tag{1}$$

is the general solution of $V_{xx} + V_{yy} = 0$ where f and g are analytic functions. The analogous result in three dimensions [4] gives the general solution of $V_{xx} + V_{yy} + V_{zz} = 0$ as an integral of the form

$$V = \int_{-\pi}^{\pi} f(z + ix \cos u + iy \sin u, u)du \tag{2}$$

from which the extraction of solutions in more explicit form is much more difficult than in the two dimensional case.

Von Beckh-Widmanstetter [1] has shown that the general solution of Laplace's equation in three dimensions can not be generated by a hypervariable with analytic functions having only three components. Various authors have constructed special solutions with the aid of hypervariables, and Ketchum [2] has obtained the general solution by using a hypervariable with analytic functions having infinitely many components. The author forms particular solutions from the components of $(\omega - 1)f(t)$ where $t = x + \omega y + \omega^2 z$, $\omega^3 = 1$ and f is an analytic function of t. Since only two of the three component functions are linearly independent and the functions are constant on any line normal to the plane $x + y + z = 0$, the harmonic functions are in essence two dimensional, and the author closes with the transformation relating them to the ordinary components of an analytic function of a complex variable.

2. Properties of analytic functions of t.

DEFINITION. *$f(t) = (u, y, z) + \omega v(x, y, z) + \omega^2 w(x, y, z)$ is said to be analytic at $t = t_0$ if $\lim_{\Delta t \to 0} [f(t + \Delta t) - f(t)]/\Delta t$ exists in a neighborhood of t_0.* The generalized Cauchy Riemann Equations

$$u_x = v_y = w_z, \quad u_y = v_z = w_x \quad \text{and} \quad u_z = v_x = w_y \tag{3}$$

obviously hold throughout a neighborhood of t_0 if $f(t)$ is analytic at t_0. If u, v and $w \in C^2$ at t_0 we have from equation (3) and the equality of cross derivatives:

$$\nabla^2 u = \nabla^2 v = \nabla^2 w \tag{4}$$

from which we conclude

THEOREM 1. *The function,*

$$F(t) = (\omega - 1)f(t) = U + \omega V + \omega^2 W \tag{5}$$

has components $U = w - u$, $V = u - v$ and $W = v - w$ which are harmonic and satisfy

the equation $U+V+W=0$.

The usual laws establishing the analyticity and derivatives of (a) analytic functions of an analytic function, (b) sums and products of an analytic function, (c) positive and negative integral powers of t and (d) all functions which can be built up by rational operations on t follow as in Ward's paper [3]. Also, we may generalize the exponential and trigonometric functions from their usual series expansion and obtain the usual formulas for their derivatives.

3. Examples of harmonic functions generated by Theorem 1.

(a) The components of $(\omega-1)t^3$ are readily seen to be

$$\begin{aligned} U &= -x^3+3x^2z+3xy^2-6xyz-y^3+3yz^2-z^3\\ V &= x^3-3x^2y+6xyz-3xz^2+y^3-3y^2z+z^3\\ W &= 3x^2y-3x^2z-3xy^2+3xz^2-3yz^2+3y^2z. \end{aligned}$$

It is to be noted that the components of $(\omega-1)t^n$ for n integral provide only two independent homogeneous harmonic polynomials of degree n whereas the general solution in series form involves linear combinations of $2n+1$ such polynomials [4] for each degree n.

(b) Consider $F(t)=(\omega-1)e^t=(\omega-1)e^xe^{\omega y}e^{\omega^2 z}$. Let

$$g_1(s) = 1/3\left[e^s - e^{-\frac{1}{2}s}\cos\frac{\sqrt{3}}{2}s - \sqrt{3}\,e^{-\frac{1}{2}s}\sin\frac{\sqrt{3}}{2}s\right] = \sum_{n=0}^{\infty}\frac{s^{3n+2}}{(3n+2)!} \tag{6}$$

$$g_2(s) = g_1'(s) = \sum_{n=0}^{\infty}\frac{s^{3n+1}}{(3n+1)!} \tag{7}$$

$$g_3(s) = g_1''(s) = \sum_{n=0}^{\infty}\frac{s^{3n}}{(3n)!}; \tag{8}$$

then

$$\begin{aligned} F(t) &= (\omega-1)e^x[g_3(y)+\omega g_2(y)+\omega^2 g_1(y)][g_3(z)+\omega g_1(z)+\omega^2 g_2(z)]\\ &= U+\omega V+\omega^2 W, \end{aligned}$$

where

$$U = e^x\{g_1(y)[g_3(z)-g_1(z)]+g_2(y)[g_1(z)-g_2(z)]+g_3(y)[g_2(z)-g_3(z)]\} \tag{9}$$

$$V = e^x\{g_1(y)[g_1(z)-g_2(z)]+g_2(y)[g_2(z)-g_3(z)]+g_3(y)[g_3(z)-g_1(z)]\} \tag{10}$$

$$W = e^x\{g_1(y)[g_2(z)-g_3(z)]+g_2(y)[g_3(z)-g_1(z)]+g_3(y)[g_1(z)-g_2(z)]\}. \tag{11}$$

With the aid of the recursion formulas for $g_k(s)$, $k=1, 2, 3$, we can readily show that U is harmonic as we know it must be from Theorem 1. Since $U+V+W=0$, $U_{xx}=U$, $U_{yy}=V$ and $U_{zz}=W$, we have immediately

$$\nabla^2 U = U+V+W = 0, \tag{12}$$

or expanding (9), (10) and (11) by means of (6), (7) and (8) we have

$$(13)\qquad U = \frac{2}{\sqrt{3}} e^{x-\frac{1}{2}y-\frac{1}{2}z} \sin\left[\frac{\sqrt{3}}{2}(y-z)+\frac{4\pi}{3}\right] = U_{xx},$$

$$(14)\qquad V = \frac{2}{\sqrt{3}} e^{x-\frac{1}{2}y-\frac{1}{2}z} \sin\left[\frac{\sqrt{3}}{2}(y-z)+\frac{2\pi}{3}\right] = U_{yy},$$

$$(15)\qquad W = \frac{2}{\sqrt{3}} e^{x-\frac{1}{2}y-\frac{1}{2}z} \sin\left[\frac{\sqrt{3}}{2}(y-z)\right] = U_{zz},$$

from which (12) also follows by addition.

4. Relation to two dimensional harmonic functions. The laws for differentiating elementary analytic functions of t correspond in a one-one fashion with those for differentiating the corresponding analytic functions of a complex variable $Z=X+iY$. In fact, we note that when

$$f(t) = \sum_{n=0} a_n t^n,$$

where the summation is finite or infinite, the computation of the expression $(\omega-1)[f(t+\Delta t)-f(t)]$ in the special case $\Delta t=\Delta x+\omega\Delta y+\omega^2\Delta z$, with $\Delta x=\Delta y=\Delta z$, leads to

$$(16)\qquad \begin{aligned}(\omega-1)&[f(t+\Delta t)-f(t)]\\ &= (\omega-1)\sum_{n=1} a_n\Delta t[(t+\Delta t)^{n-1}+(t+\Delta t)^{n-2}t+\cdots+t^{n-1}] = 0\end{aligned}$$

since $(\omega-1)\Delta t=0$ in this case. Thus the functions (5) are constant on any line normal to the plane $x+y+z=0$. Since they are harmonic in 3 dimensions and have directional derivatives zero normal to that plane, the plane section of the functions (5) in $x+y+z=0$ must be two dimensional harmonic within that plane and hence are associated with analytic functions of a complex variable in that plane.

Heretofore, only the property $\omega^3=1$ has been used in forming functions $(\omega-1)f(t)$ with harmonic components. The complex number $\omega=\frac{1}{2}(-1+i\sqrt{3})$ has the additional property that $\omega^2+\omega+1=0$, which destroys uniqueness in the expansion of $f(t)$. However, functions equivalent modulo $\omega^2+\omega+1$ to the functions (5) have harmonic components which are linear combinations of those given by Theorem 1. The decomposition into real and imaginary parts of

$$(17)\qquad Z = X+iY = t = x+\omega y+\omega^2 z = x-\tfrac{1}{2}y-\tfrac{1}{2}z+i\frac{\sqrt{3}}{2}(y-z)$$

is obtained by setting $\omega=\frac{1}{2}(-1+i\sqrt{3})$. Letting $f(Z)$ be an analytic function of Z, and using the relation $i=(\omega-\omega^2)/\sqrt{3}$, we have

$$
\begin{aligned}
(\omega - 1)f(Z) &= (\omega - 1)[U(X, Y) + iV(X, Y)] \\
&= (\omega - 1)\left[U(X, Y) + \omega \frac{V(X, Y)}{\sqrt{3}} - \omega^2 \frac{V(X, Y)}{\sqrt{3}}\right] \\
&= \left[-\frac{V(X, Y)}{\sqrt{3}} - U(X, Y)\right] + \omega\left[U(X, Y) - \frac{V(X, Y)}{\sqrt{3}}\right] \qquad (18) \\
&\quad + \omega^2\left[\frac{2V(X, Y)}{\sqrt{3}}\right] \\
&= U^*(X, Y) + \omega V^*(X, Y) + \omega^2 W^*(X, Y),
\end{aligned}
$$

where, by (17), we may write

$$
\begin{aligned}
U^*(X, Y) = -\Bigg[&\frac{1}{\sqrt{3}} V\left(x - \tfrac{1}{2}y - \tfrac{1}{2}z, \frac{\sqrt{3}}{2}(y - z)\right) \\
&+ U\left(x - \tfrac{1}{2}y - \tfrac{1}{2}z, \frac{\sqrt{3}}{2}(y - z)\right)\Bigg] \qquad (19)
\end{aligned}
$$

and corresponding formulas for V^* and W^*. Since the components of e^Z are $U_1 = e^X \cos Y$ and $V_1 = e^X \sin Y$, the expansion of U in (13) follows readily from (19). The results of (18), (19) and Theorem 1 may be combined to prove the closing

THEOREM 2. *Let $f_1(Z) = U_1(X, Y) + iV_1(X, Y)$ be an analytic function of $Z = X + iY$. The functions*

$$
U_1\left(x - \tfrac{1}{2}y - \tfrac{1}{2}z, \frac{\sqrt{3}}{2}(y - z)\right) \quad \textit{and} \quad V_1\left(x - \tfrac{1}{2}y - \tfrac{1}{2}z, \frac{\sqrt{3}}{2}(y - z)\right)
$$

are harmonic functions related to those of Theorem 1 by the equations

$$
U = -V_1/\sqrt{3} - U_1, \quad V = U_1 - V_1/\sqrt{3} \quad \textit{and} \quad W = 2V_1/\sqrt{3}.
$$

References

1. H. A. von Beckh-Widmanstetter, Laszt sich die Eigenschaft der analytischen Funktionen einer gemeinen komplexen Veranderlichen, Potentiale als Bestandteile zu liefern, auf Zahlsysteme mit drei Einheiten verallgemeinern?, Monatshefte fur Mathematik und Physik, vol. 23, 1912, pp. 257–260.

2. P. W. Ketchum, A complete solution of Laplace's equation by an infinite hypervariable, Amer. Jour. Math., vol. 51, 1929, pp. 179–188.

3. J. A. Ward, A theory of analytic functions in linear associative algebras, Duke Math. Jour., vol. 7, 1940, pp. 233–248.

4. E. T. Whittaker and G. N. Watson, A Course of Modern Analysis, 4th Edition, 1927, Cambridge University Press, pp. 388–391.

GEOMETRY OF THE TETRAHEDRON*

VICTOR THÉBAULT, Tennie (Sarthe), France

This note has as its object a new or little known theorem which may be used to find properties of tetrahedrons in which the Monge point is situated in the plane of one face. [1].

THEOREM. *If the plane of three points a, b, c, situated respectively in the planes of the faces DBC, DCA, DAB of a tetrahedron $T\equiv ABCD$, passes through the vertex D, the plane determined by the isogonal conjugates a', b', c' of a, b, c with respect to the corresponding faces also passes through the point D, and conversely.*

Proof. Let the segments $DA'=DB'=DC'$ be placed on the edges DA, DB' DC. The lines Da and Da', Db and Db', Dc and Dc' meet $B'C'$, $C'A'$, $A'B'$ in a_1 and a_1', b_1 and b_1', c_1 and c_1'. The bisectors of the angles at D of the faces DBC, DCA, DAB are perpendicular to the sides of the triangle $A'B'C'$ at their midpoints. It follows that the points a_1, b_1, c_1, situated in the plane $Dabc$, are collinear and symmetric to the points a_1', b_1', c_1' with respect to the midpoints of $B'C'$, $C'A'$, $A'B'$. In other words, the points a_1', b_1', c_1' are on the reciprocal transversal d_1' of the line $d_1\equiv a_1b_1c_1$ with respect to the triangle $A'B'C'$. Hence the plane $a'b'c'$ is the plane (D, d_1') and conversely since the sets of points (a, b, c) and (a', b', c') may be interchanged.

COROLLARY. *If the Monge point of a tetrahedron T is situated in the plane of one face, the opposite vertex and the orthocenters of the adjacent faces are coplanar and conversely. The opposite vertex and the circumcenters of the adjacent faces are also coplanar and conversely.*

Proof. If the Monge point is in the face ABC, then the foot D' of the altitude DD' of T is on the circumcircle of ABC [1] and its orthogonal projections A_1, B_1, C_1 on the edges BC, CA, AB (*i.e.*, the feet of the altitudes DA_1, DB_1, DC_1 of faces DBC, DCA, DAB) are on the Simson line S_1 of D' with respect to triangle ABC. Hence the plane (D, S_1) contains the altitudes DA_1, DB_1, DC_1 and consequently the orthocenters of the faces DBC, DCA, DAB. Conversely, if the plane of the orthocenters of the faces DBC, DCA, DAB passes through the vertex D, it contains the points A_1, B_1, C_1. The perpendiculars to BC, CA, AB drawn through A_1, B_1, C_1 intersect at the foot D' of the altitude DD'. Hence D' must be on the circumcircle of ABC as A_1, B_1, C_1 are collinear by hypothesis. As D' is on the sphere $ABCD$, the Monge point of T is in the plane ABC. Furthermore, according to the preceding theorem the vertex D of T and the circumcenters of DBC, DCA, DAB situated on the isogonals of the altitudes DA_1, DB_1, DC_1 with respect to the corresponding faces are likewise coplanar and conversely.

COROLLARY. *If the Monge point is in the plane of face ABC of T, the isotomic*

* Translated by W. E. Byrne.

conjugates a'', b'', c'' of the orthocenters of faces DBC, DCA, DAB with respect to the corresponding triangles are in a plane passing through the vertex D and conversely.

Proof. Since the points A_1, B_1, C_1 are on the Simson line S_1 of D' with respect to triangle ABC, the points of intersection A_2, B_2, C_2 of the isotomic conjugates of DA_1, DB_1, DC_1 in faces DBC, DCA, DAB, are on the reciprocal transversal S_2 of S_1 with respect to triangle ABC. S_2 coincides with the Simson line with respect to triangle ABC of the point D'' diametrically opposite D and D' on the sphere $ABCD$ and on the circle ABC. Hence the points a'', b'', c'' on DA_2, DB_2, DC_2 are coplanar with D and conversely. Note that the points a'', b'', c'' coincide in each of the triangles DBC, DCA, DAB with the anticomplementary point of the symmedian point. [2].

COROLLARY. *The planes ABC, (D, S_1), (D, S_2) meet on the nine-point circle of the triangle ABC in the orthopole of diameter $D'D''$ of the circle ABC.*

Proof. According to a known theorem [3] the Simson lines S_1 and S_2 are perpendicular and they intersect at the orthopole of the diameter $D'D''$ of circle ABC. The orthopole in question is on the nine-point circle of triangle ABC. [4]

References

1. V. Thébault, Comptes Rendus (Paris), 1944, p. 262; Les Sciences au Baccalauréat (Paris), 1946, p. 101; This MONTHLY, vol. 56, 1949, p. 6; N. A. Court, This MONTHLY, vol. 56, 1949, p. 312.

2. A. Gob, Mémoires de la Soc. Royale des Sc. de Liège, 2nd series, t. XVI; Mathesis, 1889 (Supplement).

3. R. A. Johnson, Modern Geometry, pp. 207, 328.

4. V. Thébault, Nouvelles Annales de Mathématiques, 1910, p. 272; W. Gallatly, The Modern Geometry of the Triangle, p. 48.

ON A FORMULA OF GROSSWALD*

T. S. NANJUNDIAH, Mysore University

Using known properties of the Legendre polynomials $P_n(x)$ and the hypergeometric function $F(a, b; c; x)$, Grosswald [2] has recently derived the formula

$$\sum_{k=0}^{n-r}(-2)^{-k}\binom{n}{r+k}\binom{n+r+k}{k} = \begin{cases} (-1)^{(n-r)/2}2^{-(n-r)}\binom{n}{(n-r)/2}, & r \equiv n \pmod 2 \\ 0, & r \not\equiv n \pmod 2 \end{cases} \tag{1}$$

and has remarked that (1) does not follow readily by the methods of [3] and that a direct proof seems rather difficult. Carlitz [1] has shown that the known formula [Gauss],

* *Editorial note.* Within a month after this paper had been accepted for publication, a note containing very similar results was submitted by B. S. Popov, University of Skopje, Yugoslavia.

$$F(a, b; \tfrac{1}{2}(a+b+1); \tfrac{1}{2}) = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}+\frac{1}{2}a+\frac{1}{2}b)}{\Gamma(\frac{1}{2}+\frac{1}{2}a)\Gamma(\frac{1}{2}+\frac{1}{2}b)} \tag{2}$$

yields (1) as a special case for $a=-(n-r)$, $b=n+r+1$ .We wish to point out that (1) can actually be recovered from the formula (see [3], p. 253):

$$\sum_{k=0}^{n} (-2)^k \binom{\mu}{k}\binom{2\mu-2k}{n-k} = \begin{cases} \dbinom{\mu}{n/2}, & n = 0, 2, \cdots \\ 0, & n = 1, 3, \cdots, \end{cases} \tag{3}$$

which is the result of equating the coefficients of z^n on both sides of the identity:

$$(1+z)^{2\mu}(1-2z(1+z)^{-2})^{\mu} = (1+z^2)^{\mu}.$$

Indeed, following Grosswald, one is naturally led to (3) by analogous considerations of the ultraspherical polynomials $P_n^\lambda(x)$, for which we note the explicit representation (see [4], p. 84):

$$P_n^\lambda(x) = \sum_{m=0}^{[n/2]} (-1)^m \binom{\lambda+n-m-1}{n-m}\binom{n-m}{m}(2x)^{n-2m}, \tag{3a}$$

and the hypergeometric representation (see [4], p. 80):

$$P_n^\lambda(x) = \binom{2\lambda+n-1}{n} F(-n, 2\lambda+n; \lambda+\tfrac{1}{2}; (1-x)/2). \tag{3b}$$

Thus one finds (3), without difficulty, by equating the values of $P_n^\lambda(0)$ derived from (3a) and (3b) and setting $\lambda=-\mu$. One may also verify, after Carlitz, that (2) yields (3) as a special case for $a=-n$, $b=-2\mu+n$.

Now taking into account the relation (see [4], p. 83):

$$P_n^{(r)}(x) = 1\cdot 3 \cdots (2r-1)P_{n-r}^{r+1/2}(x)$$

in which $P_n^{(r)}(x)$ is the rth derivative of $P_n(x) \equiv P_n^{1/2}(x)$ and recalling Grosswald's derivation of (1), one certainly recovers (1) from (3) by taking $n-r$ in place of n and setting $-\mu=r+\frac{1}{2}$ in (3).

That (1) is a special case of (3) as just indicated is by no means apparent without the above facts in the background. It is therefore desirable to point out a simple and direct proof of (1). To this end, we just note that, on changing the summation index k in (1) into $(n-r)-k$, (1) appears at once as a special case of the formula:

$$\sum_{k=0}^{n} (-2)^k \binom{\mu}{k}\binom{2\mu-k}{n-k} = \begin{cases} (-1)^{n/2}\dbinom{\mu}{n/2}, & n = 0, 2, \cdots \\ 0, & n = 1, 3, \cdots, \end{cases} \tag{4}$$

which is the result of equating the coefficients of z^n on both sides of the identity:

$$(1+z)^{2\mu}(1-2z(1+z)^{-1})^{\mu}=(1-z^2)^{\mu}.$$

In fact, we regain (1) from (4) by taking $n-r$ in place of n and then $\mu=n$ in (4).

References

1. L. Carlitz, Note on a formula of Grosswald, this MONTHLY, vol. 60, 1953, p. 181.
2. E. Grosswald, On sums involving binomial coefficients, this MONTHLY, vol. 60, 1953, pp. 179–181.
3. E. Netto, Lehrbuch der Combinatorik, B. G. Teubner, Leipzig und Berlin, 1927.
4. G. Szegö, Orthogonal Polynomials, Amer. Math. Soc. Colloquium Publ. v. XXIII, 1939

THE CHARACTERISTIC INITIAL VALUE PROBLEM FOR THE WAVE EQUATION AND RIEMANN'S METHOD

M. H. PROTTER, University of California, Berkeley

1. Introduction. Riemann's method has been used extensively for solving the Cauchy problem for hyperbolic equations in two independent variables. This method has been extended to the wave equation in more than two variables by Lewy [4], Martin [5], and Diaz and Martin [3]. The determination of the Riemann function in these cases hinges on the solution of a characteristic initial value problem. The ability to do this (usually by iterations for two independent variables) is presupposed, or in some cases the Riemann function can be found explicitly.

If a solution of a characteristic initial value problem is sought by Riemann's method then the natural way of determining the Riemann function leads to another characteristic initial value problem and the process becomes circular. Instead of the usual procedure, a variation, first employed by Martin for the Cauchy problem [5], will be used to overcome this difficulty for the case of certain hyperbolic equations in more than two independent variables. The argument will be carried through for the wave equation in three variables

$$u_{xx}+u_{yy}=u_{zz} \tag{1}$$

although it will be clear from the discussion that the result holds for hyperbolic equations with constant coefficients in any number ($\geqq 3$) of variables. In addition certain cases where the coefficients depend only on z can be handled without additional difficulty.

2. The characteristic initial value problem. Without loss of generality we consider the characteristic cone

$$x^2+y^2=(z-1)^2 \tag{2}$$

of equation (1) and seek a solution of this equation interior to (2), $0\leqq z\leqq 1$, satisfying the condition

$$u(x,y,1-\sqrt{x^2+y^2})=\psi(x,y), \qquad 0\leqq x^2+y^2\leqq 1. \tag{3}$$

The given function $\psi(x, y)$ is assumed to have continuous second derivatives. We transform (1) to characteristic coordinates by introducing (α, β, ϕ) as new variables according to the relations

$$x = \tfrac{1}{2}(\alpha - \beta) \cos \phi, \qquad y = \tfrac{1}{2}(\alpha - \beta) \sin \phi, \qquad z = \tfrac{1}{2}(\alpha + \beta).$$

Then equation (1) becomes

$$Lu \equiv u_{\alpha\beta} - \frac{1}{2(\alpha - \beta)} (u_\alpha - u_\beta) - \frac{u_{\phi\phi}}{(\alpha - \beta)^2} = 0. \tag{4}$$

It is easy to see that the surfaces $\alpha = \text{const.}$, $\beta = \text{const.}$ are characteristic cones with vertices on the z-axis, while $\phi = \text{const.}$ represents the half-planes containing the z-axis. Following this transformation we introduce the *associate equation* to (4):

$$Mv \equiv v_{\alpha\beta} + \frac{1}{2(\alpha - \beta)} (v_\alpha - v_\beta) = 0.$$

This equation assumes the role of the *adjoint equation* usually considered in such problems, but is essentially different in that it depends only on the variables α and β. This dependence on two variables (the characteristic variables) continues to hold for hyperbolic equations (with constant coefficients) in any number of variables.

If D is any domain in (α, β, ϕ) space with boundary B we may easily verify the identity

$$\begin{aligned} &\iiint_D [(v_\beta - v_\alpha)Lu - (u_\beta - u_\alpha)Mv] d\alpha d\beta d\phi \\ &\qquad = \iint_B \left[u_\beta v_\beta d\beta d\phi - u_\alpha v_\alpha d\alpha d\phi + u_\phi \frac{v_\alpha - v_\beta}{(\alpha - \beta)^2} d\alpha d\beta \right]. \end{aligned} \tag{5}$$

Let $P(0, 0, z_0)$, $0 \leqq z_0 \leqq 1$, be a point at which we wish to determine the solution. Once the solution is obtained at points on the z-axis we may employ a Lorentz transformation to obtain the solution at all interior points of the cone (see [2, p. 442]).

In (x, y, z) space we consider a domain D^* bounded by the portions of the two cones

$$\begin{aligned} x^2 + y^2 &= (z - 1)^2, & \tfrac{1}{2}(1 + z_0) &\leqq z \leqq 1; \\ x^2 + y^2 &= (z - z_0)^2, & z_0 &\leqq z \leqq \tfrac{1}{2}(1 + z_0). \end{aligned}$$

If (α, β, ϕ) is considered as a system of rectangular coordinates then the domain D^* when transformed to (α, β, ϕ) space is the (infinite) triangular prism bounded by the planes $\alpha = 1$, $\beta = z_0$, $\alpha = \beta$. Since we are interested only in solutions of (4) which have period 2π we limit our considerations to the section of the prism

contained between the planes $\phi=0$ and $\phi=2\pi$. We designate this domain as D and its boundary B.

If u is the desired solution of $Lu=0$ and v is a regular solution of $Mv=0$ yet to be prescribed then (5) will vanish. Since v is independent of ϕ those integrals on the right side of (5) taken over the part of B lying in $\phi=0$ and $\phi=2\pi$ will cancel. We therefore have

$$\iint [u_\beta v_\beta]_{\alpha=1} d\beta d\phi - \iint [u_\alpha v_\alpha]_{\beta=z_0} d\alpha d\phi + \iint [u_\beta v_\beta - u_\alpha v_\alpha]_{\alpha=\beta} d\alpha d\phi = 0. \tag{6}$$

We now seek a solution of $Mv=0$ which has the property that

$$v_\alpha = -1 \text{ on } \beta = z_0 \quad\text{and}\quad v_\alpha = v_\beta = 0 \text{ on } \alpha = \beta.$$

This is a problem of Goursat type for a hyperbolic equation in two independent variables and any such solution will be termed a Riemann function. It is easy to verify that the solution

$$v = -(\alpha+\beta) + 2(\alpha - z_0)^{1/2}(\beta - z_0)^{1/2}$$

satisfies the required conditions. Inserting this value for v in the above equation (6) we obtain

$$\int_0^{2\pi}\int_{z_0}^1 [u_\alpha]_{\beta=z_0} d\alpha d\phi = \int_0^{2\pi}\int_{z_0}^1 [u_\beta]_{\alpha=1}\left[\left(\frac{1-z_0}{\beta-z_0}\right)^{1/2} - 1\right] d\beta d\phi$$

or

$$\frac{1}{2\pi}\int_0^{2\pi} [u]_{\alpha=1,\beta=z_0} d\phi - \frac{1}{2\pi}\int_0^{2\pi} [u]_{\beta=z_0,\alpha=z_0} d\phi = \frac{1}{2\pi}\int_0^{2\pi}\int_{z_0}^1 [u_\beta]_{\alpha=1}\left[\left(\frac{1-z_0}{\beta-z_0}\right)^{1/2} - 1\right] d\beta d\phi.$$

When transformed to (x, y, z) coordinates the second integral on the left above is seen to be the value of u at the point P. In the integral on the right u is known $(=\psi)$ in the plane $\alpha=1$ and since

$$\frac{\partial}{\partial\beta} = -\frac{1}{2}\left(\frac{x}{\sqrt{x^2+y^2}}\frac{\partial}{\partial x} + \frac{y}{\sqrt{x^2+y^2}}\frac{\partial}{\partial y} - \frac{\partial}{\partial z}\right)$$

is an inner derivative (*i.e.*, derivative in a characteristic direction) we know the value of u_β in this plane. Hence we finally find

$$u(P) = \frac{1}{2\pi}\int_0^{2\pi} [\psi]_{\alpha=1,\beta=z_0} d\phi - \frac{1}{2\pi}\int_0^{2\pi} [\psi_\beta]_{\alpha=1}\left[\left(\frac{1-z_0}{\beta-z_0}\right)^{1/2} - 1\right] d\beta d\phi. \tag{7}$$

The fact that u given by (7) on the z-axis and by Lorentz transformations of (7) at other points actually satisfies the differential equation may be established by differentiation. The justification of the differentiations of the various integrals involved has been given by d'Adhémar [1] in the process of obtaining a solution to the same problem considered here. Equation (7) gives an explicit formula for the solution of this problem and the method seems much simpler than that given in [2] which depends on a mean value theorem of Asgeirsson and the solution of an Abel integral equation.

References

1. R. d'Adhémar, Sur une Équation aux Derivées Partielles du Type Hyperbolique, Rendiconti del Circolo Matematico di Palermo, vol. 20, 1905.

2. R. Courant and D. Hilbert, Methoden der Mathematischen Physik, vol. II, Interscience, New York.

3. J. Diaz and M. H. Martin, Riemann's method and the problem of Cauchy II, Proc. Amer. Math. Soc., vol. 3, 1952.

4. H. Lewy, Verallgemeinerung der Riemannschen Methode auf mehr Dimensionen, Nachr. Ges. Wiss., Gottingen, 1928, pp. 118–123.

5. M. H. Martin, Riemann's method and the problem of Cauchy, Bull. Amer. Math. Soc., vol. 57, 1951.

CLASSROOM NOTES

EDITED BY G. B. THOMAS, Massachusetts Institute of Technology

All material for this department should be sent to G. B. Thomas, Department of Mathematics, Massachusetts Institute of Technology, Cambridge 39, Mass.

ON THE VECTOR TRIPLE PRODUCT

M. S. KLAMKIN, Polytechnic Institute of Brooklyn

In this note, different methods of expanding the vector triple product will be reviewed. In addition, two new variations will be presented.

A. A direct method [1] is to expand $\mathbf{N}\times(\mathbf{B}\times\mathbf{C})$ and $[(\mathbf{N}\cdot\mathbf{C})\mathbf{B}-(\mathbf{N}\cdot\mathbf{B})\mathbf{C}]$ into their $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ components and compare. Although this method is direct it is inelegant.

B. Another direct method [2] but with little calculation is to use the permutation symbol ϵ_{rst}. Thus for the rth component of $\mathbf{N}\times(\mathbf{B}\times\mathbf{C})$ we have

$$(1)\qquad [\mathbf{N}\times(\mathbf{B}\times\mathbf{C})]_r = \epsilon_{rst}\mathbf{N}_s[\epsilon_{tuv}\mathbf{B}_u\mathbf{C}_v] = \mathbf{N}_s\delta_{rs}^{uv}\mathbf{B}_u\mathbf{C}_v = (\mathbf{N}_s\mathbf{C}_s)\mathbf{B}_r - (\mathbf{N}_s\mathbf{B}_s)\mathbf{C}_r.$$

C. Geometric methods [3] and [4].

D. It follows from the definition of the vector product that $\mathbf{N}\times(\mathbf{B}\times\mathbf{C})$ is a vector parallel to the plane of $\mathbf{B}$ and $\mathbf{C}$. Consequently,

$$(2)\qquad \mathbf{N}\times(\mathbf{B}\times\mathbf{C}) = k_1\mathbf{B} + k_2\mathbf{C},$$

where k_1, and k_2 are scalars to be determined. On taking the scalar product of (2) with $\mathbf{N}$, we find that

$$0 = k_1\mathbf{N}\cdot\mathbf{B} + k_2\mathbf{N}\cdot\mathbf{C}. \tag{3}$$

Therefore,

$$\mathbf{N} \times (\mathbf{B} \times \mathbf{C}) = \lambda(\mathbf{N}, \mathbf{B}, \mathbf{C})[(\mathbf{N}\cdot\mathbf{C})\mathbf{B} - (\mathbf{N}\cdot\mathbf{B})\mathbf{C}], \tag{4}$$

where $\lambda(\mathbf{N}, \mathbf{B}, \mathbf{C})$ is a scalar function of $\mathbf{N}$, $\mathbf{B}$, and $\mathbf{C}$, to be determined. It will be assumed here that none of the vectors $\mathbf{N}$, $\mathbf{B}$, or $\mathbf{C}$ is zero, and that $\mathbf{N}$ is not parallel to $\mathbf{B}\times\mathbf{C}$, since in these cases (4) will hold regardless of the value of $\lambda(\mathbf{N}, \mathbf{B}, \mathbf{C})$. The six following proofs all start from this point but differ in the method of proving that $\lambda(\mathbf{N}, \mathbf{B}, \mathbf{C}) = 1$.

D_1. In this method, [5], $\mathbf{C}$ is resolved into two components $\mathbf{D}$ and $\mathbf{E}$ such that $\mathbf{B}$, $\mathbf{D}$, and $\mathbf{E}$ are not coplanar. Then by (4) it follows that

$$\begin{aligned}\mathbf{N} \times (\mathbf{B} \times [\mathbf{D} + \mathbf{E}]) &= \lambda(\mathbf{N}, \mathbf{B}, \mathbf{C})[\mathbf{N}\cdot(\mathbf{D} + \mathbf{E})\mathbf{B} - (\mathbf{N}\cdot\mathbf{B})(\mathbf{D} + \mathbf{E})] \\ &= \lambda(\mathbf{N}, \mathbf{B}, \mathbf{C})[(\mathbf{N}\cdot\mathbf{D})\mathbf{B} - (\mathbf{N}\cdot\mathbf{B})\mathbf{D}] \\ &\quad + \lambda(\mathbf{N}, \mathbf{B}, \mathbf{C})[(\mathbf{N}\cdot\mathbf{E})\mathbf{B} - (\mathbf{N}\cdot\mathbf{B})\mathbf{E}],\end{aligned} \tag{5}$$

but also

$$\begin{aligned}\mathbf{N} \times (\mathbf{B} \times [\mathbf{D} + \mathbf{E}]) &= \mathbf{N} \times (\mathbf{B} \times \mathbf{D}) + \mathbf{N} \times (\mathbf{B} \times \mathbf{E}) \\ &= \lambda(\mathbf{N}, \mathbf{B}, \mathbf{D})[(\mathbf{N}\cdot\mathbf{D})\mathbf{B} - (\mathbf{N}\cdot\mathbf{B})\mathbf{D}] \\ &\quad + \lambda(\mathbf{N}, \mathbf{B}, \mathbf{E})[(\mathbf{N}\cdot\mathbf{E})\mathbf{B} - (\mathbf{N}\cdot\mathbf{B})\mathbf{E}].\end{aligned}$$

Whence it follows that

$$\lambda(\mathbf{N}, \mathbf{B}, \mathbf{C}) = \lambda(\mathbf{N}, \mathbf{B}, \mathbf{D}) = \lambda(\mathbf{N}, \mathbf{B}, \mathbf{E}). \tag{6}$$

The conclusion then improperly drawn in [5] was that $\lambda(\mathbf{N}, \mathbf{B}, \mathbf{C})$ is independent of $\mathbf{N}$, $\mathbf{B}$, and $\mathbf{C}$, and thus is a constant. Then by choosing $\mathbf{N}=\mathbf{k}$, $\mathbf{B}=\mathbf{j}$, and $\mathbf{C}=\mathbf{k}$, $\lambda(\mathbf{N}, \mathbf{B}, \mathbf{C})$ is determined to be equal to 1. The conclusion to be drawn, however, is not that $\lambda(\mathbf{N}, \mathbf{B}, \mathbf{C})$ is independent of $\mathbf{N}$, $\mathbf{B}$, and $\mathbf{C}$, but that it is independent of $\mathbf{C}$. (This latter conclusion will still hold even if one of the two vectors $[(\mathbf{N}\cdot\mathbf{D})\mathbf{B}-(\mathbf{N}\cdot\mathbf{B})\mathbf{D}]$, $[(\mathbf{N}\cdot\mathbf{E})\mathbf{B}-(\mathbf{N}\cdot\mathbf{B})\mathbf{E}]$ is zero). To complete the proof, we note that since

$$\lambda(\mathbf{N}, \mathbf{B}, \mathbf{C}) = \lambda(\mathbf{N}, \mathbf{C}, \mathbf{B}) \qquad [\text{from } (4)], \tag{7}$$

it follows that $\lambda(\mathbf{N}, \mathbf{B}, \mathbf{C})$ is also independent of $\mathbf{B}$. By choosing $\mathbf{N}=\mathbf{i}n_1+\mathbf{j}n_2+\mathbf{k}n_3$, $\mathbf{B}=\mathbf{j}$, $\mathbf{C}=\mathbf{k}$, we find that $\lambda(\mathbf{N}, \mathbf{B}, \mathbf{C})=1$.

D_2. Since $\mathbf{N}\times(\mathbf{B}\times\mathbf{C})$ and $[(\mathbf{N}\cdot\mathbf{C})\mathbf{B}-(\mathbf{N}\cdot\mathbf{B})\mathbf{C}]$ are linear distributive functions of each of the vectors $\mathbf{N}$, $\mathbf{B}$, and $\mathbf{C}$, it follows that if $\lambda(\mathbf{N}, \mathbf{B}, \mathbf{C})$ is equal to 1 for every combination of the unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, it is also equal to 1 for all vectors $\mathbf{N}$, $\mathbf{B}$, and $\mathbf{C}$, [6]. This is easily verified.

D_3. In this method, [7], it is first shown that $\lambda(\mathbf{N}, \mathbf{N}, \mathbf{C})=1$. Then using

this result it is shown that $\lambda(\mathbf{N}, \mathbf{B}, \mathbf{C}) = 1$.

D_4. Here, [8], a special coordinate system is set up such that

$$\mathbf{B} = b_1\mathbf{i}, \qquad \mathbf{C} = c_1\mathbf{i} + c_2\mathbf{j}, \qquad \mathbf{N} = \mathbf{i}n_1 + \mathbf{j}n_2 + \mathbf{k}n_3. \tag{8}$$

It then follows that after substituting (8) in (4) that $\lambda(\mathbf{N}, \mathbf{B}, \mathbf{C}) = 1$. However, we should add the statement, [9], that if a tensor equation is true in one coordinate system it is true in all coordinate systems.

In methods D_5 and D_6 which follow, the proofs presented are believed to be new.

D_5. Since $\mathbf{A}\cdot\mathbf{B}\times\mathbf{C} = \mathbf{A}\times\mathbf{B}\cdot\mathbf{C}$, it follows that

$$\mathbf{M}\cdot\mathbf{N}\times(\mathbf{B}\times\mathbf{C}) = -\mathbf{C}\cdot\mathbf{B}\times(\mathbf{M}\times\mathbf{N}) = -\mathbf{N}\cdot\mathbf{M}\times(\mathbf{B}\times\mathbf{C}), \tag{9}$$

where $\mathbf{M}$ is chosen such that $\mathbf{M}\times(\mathbf{B}\times\mathbf{C}) \neq 0$. Expanding (9) by means of (4) we get

$$\begin{aligned} \lambda(\mathbf{N}, \mathbf{B}, \mathbf{C})&[(\mathbf{N}\cdot\mathbf{C})(\mathbf{M}\cdot\mathbf{B}) - (\mathbf{N}\cdot\mathbf{B})(\mathbf{M}\cdot\mathbf{C})] \\ &= \lambda(\mathbf{B}, \mathbf{M}, \mathbf{N})[(\mathbf{N}\cdot\mathbf{C})(\mathbf{M}\cdot\mathbf{B}) - (\mathbf{N}\cdot\mathbf{B})(\mathbf{M}\cdot\mathbf{C})] \\ &= \lambda(\mathbf{M}, \mathbf{B}, \mathbf{C})[(\mathbf{N}\cdot\mathbf{C})(\mathbf{M}\cdot\mathbf{B}) - (\mathbf{N}\cdot\mathbf{B})(\mathbf{M}\cdot\mathbf{C})]. \end{aligned} \tag{10}$$

From this latter equation and (7) it follows that $\lambda(\mathbf{N}, \mathbf{B}, \mathbf{C})$ is constant. By choosing $\mathbf{N} = \mathbf{k}$, $\mathbf{B} = \mathbf{j}$, and $\mathbf{C} = \mathbf{k}$, we find that $\lambda(\mathbf{N}, \mathbf{B}, \mathbf{C}) = 1$. (For the case $\mathbf{M}\times(\mathbf{B}\times\mathbf{C}) = 0$, (4) with $\mathbf{N}$ replaced by $\mathbf{M}$ will hold for any value of $\lambda(\mathbf{M}, \mathbf{B}, \mathbf{C})$).

D_6. If we let

$$\mathbf{P} = \mathbf{A}\times(\mathbf{B}\times\mathbf{C}) + \mathbf{B}\times(\mathbf{C}\times\mathbf{A}) + \mathbf{C}\times(\mathbf{A}\times\mathbf{B}), \tag{11}$$

then $\mathbf{P}\cdot\mathbf{A} = \mathbf{P}\cdot\mathbf{B} = \mathbf{P}\cdot\mathbf{C} = 0$. Thus if $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are noncoplanar vectors, $\mathbf{P} = 0$. However, if $\mathbf{A} = m\mathbf{B} + n\mathbf{C}$, then by direct substitution into (11), $\mathbf{P} = 0$. If we expand (11) by means of (4) we find that

$$\lambda(\mathbf{A}, \mathbf{B}, \mathbf{C}) = \lambda(\mathbf{B}, \mathbf{C}, \mathbf{A}) = \lambda(\mathbf{C}, \mathbf{A}, \mathbf{B}), \tag{12}$$

provided that $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) \neq 0$. (If $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = 0$, then (4) with $\mathbf{N}$ replaced by $\mathbf{A}$ will hold for any value of $\lambda(\mathbf{A}, \mathbf{B}, \mathbf{C})$). Also, since $\mathbf{N}\cdot\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = -\mathbf{A}\cdot\mathbf{N}\times(\mathbf{B}\times\mathbf{C})$, it follows that $\lambda(\mathbf{A}, \mathbf{B}, \mathbf{C}) = \lambda(\mathbf{N}, \mathbf{B}, \mathbf{C})$ $[\mathbf{N}\cdot\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) \neq 0]$. Thus $\lambda(\mathbf{N}, \mathbf{B}, \mathbf{C})$ is a constant and thus equal to 1.

References

1. Rutherford, D. E., Vector Methods, Oliver and Boyd, 1948, p. 9.
2. Craig, H. V., Vector and Tensor Analysis, McGraw-Hill, 1943, p. 156.
3. Coe and Rainich, this MONTHLY, vol. 56, 1949, p. 175.
4. Lubin, C. I., this MONTHLY, vol. 61, 1954, p. 42.
5. Coffin, B. S., Vector Analysis, John Wiley, 1938, p. 53.
6. Phillips, H. B., Vector Analysis, John Wiley, 1933, p. 17.
7. Lass, H., Vector and Tensor Analysis, McGraw-Hill, 1950, p. 24.
8. Brand, L., Vector and Tensor Analysis, John Wiley, 1948, p. 40.
9. Synge, J. L. and Schild, A., Tensor Calculus, Toronto, 1949, p. 13.

A CONSIDERATION OF ANNUITY FORMULAS

ROGER OSBORN, University of Texas

One of the mathematics courses taught at the freshman level in many colleges is Mathematics of Finance (or Mathematics of Investment). It is highly desirable for the student to have an understanding of the fundamental operations, procedures, and techniques in the course, but it should also be desirable for him to have a set of formulas, which if applied intelligently, would enable him to work problems rapidly. This is especially true in the treatment of annuities certain. It is toward a successful way to present these formulas that this note is directed.* The author has found the following table of formulas to be of very considerable help to the students in his classes. The formulas as presented in the following table have these advantages.

(1) All formulas needed for the treatment of annuities certain in the ordinary freshman course are collected in one place. (The formulas as given here do not make allowances for perfectly general annuities, but these formulas do cover the three primary cases.)

(2) In any one of the six classifications in which both sum and present value formulas are given in the table below, the formulas possess complete similarity. That is, the only difference between formulas is in the $a_{\overline{n}|i}$ or $s_{\overline{n}|i}$ symbol.

(3) In every instance the formulas are the basic formulas for Case I, ordinary annuities (defined later) with additional factors being included because of different data. The particular arrangement of factors is an aid to the student's memory.

(4) In every instance the formulas use the same basic symbols with which the student is already familiar, with the exception of j_p (which is defined below and to which the student may have been introduced in considering nominal rates of interest) and $a_{\overline{n}|i}$ and $s_{\overline{n}|i}$ which are defined early in the treatment of Case I annuities. This use of basic symbols eliminates the necessity for the student to memorize a new set of symbols for every new set of data, as is true in some treatments of the subject.

(5) Transition from row to row or column to column of the table is quite simply performed, which is a definite aid to memory.

(6) The formulas for deferred annuities are based on data which enables the student to treat every deferred annuity as an ordinary annuity.

(7) To be sure, other forms of these formulas may lead to simpler computation in certain types of problems, but the formulas as given here lend themselves easily to solution for the periodic payment by a method involving multiplication instead of division. It seems that this more than compensates for the additional work necessitated in certain other types of problems.

* W. L. Hart (vol. 35, p. 358) and C. C. Wylie (vol. 36, p. 327) have both written on this topic in this MONTHLY, but the author has found the system herein presented to be more satisfactory in his classes.

There are textbooks available which make some effort in the direction of this note, but the author has found none which goes as far as giving a complete table of formulas,* and those which have a summary of formulas seem to give the impression that a summary is needed due to the very large number of different symbols used.

For the formulas in the table the following definitions are used:

(1) Case I annuity—an annuity in which payment periods and interest conversion periods coincide;

(2) Case II annuity—an annuity in which there are k interest conversion periods (k an integer greater than 1—this defines k) in one payment period;

(3) Case III annuity—an annuity in which there are p payment periods (p an integer greater than 1—this defines p) in one interest conversion period.

As a final word before giving the table of formulas, it may be noted that an added advantage is that the quantities involved in these formulas are explicitly stated and may be found in any standard set of tables included in most textbooks which treat this subject.

Table of Annuity Formulas

	Ordinary Annuity	Annuity Due	Deferred Annuity
Case I	$A = Ra_{\overline{n}\|i}$ $S = Rs_{\overline{n}\|i}$	$A = Ra_{\overline{n}\|i}(1+i)$ $S = Rs_{\overline{n}\|i}(1+i)$	$A = Ra_{\overline{n}\|i}(1+i)^{-m}$
Case II	$A = Ra_{\overline{n}\|i} \cdot \dfrac{1}{s_{\overline{k}\|i}}$ $S = Rs_{\overline{n}\|i} \cdot \dfrac{1}{s_{\overline{k}\|i}}$	$A = Ra_{\overline{n}\|i} \cdot \dfrac{1}{s_{\overline{k}\|i}}(1+i)^{k}$ $S = Rs_{\overline{n}\|i} \cdot \dfrac{1}{s_{\overline{k}\|i}}(1+i)^{k}$	$A = Ra_{\overline{n}\|i} \cdot \dfrac{1}{s_{\overline{k}\|i}}(1+i)^{-m}$
Case III	$A = Ra_{\overline{n}\|i} \cdot p \cdot \dfrac{i}{j_p}$ $S = Rs_{\overline{n}\|i} \cdot p \cdot \dfrac{i}{j_p}$	$A = Ra_{\overline{n}\|i} \cdot p \cdot \dfrac{i}{j_p}(1+i)^{1/p}$ $S = Rs_{\overline{n}\|i} \cdot p \cdot \dfrac{i}{j_p}(1+i)^{1/p}$	$A = Ra_{\overline{n}\|i} \cdot p \cdot \dfrac{i}{j_p}(1+i)^{-m}$

* A table similar in a few respects to the one given here is to be found in Simpson, Pirenian, and Crenshaw, Mathematics of Finance (3rd ed.), Prentice-Hall, Inc., 1951, p. 225. For convenience, the notation used in this paper corresponds to that used in the foregoing book, pp. 224, 225, with the following elaborations: R = the periodic payment (not rent); n = the number of interest conversions in the term of the annuity; p, as defined in Case III; and $j_p = p[(1+i)^{1/p} - 1]$.

THE PROBABILITY INTEGRAL

A. J. COLEMAN, University of Toronto

The author was recently faced with the problem of evaluating the probability integral for a class which had not studied double integration, so the well-known method reproduced on p. 129 of Feller's *Probability Theory and its Applications* was not suitable. The following is the author's solution of his problem. He would be glad to learn of more elementary ones.

Consider the integral,

$$I_m = \int_0^\infty x^m e^{-nx^2/2} dx$$

where m and n are integers. Integrating by parts, we see that

$$I_m = \frac{m-1}{n} I_{m-2}, \qquad (m > 1, n > 0). \tag{1}$$

For any real t,

$$\int_0^\infty x^m (x-t)^2 e^{-nx^2/2} dx > 0.$$

Therefore, $I_{m+2} - 2tI_{m+1} + t^2 I_m > 0$ for all t. Hence,

$$I_{m+1} < \sqrt{I_m I_{m+2}}.$$

If we set m equal to n and $n-1$, and use (1) for m equal to $n+1$ and $n+2$, we obtain the inequalities

$$I_{n+2} > I_{n+1} > I_n. \tag{2}$$

For $n = 2k+1$, where k is any positive integer, the recursion formula (1) easily gives

$$I_{2k+1} = \frac{2 \cdot 4 \cdots (2k)}{(2k+1)^{k+1}}$$

$$I_{2k+2} = \frac{1 \cdot 3 \cdot 5 \cdots (2k+1)}{(2k+1)^{k+1}} I_0$$

$$I_{2k+3} = \frac{2 \cdot 4 \cdots (2k)(2k+2)}{(2k+1)^{k+2}} .$$

A change of variables shows that

$$I = \int_0^\infty e^{-u^2/2} du = \sqrt{2k+1}\, I_0.$$

Substituting these expressions in (2) gives,

$$\left(1+\frac{1}{2k+1}\right)\frac{2\cdot 4\cdots(2k)}{1\cdot 3\cdots(2k-1)}\frac{1}{\sqrt{2k+1}} > I > \frac{2\cdot 4\cdot 6\cdots(2k)}{1\cdot 3\cdots(2k-1)}\frac{1}{\sqrt{2k+1}}.$$

But Wallis' formula* states that the final expression above approaches $\sqrt{\pi/2}$ as k approaches infinity. The expression on the left approaches the same limit. Hence,

$$\int_0^\infty e^{-u^2/2}du = \sqrt{\frac{\pi}{2}}.$$

* See Courant and Robbins, What is Mathematics? pp. 509–510, for details.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1141. *Proposed by Leon Bankoff, Los Angeles, Calif.*

Find the radius of the circle inscribed in the mixtilinear triangle formed by the two legs of a given right triangle ABC and the semi-circumference described externally upon the hypothenuse AB.

E 1142. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn, N. Y.*

Find the semi-vertical angle of a right circular cone if three generating lines make angles of 2α, 2β, 2γ, with each other.

E 1143. *Proposed by C. D. Olds, San Jose State College, Calif.*

If the roots of the equation

$$a_0x^n - na_1x^{n-1} + \tfrac{1}{2}n(n-1)a_2x^{n-2} - \cdots + (-1)^n a_n = 0$$

are positive and distinct, prove that $a_r a_{n-r} > a_0 a_n$, $r = 1, 2, \cdots, n-1$.

E 1144. *Proposed by A. S. Hendler, Rensselaer Polytechnic Institute, N. Y.*

For what positive values of a is $\log_a b < b$ for all positive b?

E 1145. *Proposed by B. A. Hausmann, West Baden College, Indiana*

Let $I=(m+n-1)!/m!n!$, where m and n are natural numbers. Find a necessary and sufficient condition for I to be an integer.

SOLUTIONS

A "Faded Document" Division

E 1111 [1954, 258]. *Proposed by P. L. Chessin, Westinghouse Electric Corporation*

Our good friend and eminent numerologist, Professor Euclide Paracelso Bombasto Umbugio, has been busily engaged testing on his desk calculator the $81\cdot 10^9$ possible solutions to the problem of reconstructing the following exact long division in which the digits indiscriminately were each replaced by x save in the quotient where they were almost entirely omitted.

```
                  8
        _______________
x x x ) x x x x x x x x
          x x x
          _____
            x x x x
              x x x
            _______
                x x x x
                x x x x
                _______
```

Deflate the Professor! That is, reduce the possibilities to $(81\cdot 10^9)^0$.

Solution by W. B. Carver, Cornell University. If we denote the divisor by d, we have $8d<1000$, $d<125$. Then, since $7d<900$, it follows that the first digit in the quotient is 8, and that the quotient is 80809. Then, since $(80809)d$ must be greater than 10,000,000, we have $d>123$. Hence $d=124$.

Also solved by P. M. Berry, D. G. Brennan, R. L. Caskey, Monte Dernham, William Douglas, S. H. Eisman, P. C. Fife, Jean Gamzon, W. V. Gamzon, H. W. Gould, W. T. Grant, R. E. Greenwood, T. N. E. Greville, Fred Gruenberger, R. K. Guy, J. D. Haggard, J. W. Hamblen, H. J. Hauer, Joy Heller, Boyd Henry, H. K. Hilton, R. T. Hood, R. E. Horton, A. R. Hyde, Virginia Johnson, Edgar Karst, J. M. Kingston, H. R. Leifer, Octave Levenspiel, D. C. B. Marsh, J. H. Means, B. E. Mitchell, George Mott, T. F. Mulcrone, John Murtha, E. J. Musch, David Muskat, J. B. Muskat, C. S. Ogilvy, H. J. Osner, J. V. Pennington, Walter Penney, P. A. Piza, Alan Plait, N. Popiel, Jr., L. A. Ringenberg, Max Rosenberg, Azriel Rosenfeld, David Rothman, E. P. Rozycki, W. M. Sanders, William Saul, R. E. Shafer, S. W. Shelton, N. Shklov, Malcolm Smith, R. A. Spong, R. B. Stein, R. P. Tapscott, A. J. Tingley, C. W. Trigg, George Walker, Robert Wallach, E. G. Walsh, W. D. Ward, R. H. Wilson, Jr., Monica Wyzalek, and the proposer. Late solution by E. S. Johnson.

Several solvers located the problem and the answer, with no analysis or proof of uniqueness, as Problem 33 in Joseph de Grazia, *Math is Fun*, Gresham Press, New York (1948), pp. 34, 35, 119. Apparently the problem is considerably older, however, for a number of solvers recalled it from 15 to 20 years back.

J. B. Muskat remarked that if the middle 8 in the quotient had not been given, there would have been 163 possible solutions. Plait remarked that if the given 8 in the quotient had been the last digit of the quotient, instead of the middle one, there would have been exactly 10 different solutions.

The proposer stated that if Professor Umbugio were to perform one long division calculation a second and work an eight hour day every day of the week, without summer vacations, his technique would take about 77 years!

Trigg observed that E 1111 contains as many ones as the Professor has names, 4, and this times the number of letters in the Professor's complete name, 31, gives the divisor of the problem. Thus the number of the Professor's names and the number of letters in these names are congruent modulo 9. Then the sum, 25, of the digits of the quotient, the last two digits of the dividend, the modulus, the number of the Professor's names, and the Professor are consecutive squares. See E 961 [1951, 700].

The Tac-locus of Two Parabolic Pencils of Circles

E 1112 [1954, 259]. *Proposed by L. C. Graue, Sacramento State College*

Consider two families of circles, one tangent at the origin to the x-axis and the other tangent at the point (1, 1) to a line of slope m. Find the locus of the points of tangency of the two families.

I. *Solution by Hüseyin Demir, Zonguldak, Turkey.* Let (C) be the circle tangent to Ox at O, d the line containing $A(1, 1)$, (U_1) and (U_2) the circles tangent to d at A and to (C) at T_1 and T_2 respectively. Invert the figure with respect to O as center. Under the inversion (C), d, A, (U_1), (U_2), T_1, T_2 invert into c, (D), A', (U_1'), (U_2'), T_1', T_2' respectively. Evidently circle (D) passes through O, and line c is parallel to Ox. Also, (U_1') and (U_2') are tangent to (D) at A', and to c at T_1' and T_2'. Since the tangent c to (U_1'), $U_2')$ at T_1', T_2' keeps a fixed direction, the loci of T_1', T_2' are straight lines $A'T_1'$, $A'T_2'$. Hence the required loci of T_1 and T_2 are two circles. Each of these circles passes through A and the center of inversion O.

We now show that these two circles are orthogonal by showing that $A'T_1'$, $A'T_2'$ are perpendicular to each other. Consider the common tangent to the circles (D), (U_1'), (U_2'); it bisects $T_1'T_2'$ at I. Since $IA'=IT_1'=IT_2'$, triangle $T_1'A'T_2'$ is a right triangle and $A'T_1'$, $A'T_2'$ are perpendicular.

The result may be stated as a theorem: *The locus of the points of tangency of the circles of any two parabolic pencils of circles consists of two orthogonal circles each passing through the base points of the two pencils.*

II. *Solution by Chih-yi Wang, University of Minnesota.* The general equations of the two families are, respectively,

$$(1) \qquad x^2 + y^2 - 2ay = 0,$$

$$(2) \qquad (x - m - 1 + mk)^2 + (y - k)^2 = (1 + m^2)(1 - k)^2,$$

where a and k are parameters. Since the points of tangency must lie on the two circles and on the lines joining the centers, we must also have

$$(3) \qquad (k - a)x - (m + 1 - mk)(y - a) = 0.$$

Eliminating a and k from (1), (2), (3) we obtain, after simplification,

$$(4) \quad m[(x^2 + y^2)^2 - 2(x + y)(x^2 + y^2) + 4xy] + 2(x - y)(x^2 + y^2 - x - y) = 0.$$

Case I ($m \neq 0$). We obtain the equations of two orthogonal circles:

$$(5) \quad x^2 + y^2 - (1 - 1/m \pm \sqrt{1 + m^2}/m)x - (1 + 1/m \mp \sqrt{1 + m^2}/m)y = 0.$$

If the line passing through (1, 1) is parallel to the y-axis, we let $m \to \infty$ in (5).

Case II ($m = 0$). We obtain from (4) equations of a circle and one of its diametral lines:

$$x^2 + y^2 - x - y = 0 \quad \text{and} \quad x - y = 0.$$

The same results hold if we eliminate a and k from (1) and (2) by differentiating and equating dy/dx, without using (3).

Also solved by B. H. Bissinger, W. B. Carver, Michael Goldberg, M. S. Klamkin, Beckham Martin, C. S. Ogilvy, O. J. Ramler, R. E. Shafer, N. Shklov, and the proposer.

Bounds on a Solution of a Special Riccati Equation

E 1113 [1954, 259]. *Proposed by Peter Treuenfels, Ballistic Research Laboratories, Aberdeen Proving Ground*

Let $y(x)$ denote that solution of the differential equation $dy/dx = x^2 + y^2$ which passes through the origin. Show that $y(1) < 23/53$.

Solution by R. E. Shafer, University of California. We can evaluate upper and lower bounds of $y(x)$ by successive approximation. Clearly $dy/dx \geqq 0$ and, for a fixed value of x, dy/dx is an increasing function of y. Let

$$(1) \qquad dy_n(x)/dx = y_{n-1}^2(x) + x^2.$$

For generating a set of lower bounds $y_0 < y_1 < y_2 < \cdots$ for $0 \leqq x \leqq 1$ let $y_0(x) = 0$. Using equation (1) and imposing the condition $y(0) = 0$, we obtain

$$y_1 = x^3/3, \qquad y_2 = x^3/3 + x^7/63.$$

Hence we may use $y_2(1) = 22/63$ as a lower bound to $y(1)$.

For generating a set of upper bounds $y_0 > y_1 > y_2 > \cdots$ for $0 \leqq x \leqq 1$ let

$y_0(x)=x$. Using equation (1) and imposing the condition $y(0)=0$, we obtain

$$y_1 = 2x^3/3, \qquad y_2 = x^3/3 + 4x^7/63,$$
$$y_3 = x^3/3 + x^7/63 + 8x^{11}/11\cdot 189 + 16x^{15}/15\cdot 63^2.$$

Hence we may use $y_3(1) < 23/63$ as an upper bound to $y(1)$.

It follows that $22/63 < y(1) < 23/63$.

Also solved by A. N. Aheart, B. H. Bissinger, Christian Blatter, C. N. Campopiano, W. B. Carver, J. E. Darraugh, William Dite and Malcolm Smith (jointly), I. A. Dodes, P. C. Fife, E. K. Greatrix, Louisa Grinstein, P. G. Hodge, Jr., A. R. Hyde, P. G. Kirmser, M. S. Klamkin, D. C. B. Marsh, T. F. Mulcrone, C. S. Ogilvy, F. D. Parker, L. A. Ringenberg, David Rothman, D. C. Russell, W. M. Sanders, O. E. Stanaitis, Chih-yi Wang, J. E. Wilkins, Jr., the proposer, and someone with an illegible signature.

Marsh and Stanaitis showed that $y(1) < 4/11$, which is a closer upper bound than $23/63$. Actually $y(1) = 0.350168^+$. In view of the fact that $23/63$ is an upper bound it might be thought that the $23/53$ of the proposed problem was a misprint; this was not the case.

Pie and Plate Problem

E 1114 [1954, 259]. *Proposed by Viktors Linis, University of Saskatchewan*

What is the diameter of the smallest circular plate on which a semicircular pie can be placed if the pie is allowed to be cut in sectors of the same radius as the pie?

Solution by W. B. Carver, Cornell University. Let the radius of the semicircular pie be r, its area $\pi r^2/2$. Cut the pie into a very large number of equal sectors. By laying a number of sectors adjacent to one another, with radial sides coincident but oppositely directed, and by removing half of one end-sector to the other end of the array, an approximate rectangle can be formed. Let us fit the sectors of the pie into three such rectangles, one rectangle r by $r+2x$ and two others r by x. These rectangles can now be fitted together to form an approximate Greek cross (gotten by cutting squares of side x from each corner of a larger square of side $r+2x$), which can be placed on a circular plate whose diameter is equal to a diagonal of the big rectangle. This diameter is easily calculated to be

$$d = (r/4)\sqrt{\pi^2 + 4\pi + 20} = 0.6286r, \text{ approximately.}$$

It is clear that by using a sufficiently large number of sectors we can, in this fashion, place the pie on a plate of diameter

$$(r/4)\sqrt{\pi^2 + 4\pi + 20} + \epsilon,$$

where ϵ is an arbitrarily small positive number.

It is of course possible that some entirely different method of cutting and arranging the pieces of the pie would require a plate of diameter less than the limiting diameter of this method.

Also solved by R. K. Guy, A. R. Hyde, C. S. Ogilvy, and the proposer.

The proposer expressed interest in the best solution for a given (small) number n of pieces (equal or not?). He found results for a few cases:

$$
\begin{array}{ll}
n = 1, 2, 3 & d = 2r \\
n = 4 & d = \sqrt{3}r = 1.7320r \\
n = 5 & d = 1.7016r \text{ (the best?)} \\
n = 6 & d = 1.7390r \\
n = 7 & d = 1.8080r \\
n = 8 & d = 1.7320r \text{ (as for } n = 4).
\end{array}
$$

Carver, by arranging 90 equal sectors into a Greek cross, as in his solution above, found for this case $d=1.6336r$.

A Poristic Property

E1115 [1954, 259]. *Proposed by R. M. Gordon, China Lake, California*

(1) Let $Q_1Q_2Q_3Q_4$ be a (not necessarily convex) plane quadrilateral. On its sides construct similar isosceles triangles $Q_vP_vQ_{v+1}$ (with $Q_5 \equiv Q_1$), having arbitrary base angles θ. The angles $Q_{v+1}Q_vP_v(=\theta)$ are oriented alternately clockwise and counterclockwise from the adjacent sides, $Q_{v+1}Q_v$, of the quadrilateral. Show that P_1, P_2, P_3, P_4, the vertices of the isosceles triangles, are the vertices of a parallelogram.

(2) Let $P_1P_2P_3P_4$ be a plane quadrilateral. On its vertices construct similar isosceles triangles $Q_vP_vQ_{v+1}$ (with $Q_5 \equiv Q_1$) having vertex angles $Q_vP_vQ_{v+1}$ and base angles θ. The angles $Q_{v+1}Q_vP_v$ $(=\theta)$ are oriented alternately clockwise and counterclockwise in adjacent triangles. Show that, for arbitrary θ, the bases of the isosceles triangles are the sides of infinitely many quadrilaterals $Q_1Q_2Q_3Q_4$ provided that $P_1P_2P_3P_4$ is a parallelogram, and that if $P_1P_2P_3P_4$ is not a parallelogram then there exists a unique quadrilateral $Q_1Q_2Q_3Q_4$.

Solution by W. B. Carver, Cornell University. The final statement in paragraph (2) of the problem should read, "if $P_1P_2P_3P_4$ is not a parallelogram then there exists *no* quadrilateral $Q_1Q_2Q_3Q_4$."

(1) Let the points Q_i and P_i of the problem correspond respectively to the complex numbers q_i and p_i in the usual representation of complex numbers in the plane; and let $t=e^{2i\theta}$, where θ is the base angle of the problem. Then since $Q_1P_1Q_2$ is an isosceles triangle with base angles θ, we must have

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known text books or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4613. *Proposed by Joseph Lehner and G. Milton Wing, Los Alamos Scientific Laboratory*

Let $\phi(t)$ be integrable over (0, 1) and suppose, for $z=re^{i\theta}$, that

$$F(z) = \int_0^1 e^{\alpha z t}\phi(t)dt, \qquad R\alpha > 0$$

satisfies

$$\int_0^\infty |F(r)|^2 dr < \infty.$$

Show that $\phi=0$ almost everywhere.

4614. *Proposed by H. F. Sandham, Institute for Advanced Studies, Ireland*

Prove that

$$\int_0^1 (1-x)(1-x^2)(1-x^3)\cdots dx = \pi\sqrt{\frac{48}{23}}\,\frac{\sinh\frac{1}{3}\pi\sqrt{23}}{\cosh\frac{1}{2}\pi\sqrt{23}}\cdot$$

4615. *Proposed by D. J. Newman, Republic Aviation Corporation, Farmingdale, N. Y.*

Define

$$F(x) = \cos x \cos\frac{x}{2}\cos\frac{x}{3}\cos\frac{x}{4}\cdots.$$

Prove that $F(x)\to 0$ and $x\to+\infty$. Prove in fact that $F(x)$ falls off exponentially.

4616. *Proposed by A. C. Cohen, Jr., University of Georgia*

Show that the compound normal distribution function

$$f(x) = \frac{1}{2\sigma\sqrt{2\pi}}\left[\exp\left\{-\tfrac{1}{2}\left(\frac{x-m_1}{\sigma}\right)^2\right\} + \exp\left\{-\tfrac{1}{2}\left(\frac{x-m_2}{\sigma}\right)^2\right\}\right]$$

is bimodal if $|m_1-m_2|>2\sigma$ and unimodal otherwise.

4617. *Proposed by Albert Wilansky, Lehigh University*

Let $\{u_n\}$, $\{v_n\}$ be real sequences, $n=2, 3, 4, \cdots$, such that

$$\sum (u_n^2 + v_n^2) < 1.$$

Let f, g be real continuous functions of two real variables x, y, $x^2+y\leqq 1$; such that

$$f(\cos\theta, \sin\theta) = \cos\theta - \sum (u_n \cos n\theta + v_n \sin n\theta)$$

$$g(\cos\theta, \sin\theta) = \sin\theta - \sum (v_n \cos n\theta - u_n \sin n\theta).$$

Show that f, g vanish simultaneously at some point (x, y) for which $x^2+y^2<1$.

SOLUTIONS

Derangement of a Familiar Series

4552 [1953, 482]. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn, New York*

What derangement of terms of

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

will produce a sum which is rational?

Solution by V. C. Harris, San Diego State College, California. If neither the order in which the positive terms occur nor the order in which the negative terms occur is changed, but if the terms are rearranged so that k is the limit of the ratio of the number of positive terms to the number of negative terms in the first n terms, then the alteration L in the sum of the series is $L=\frac{1}{2}\log k$. (This result is due to Pringsheim; see Bromwich, *An Introduction to the Theory of Infinite Series*, Second Edition, Revised, p. 76.) Since the sum of the given series is $\log 2$, the sum after alteration is $\log 2+\frac{1}{2}\log k$. Setting $k=K^2/4$, the sum becomes $\log K$. If this is rational, say m/n, then $k=K^2/4=(1/4)\exp(2m/n)$. To effect the derangement, set up any sequence of rationals which has limit $(1/4)\exp(2m/n)$, *e.g.* convergents of its continued fraction expansion, and take a number of positive terms equal to the numerator followed by a number of negative terms equal to the denominator, of the successive terms of the sequence. In particular, for $k=1/4$, we have

$$0 = 1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \frac{1}{8} + \frac{1}{3} - \frac{1}{10} - \frac{1}{12} - \frac{1}{14} - \frac{1}{16} + \cdots.$$

Also solved by L. C. Graue, Erich Michalup, C. D. Olds, S. Parameswaran, L. A. Ringenberg, and the Proposer.

Inequality

4553 [1953, 554]. *Proposed by D. J. Newman, Republic Aviation Corporation, Farmingdale, N. Y.*

Consider any sequence $\{a_n\}$ of real numbers. Prove

$$\sum_{n=1}^{\infty} a_n \leqq \sqrt{2}\sum_{n=1}^{\infty} \sqrt{\frac{a_n^2 + a_{n+1}^2 + \cdots}{n}}.$$

Solution by Chih-yi Wang, University of Minnesota. In Hardy, Littlewood and Pólya, *Inequalities* (2nd ed.), p. 255, theorem 345 states that if $0<p<1$, then

$$\sum_{n=1}^{\infty}\left(\frac{\alpha_n + \alpha_{n+1} + \cdots}{n}\right)^p > p^p \sum_{n=1}^{\infty} \alpha_n^p.$$

The given problem is a special case with $\alpha_n = a_n^2$ for all n and $p=\frac{1}{2}$. Equality holds if $a_n = 0$ for all n.

Also solved by O. E. Stanaitis and the Proposer.

Solution of a Differential Equation

4554 [1953, 554]. *Proposed by M. R. Spiegel, Rensselaer Polytechnic Institute, Troy, New York.*

Find the general solution of the differential equation

$$\left(\frac{dy}{dx} + Py\right)^P = R\left(\frac{dy}{dx} + Qy\right)^Q$$

where P, Q and R are constants.

Solution by Omar Ali Siddiqi, Aligarh University, India. Taking logarithms and differentiating and simplifying, we have, if $P \neq Q$,

$$y'\{y'' + (P+Q)y' + PQy\} = 0,$$

giving the general solution $y = Ae^{-Px} + Be^{-Qx}$. The constants A, B are not independent but, on substituting in the differential equation,

$$(P-Q)^P B^P = R(Q-P)^Q A^Q.$$

If $P = Q$, and for certain other special cases in which the above does not apply, the solution is immediate.

With little additional difficulty the equation

$$(y' + P_1 y)^P = R(y' + Q_1 y)^Q$$

can be solved, where no condition on P, Q, P_1, Q_1 is specified. Logarithmic differentiation gives

$$a_1 y'' y' + a_2 y'' y + a_3 y'^2 + a_4 y' y = 0,$$

the a_i being constants expressible in terms of P, Q, P_1, Q_1. If now the dependent variable is changed to $v = y'/y$, the equation becomes

$$(a_1v + a_2)(v' + v^2) + a_3v^2 + a_4v = 0,$$

so that

$$x + A = -\int \frac{(a_1v + a_2)dv}{a_1v^3 + (a_2 + a_3)v^2 + a_4v}.$$

By performing the integration in finite terms or in series and determining the second arbitrary constant, the solution can be completed.

Also solved by W. J. Blundon, L. F. Boron, Emil Grosswald, Karl Guderley and Henry Fettis, R. T. Herbst, M. S. Klamkin, M. Morduchow, Celestine O'Callaghan, M. J. Pascual, L. B. Robinson, R. I. Scibor-Marchocki, O. E. Stanaitis, B. Stanović, F. Underwood, Chih-yi Wang, and the Proposer.

A Ballot Problem Related to Lattice Paths

4556 [1953, 555]. *Proposed by H. D. Grossman, New York City*

Suppose an election results in km votes for A and kn for B. In how many orders may votes be cast so that A's vote is always at least m/n times B's? Prove the following formula

$$p_k = \sum \frac{F_1^{k_1}F_2^{k_2}\cdots}{k_1!k_2!\cdots}$$

in which $k_1 + 2k_2 + \cdots = k$ and $F_j = \binom{jm+jn}{jm}/j(m+n)$ and the summation is over all partitions of k.

Editorial Note. In a paper with the title, New formula for the number of minimal lattice paths, *The Journal of the Institute of Actuaries*, London, vol. 80, pp. 55–62, M. T. L. Bizley considers the problem in the form: Determine the number of lattice paths from (0, 0) to (km, kn) which do not rise above the line $my = nx$. (A lattice path is a path joining (0, 0) to (x, y) say, composed of horizontal and vertical steps of unit length, the total number of steps being $x + y$.) He provides a proof of Grossman's formula and in addition gives (with proof) the number of paths which do not meet $my = nx$ at all between the end points, and also the number which meet it at a specified number of points.

Summation

4557 [1953, 555]. *Proposed by D. H. Browne, Buffalo, N. Y.*

Let $f(n)$ be the number of ways the integer n can be decomposed into dissimilar factors [*e.g.*, $f(1)=1, f(45)=3$ because $1\cdot 45=3\cdot 15=5\cdot 9$ but not $3\cdot 3\cdot 5$]. Evaluate

$$\sum_{n=0}^{\infty} \frac{f(2n+1)}{(2n+1)^2}.$$

Solution by W. J. Blundon, Memorial University of Newfoundland. For every factorization $k_1k_2k_3 \cdots$ of $2n+1$ such that $k_1<k_2< \cdots$, there is associated the term a_n, where $a_n=(2n+1)^{-2}$. Thus the given expression is equal to

$$\begin{aligned} 1+\sum a_1+\sum a_1a_2+\sum a_1a_2a_3+\cdots &= \prod_{n=1}^{\infty}(1+a_n) = \prod_{n=1}^{\infty}[1+(2n+1)^{-2}] \\ &= \frac{1}{2}\prod_{n=0}^{\infty}[1+(2n+1)^{-2}] = \frac{1}{2}\cosh\frac{1}{2} \end{aligned}$$

from the well-known infinite product.

Also solved by W. E. Briggs, N. J. Fine, and the Proposer.

An Extension of Pascal's Theorem

4558 [1953, 631]. *Proposed by V. F. Ivanoff, San Francisco*

If an octagon $l_1l_2 \cdots l_8$ is inscribed in a conic, then the eight points of intersection of the sides l_i and l_j ($j\equiv i+3$, mod 8) lie on another conic.

I. *Solution by Harley Flanders, University of California, Berkeley.* We denote the vertices of the simple octagon by

$$P_{12}=l_1\cap l_2,\ P_{23}=l_2\cap l_3,\ \cdots,\ P_{81}=l_8\cap l_1,$$

and we set

$$Q_{14}=l_1\cap l_4,\ Q_{25}=l_2\cap l_5,\ \cdots,\ Q_{83}=l_8\cap l_3.$$

The problem is to prove that the Q's are coconical. Evidently it suffices to prove that enough sets of six of these points are on a conic.

Consider the simple hexagon $l_1l_2l_3l_4l_5l_6'$, where $l_6'=P_{56}\cup P_{81}$. By Pascal's theorem $Q'=l_6'\cap l_3$, Q_{14} and Q_{25} are collinear. Next, consider the simple hexagon $Q_{14}Q_{36}Q_{58}Q_{83}Q_{25}Q_{61}$. We have

$$\begin{aligned} (Q_{14}\cup Q_{25})\cap(Q_{36}\cup Q_{83}) &= (Q_{14}\cup Q_{25})\cap l_3 = Q', \\ (Q_{14}\cup Q_{61})\cap(Q_{83}\cup Q_{58}) &= l_1\cap l_8 = P_{81}, \\ (Q_{36}\cup Q_{61})\cap(Q_{25}\cup Q_{58}) &= l_6\cap l_5 = P_{56}, \end{aligned}$$

all points on the line l_6'. Hence, by Pascal's theorem again, the vertices of the hexagon are on a conic.

By symmetry, the points Q_{47}, Q_{61}, Q_{83}, Q_{36}, Q_{58}, Q_{14} are also coconical. Combined with the last result, this gives us seven of the Q's on a conic. One more such manipulation and we have all eight of the Q's on this conic.

II. *Solution by H. S. M. Coxeter, University of Toronto.* This is a nice exercise on the principle of coresiduation (H. F. Baker, *Principles of Geometry*, Vol. V, Cambridge, 1933, p. 67). The sixteen common points of the two degenerate quartic curves,

$$l_1l_3l_5l_7 = 0, \qquad l_2l_4l_6l_8 = 0,$$

are "associated" in the sense that every quartic through thirteen of them goes through the remaining three also. One such quartic is

$$ss' = 0,$$

where $s=0$ is the given conic and $s'=0$ is the conic determined by five of the eight intersections described. The remaining three, not lying on $s=0$, must lie on $s'=0$.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to E. P. Vance, Oberlin College, Oberlin, Ohio.

Theory of equations. By C. C. MacDuffee. New York, John Wiley and Sons, Inc., 1954. 7+120 pages. $3.75.

As the author mentions in his preface, this book is the outgrowth of a one semester course given to junior and senior students. In a little over a hundred pages the author very competently covers the standard topics in the theory of equations. A small amount of abstract algebra is incorporated in the development of the subject.

Chapter 1. Linear systems. The feature of this chapter is the treatment of linear systems without the use of determinants. Indeed determinants are not included in the book. Chapter 2. Rational solutions. After a short introduction to number-theory, the familiar theorems on rational roots of polynomials is developed. Chapter 3. Polynomials. The chapter opens with the abstract definition of (commutative) ring and field. It includes a section on partial fractions. Chapter 4. Real roots. This chapter includes the Budan-Fourier theorem, Descartes' rule of signs, the Sturm theorem, the rule of false position, Newton's method and Horner's method. Chapter 5. Complex roots. The "fundamental theorem of algebra" is proved for polynomials of degree $\leqq 5$. Chapter 6. Relations among the roots. This chapter contains a brief discussion of symmetric polynomials, including the fundamental theorem, Newton's identities and the definition of the discriminant, and the solution of the general cubic and quartic.

Chapter 7. Systems of higher degree. Beginning with Euclidean rings, the chapter includes the following section headings: Eisenstein's criterion, content and primitive part, greatest common divisor, unique factorization, systems of symmetric equations, a general method (for systems), the resultant, the discriminant. Presumably this last chapter will seem a bit difficult to the immature reader.

This is a pleasant and readable little book. The reviewer regrets that the author has not included more material even though there is ample material in the book for a one-semester course. Also a set of interesting miscellaneous problems would have helped round out the book.

L. CARLITZ
Duke University

Introduction to College Mathematics. By C. V. Newsom and Howard Eves. New York, Prentice-Hall, Inc. 1954. 8+408 pages. $5.75.

This book is a second edition of an earlier edition published in 1946 with Newsom as the sole author. It was written with a view to answering the needs of the ever increasing number of college students who are destined to take but one course in mathematics and for whom many, including this reviewer, believe the traditional freshman courses are not suitable.

The material covered is indicated by the following chapter headings: 1. Number and the Operations of Arithmetic. 2. The Arithmetic of Numbers in the Exponential Form. 3. The Arithmetic of Measurement. 4. Logarithms. 5. Progressions of Numbers. 6. Statistics. 7. Combinations and Probability. 8. Functional Relationships. 9. Variation. 10. The Circular Functions. 11. Trigonometry. 12. The Equation. 13. Some Common Curves. 14. A Glimpse of the Calculus.

One of the outstanding features of the book is the abundance of varied and interesting exercises which often are as enlightening as the text itself, as for example, the highly informative exercise on Fibonacci progressions on page 94.

Of especial interest, both as to novelty of treatment and pedagogical significance, is the introduction in Chapter 10 of circular functions as functions of a real variable (directed arc length) rather than as functions of angles. The derivations of the formulas for sine and cosine of sums and differences of arcs are based on an interesting method of analysis attributed to Cauchy. A brief descriptive discussion of harmonic analysis is also given in Chapter 10. Applications of the circular functions to angles are treated in Chapter 11.

An excellent footnote on equivalent equations occurs on page 301 in the section on the parabola. The reviewer is inclined to think that the value to the students of this and several other illuminating footnotes would be materially increased if they were made a part of the text.

The clear treatment of the concept of instantaneous speed and instantaneous rate of change on page 339 is noteworthy.

It appears pedagogically sounder, as well as more in keeping with the logical development, to explain the definition of the multiplication of two negative numbers on the basis of the distributive law, as suggested in example 5 on page 13, rather than on the basis of division by equals.

In the interest of clarity, it would seem advisable to state what geometrical property of a straight line is being taken as a definition in the analytical treatment of the straight line in Chapter 13.

It appears to the reviewer that at least a partial explanation should be given of the basis of the procedure for finding the line of best fit. Such an explanation, for example, as that given on pages 120–122 of Richardson's *Introduction to Statistical Analysis* would seem entirely within the scope of the book and would add to its usefulness.

Everything considered, the authors are to be congratulated on writing a book so admirably adapted to a terminal course in college mathematics.

A. H. DIAMOND
Stevens Institute of Technology

Elements of Statistics. By H. C. Fryer. New York, John Wiley and Sons, Inc., 1954. vii+262 pages.

The author states that the purpose of this book is to provide the college and university student with an introductory course in statistics and that college algebra is the only prerequisite.

After an introduction, methods of summarizing data are studied. A discussion of elementary probability then prepares the student for the study of the binomial and normal distributions and sampling from these distributions. Certain aspects of confidence intervals and statistical hypotheses are then considered. The book concludes with a chapter on linear regression and correlation.

Since the book presupposes only college algebra, it is not written as a mathematical text, and proofs have not been presented. There are numerous exercises, and the author gives some evidence for the truth of certain theorems. However, some of the earlier chapters might have included more precise definitions. For example, there should be a more clear cut distinction between population parameters and sample characteristics. Perhaps the footnote on page 13 was considered adequate for this purpose. Again on page 72, the definition of mathematical expectation might be interpreted by the student as a theorem. Further there is no distinction between a cumulative distribution function and its sample image.

In dealing with tests of simple hypotheses, pages 130 to 135, the author properly states that the hypothesis and its test should be determined before the sample is observed. However, he is not as careful in his discussion of composite hypotheses. He even suggests on pages 142 and 143 that an hypothesis can be formulated after the sample values have been observed, and that the hypothesis can then be tested with that same sample. This practice is used all too frequently now, and the elementary books should stress the fact that if ob-

served data are to be used to suggest an hypothesis, the test should be made, not with those data, but with new data independently obtained.

Finally there are a few minor points. The subscript on s on page 149 should be $\bar{x}$, not $\bar{\bar{x}}$. On page 158, Problem 6.21, "five random samples" should read "a sample of five random values." An equality sign on page 207 is actually a minus sign. Tables IX and X are probability distributions of $|G|$, not G as defined on page 182. The left braces of the signs of aggregation on page 125 should follow the plus signs and not precede them. This reviewer believes the problems following sections 4.3 and 4.4 were inadvertently interchanged.

R. V. HOGG
State University of Iowa

Universal Mathematics, Part I. By the 1954 Summer Writing Group. Lawrence, Kansas, Student Union Book Store, 1954. x+310 pages. $2.75.

This preliminary edition is a planographed work which includes an experimental approach to a universal Freshman course. The motivation for writing this material has come from the Committee on the Undergraduate Program of the Mathematical Association of America, but the book is not an official publication of this committee. This volume is subtitled "Functions and Limits" and has the following chapter headings: I Coordinate Systems, II Scientific Measurement, III Functions, IV Limits, V Derivatives, VI Integrals, VII The Logarithm and Exponential Functions, VIII Summary. In each chapter the material is presented twice, first from an intuitive approach and then in a formal fashion.

Although intended for experimental teaching in a few institutions, this is not a textbook in the usual sense. It is best described in the words of the authors as "A book of experimental text materials." In the interest of speedy notice concerning this book, this brief review is being published in this issue; a more complete review will appear later.

C. B. ALLENDOERFER
University of Washington

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

THE TECHNICAL PUBLISHING SOCIETY

The Technical Publishing Society, a professional group to encourage the interchange of information among individuals engaged in preparing, editing, or

publishing technical and scientific documents, has been organized in Los Angeles. Membership includes editors, writers, illustrators, management and production personnel from industrial and governmental research and development laboratories. A Board of Directors is studying standards which will be used to accredit individual and chapter memberships.

For information write to Technical Publishing Society, P. O. Box 607, Tarzana, California.

PERSONAL ITEMS

The American Mathematical Society announces the appointment of Dr. J. H. Curtiss as Executive Director. He was formerly Adjunct Professor of Mathematics at New York University and Senior Scientist at the Institute of Mathematical Sciences of New York University.

Boston University announces: Dr. R. H. Brown of Columbia University has been appointed to an assistant professorship; Dr. S. L. Ross of Northeastern University has accepted an instructorship.

Central Michigan College of Education reports the following: Associate Professor L. G. Woodby of Mankato Teachers College, Minnesota, has been appointed to an associate professorship; Mr. D. R. Sudborough has been appointed Business Manager of the *Pentagon*; Miss Nikoline A. Bye has been elected President of the Michigan Council of Teachers of Mathematics.

Cornell University announces the following: Professor J. B. Rosser is Acting Chairman of the Department of Mathematics during 1954–55 while Professor R. J. Walker is on sabbatical leave and is at the University of California at Los Angeles; Dr. L. A. Rubel, previously a teaching assistant at the University of Wisconsin, has been appointed to an instructorship.

Massachusetts Institute of Technology reports: Dr. G. E. Baxter of the University of Minnesota, Mr. W. C. Fox, formerly an instructor at the University of Michigan, and Dr. Sigurdur Helgason of Princeton University have been appointed to C. L. E. Moore instructorships; Associate Professor R. J. Levit of the University of Georgia is on leave of absence from the University and has been appointed Visiting Assistant Professor; Mr. Bayard Rankin of the University of California has accepted a position as a research associate; Professor I. S. Cohen is on leave for the year 1954–55 and is a visiting member of the faculty of Columbia University; Professor C. C. Lin is a Guggenheim fellow at Cornell University and the California Institute of Technology.

At Syracuse University: Dr. Cyrus Derman, formerly of Columbia University, has been appointed to an instructorship; Assistant Professors W. R. Baum, P. W. Gilbert, and Erik Hemmingsen have been promoted to associate professorships.

United States Naval Academy announces: Assistant Professors R. W. Rector and H. K. Sokl have been promoted to associate professorships; Associate Professors N. H. Ball and A. E. Currier have been promoted to professorships;

Professor Ernest Hawkins has been promoted to the position of Senior Professor.

The University of Delaware announces that Mr. R. M. Lauer and Mr. W. G. Spohn have been appointed to instructorships.

University of Georgia reports the following: Dr. A. C. Cohen, Jr. was granted the $500 Michael Research Award for his work on truncated frequency distributions; Assistant Professor T. R. Brahana of Dartmouth College has been appointed to an assistant professorship; Miss Lola F. Kiser, previously a teaching assistant at the University, has been appointed to an instructorship.

University of Minnesota reports: Dr. Steven Orey of Cornell University and Dr. Irwin Fischer of Harvard University have been appointed to instructorships.

University of Mississippi announces the following: Dr. N. A. Childress and Mr. R. D. Sheffield are returning from graduate study to the University's Department of Mathematics as Assistant Professors; Assistant Professor L. L. Scott has been promoted to an associate professorship; Professor W. H. Spragens, Jr., is on leave from the University for an internship in General Education at the University of Chicago.

At the University of Pennsylvania: Dr. Wilhelm Stoll of the University of Tubingen, Germany, has been appointed Visiting Lecturer for one year beginning September 1954; Dr. Emil Grosswald, formerly an associate, has been promoted to an assistant professorship.

University of Western Ontario reports the following: Assistant Professor G. P. Henderson received a research award from the National Research Council of Canada; Mr. Arwel Evans, formerly a research student at Cambridge University, England, has been appointed to an instructorship.

Assistant Professor R. W. Ball of the University of Washington has accepted a position as Assistant Professor at Alabama Polytechnic Institute.

Assistant Professor Max Beberman of the University High School, University of Illinois, has been appointed to an associate professorship at Florida State University.

Professor David Blackwell of Howard University is on leave for the academic year of 1954–55 and is Visiting Professor at the Statistical Laboratory, University of California.

Mr. C. L. Bradshaw, formerly a mathematical analyst at Lockheed Aircraft, Marietta, Georgia, is now a member of the mathematics panel at Oak Ridge National Laboratory, Oak Ridge, Tennessee.

Dr. C. C. Bramble of the Naval Proving Ground, Dahlgren, Virginia, has accepted a position on the technical staff of the Research and Development Division, Norden Laboratories, White Plains, New York.

Assistant Professor K. A. Bush of the University of Illinois has been appointed Head of the Mathematics Department of the University of Idaho.

Dr. Buchanan Cargal, formerly a senior aerophysics engineer at Consolidated Vultee Aircraft, Fort Worth, Texas, has accepted a position as a research engineer at Hughes Aircraft Company, Culver City, California.

Mr. R. E. Carroll, previously a dynamics engineer at Bell Aircraft Company, Niagara Falls, New York, is now a leader of the I.B.M. Computation Unit of the Company.

Dr. H. E. Chrestenson of Purdue University has been appointed to an assistant professorship at Whitman College.

Mr. R. C. Clelland, formerly an instructor at Hamilton College, has received an appointment as a research assistant at the University of Pennsylvania.

Mr. R. J. Cormier, recently in military service, is now a graduate student at University of Tennessee.

Mr. H. F. Davis, II, recently a part-time instructor at Massachusetts Institute of Technology, has been appointed Assistant Professor at Miami University.

Mr. W. R. Ferrante of Lafayette College has been promoted to an assistant professorship.

Dr. L. R. Ford, Jr., of Duke University has accepted a position with the Rand Corporation, Santa Monica, California.

Dr. G. E. Forsythe, formerly a mathematician with the National Bureau of Standards, Los Angeles, California, has a position as a research mathematician at the University of California at Los Angeles.

Mr. D. E. Freeland, recently a research assistant at the Statistical Laboratory of Purdue University, has accepted a position as a teaching assistant at Illinois Institute of Technology.

Dr. N. A. Goldsmith, previously head of the Mathematics Department of Henderson State Teachers College, Arkansas, has been appointed to a professorship at Chicago Teachers College.

Mr. Fred Gruenberger, formerly a computing service project supervisor at the University of Wisconsin, has a position as an operations analyst at the General Electric Company, Richland, Washington.

Mr. L. W. Gunter, previously a student at the University of Wisconsin, has been appointed to an instructorship at Western Michigan College.

Assistant Professor F. E. Hohn of the University of Illinois has been promoted to an associate professorship.

Dr. J. W. Hollingsworth, formerly an assistant in computing at the University of Wisconsin, has accepted a position as an engineer at the General Electric Company, Schenectady, New York.

Dr. C. W. Huff, teaching assistant at the University of Georgia, has accepted a position as Assistant Professor at Alabama Polytechnic Institute.

Mr. D. G. Johnson, recently a graduate assistant at Purdue University, is now a private in the U. S. Army.

Dr. Hyman Kamel of Cornell University has accepted an assistant professorship at Rensselaer Polytechnic Institute.

Mr. P. G. Kirmser of the University of Minnesota has been appointed to an associate professorship in the Department of Applied Mechanics, Kansas State College.

Assistant Professor S. R. Knox of Millsaps College has been promoted to an associate professorship.

Mr. E. L. Kretschmar, Jr., has a position as a mathematics teacher at Pasco High School, Dade City, Florida.

Mr. M. H. Lane, formerly a mathematical statistician for the Air Proving Ground, Eglin Air Force Base, Florida, is a graduate student at Alabama Polytechnic Institute.

Mr. H. S. Leonard, Jr., formerly a graduate student at Harvard University, has a position as a teaching fellow at the University.

Dr. Viktors Linis of the University of Saskatchewan has been appointed to an assistant professorship at the University of Ottawa.

Mr. J. P. Lipp, recently a student at the University of Oklahoma, is employed as an electrical engineer in the Advanced Electronic Center, General Electric Company, Ithaca, New York.

Mr. K. L. Loewen, previously an administrative assistant for Mennonite Aid Section, Mennonite Central Committee, Akron, Pennsylvania, has been appointed to an instructorship at Freeman Junior College.

Mr. Angelo Margaris, formerly a teaching assistant at Cornell University, has accepted an instructorship at Oberlin College.

Dr. D. C. B. Marsh, who has been a mathematics assistant at the University of Colorado, has been appointed to an assistant professorship at Texas Technological College.

Mr. H. H. Martens, previously a tester for Consolidated Edison Company of New York, New York City, is employed as a technical assistant with the Bell Telephone Laboratories, New York City.

Associate Professor W. S. Massey of Brown University is Visiting Associate Professor at Princeton University.

Associate Professor H. F. Mathis of the University of Oklahoma is now associated with Goodyear Aircraft Corporation, Akron, Ohio.

Mr. F. H. McGar, Jr., previously an instructor at Sweet Briar College, is now a demonstrator in the Department of Physics, Bryn Mawr College.

Professor W. G. McGavock is on leave of absence from Davidson College and is at the University of Wisconsin on a Ford Fellowship.

Mr. W. H. Mead, Jr., recently a student at the University of Southern California, is now a graduate student at the University of California.

Associate Professor E. J. Miles of Yale University has retired.

Dr. C. T. Molloy of the Vitro Corporation of America has accepted a position as staff engineer with the Lockheed Aircraft Corporation, Burbank, California.

Professor E. P. Northrop of the University of Chicago has resigned as Associate Dean of the College but retains his professorship. He will be on assignment from January 1, 1954 to July 1, 1955 by the University to the National

Science Foundation as consultant in the area of education in the sciences, and concurrently on assignment as consultant part-time to the Fund for the Advancement of Education.

Mr. F. P. Palermo, formerly a graduate assistant at Brown University, has been appointed to an instructorship at Princeton University.

Mr. D. S. Park, previously a mathematician at David Taylor Model Basin, Washington, D. C., is a mathematician with the Ordnance Corps, Letterkenny Ordnance Depot, Chambersburg, Pennsylvania.

Miss Mary Pettus of Sue Bennett College has accepted an assistant professorship at Union College, Barbourville, Kentucky.

Mr. R. S. Pieters, recently an instructor at Phillips Academy, has a position as Visiting Lecturer at Princeton University.

Mr. R. E. Priest, previously an assistant at the University of Illinois, is employed by the Bell Telephone Laboratories, New York City.

Mr. J. D. Reid, recently a mathematician for I.B.M. Corporation, Endicott, New York, is now at the U. S. Naval Training Center, Great Lakes, Illinois.

Mr. C. B. Rogers, formerly with the U. S. Navy, has a position as a hydrologic field assistant for the U. S. Geological Survey, University of New Mexico.

Mr. R. S. Roth, former graduate student at Carnegie Institute of Technology, is at Aberdeen Proving Ground, Maryland, as a mathematician.

Assistant Professor S. G. Roth, assistant to the dean of New York University's Division of General Education, has been named Assistant Director of the Office of the Budget of New York University.

Mr. Arthur Saastad, previously an industrial engineer for the U. S. Steel Company, Chicago, Illinois, has accepted a position as a research engineer at Convair Aircraft, San Diego, California.

Mr. C. T. Salkind, previously a teacher at S. J. Tilden High School, Brooklyn, New York, has been appointed to an assistant professorship at the Polytechnic Institute of Brooklyn.

Miss Ellen A. Sanders, formerly a teacher at Taylor County High School, Campbellsville, Kentucky, has been appointed Professor of Science at Montreat College.

Assistant Professor Clarence Schilling of the University of Tampa has accepted a professorship at Canisius College.

Mr. W. P. Shulevitz, recently a student at Wayne University, has joined the U. S. Army and is stationed at Fort Monmouth, New Jersey.

Rabbi Harold Shulman of Queens College has accepted a position as Mathematician with the Institute of Mathematical Sciences, New York University.

Mr. M. R. Simonson, formerly with the General Electric Company at Schenectady, New York, is now a technical engineer for General Electric Company, Evendale, Ohio.

Associate Professor M. B. Sledd of Georgia Institute of Technology has returned after spending a year's leave of absence at Massachusetts Institute of Technology.

Mr. E. C. Smith, Jr., of the University of Oregon has been appointed to a position as Assistant Professor at the University of Utah.

Professor Ernst Snapper of the University of Southern California is Visiting Professor at Princeton University.

Mr. A. D. Sollins, formerly a mathematician with the U. S. Coast and Geodetic Survey, Washington, D. C., has a position as Visiting Professor of Civil Engineering at the College of Agriculture and Mechanical Arts, University of Puerto Rico.

Dr. J. J. Sopka, previously a mathematician at Johns Hopkins University, is employed as Applied Science Representative, I.B.M. Corporation, New York City.

Assistant Professor O. E. Stanaitis of St. Olaf College has been promoted to an associate professorship.

Dr. C. J. Standish of Cornell University has been appointed Assistant Professor of Mathematics at Union College.

Mr. W. C. Styslinger, Jr., recently a student at Duquesne University, has accepted a position as a junior salesman with the I.B.M. Corporation, Pittsburgh, Pennsylvania.

Associate Professor S. L. Thorndike, head of the Mathematics Department of the College of Emporia, has been appointed Professor and Head of the Mathematics Department of Alma College.

Associate Professor W. J. Thron of Washington University has been appointed to an associate professorship at the University of Colorado.

Assistant Professor C. J. Tremblay of Bard College has been promoted to an associate professorship.

Mr. G. P. Weeg, a graduate assistant at Iowa State College, is employed as a mathematician by Engineering Research Associates, St. Paul, Minnesota.

Dr. C. G. Werner has been appointed to an instructorship at the University of Maine.

Assistant Professor R. H. Wilson, Jr., of the University of Louisville has accepted a position at the Naval Research Laboratory, Washington, D. C.

Mr. A. W. Yonda, formerly a graduate assistant at the University of Alabama, is at Ballistics Research Laboratory, Aberdeen Proving Ground, Maryland, as an assistant programmer.

Mr. B. K. Youse, recently an instructor at the University of Georgia, has been appointed to an assistant professorship at Emory University.

President Emeritus R. H. Reece of New Mexico Institute of Mining and Technology died on August 4, 1954. He was a charter member of the Association.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

ITINERARIES OF VISITING LECTURERS, 1954–55

The following itineraries have been arranged for the visiting lecturers under the program sponsored by the National Science Foundation and the Mathematical Association of America. The dates given are subject to slight adjustments. In case of regional arrangements, no attempt has been made to list all participating institutions or subdivide the time allocated.

R. H. Bing

Oberlin College	Oberlin, Ohio	Oct. 4–9
Univ. of Conn. and	Storrs, Conn.	Oct. 11–16
Trinity College	Hartford, Conn.	
Ohio Univ. and	Athens, Ohio	Nov. 1–3
Denison U.	Granville, Ohio	
So. Illinois U.	Carbondale, Ill.	Nov. 4–6
U. of Kansas and others	Lawrence, Kan.	Nov. 8–13
Vanderbilt U., Fisk U. & Geo. Peabody Tchs. Coll.	Nashville, Tenn.	Nov. 29–Dec. 3
The Florida State U.	Tallahassee, Fla.	Dec. 6–10
U. of Wisconsin (Milwaukee Ext.)	Milwaukee, Wis.	Jan. 3, 4
Carroll College	Waukesha, Wis.	Jan. 5, 6
Emmanuel Missionary College	Berrien Springs, Mich.	Jan. 10–12
State U. of Iowa	Iowa City, Ia.	Apr. 18, 19
U. of N. Dakota	Grand Forks, N. D.	Apr. 21–23

W. L. Duren, Jr.

Vassar College	Poughkeepsie, N. Y.	Oct. 19–21
U. of Buffalo	Buffalo, N. Y.	Oct. 25–30
U. of Alabama	University, Ala.	Nov. 1
Randolph-Macon, Sweet Briar and Lynchburg Coll.	Lynchburg, Va.	Nov. 9–12
Knox College	Galesburg, Ill.	Nov. 15–20
Southwestern at Memphis	Memphis, Tenn.	Jan. 11–15

T. Fort

Queens College	Charlotte, N. Car.	Feb. 1–2
North Carolina College	Durham, N. Car.	Feb. 4–5
Atlanta University and others	Atlanta, Ga.	Feb. 8, 9
College of William and Mary	Williamsburg, Va.	Feb. 10, 11
U. of Maryland	College Park, Md.	Feb. 14, 15, 16
State Teachers College	Shippensburg, Pa.	Feb. 18
Long Island U.	Brooklyn, N. Y.	Feb. 23, 24, 25
Mary Washington College	Fredericksburg, Va.	Feb. 28

G. Pólya

University of Redlands	Redlands, Calif.	Jan. 7–13
Los Angeles City College	Los Angeles, Calif.	Jan. 14
Arizona State College	Tempe, Ariz.	Jan. 27–Feb. 4
New Mexico A. and M.A.	State College, N. M.	Feb. 7–9
Utah State Ag. Coll.	Logan, Utah	Feb. 28–Mar. 2
U. of Colorado and environs	Boulder, Col.	Mar. 3–11
U. of Nebraska	Lincoln, Neb.	Mar. 14–18
St. Louis U.	St. Louis, Mo.	Mar. 21–23
U. of Oklahoma	Norman, Okla.	Mar. 24–30
Central State College	Edmond, Okla.	
Montana State College	Bozeman, Mont.	Apr. 10–17
Montana State U.	Missoula, Mont.	Apr. 17–20
U. of Idaho and	Moscow, Idaho	Apr. 21–30
Washington State College	Pullman, Wash.	
U. of Washington and environs	Seattle, Wash.	May 1–7
Williamette U.	Salem, Ore.	May 8–11
Oregon State College	Corvallis, Ore.	May 12–17
U. of Oregon	Eugene, Ore.	May 18–20

D. V. Widder

Syracuse U.	Syracuse, N. Y.	Mar. 28–31
U. of Kentucky	Lexington, Ky.	Apr. 1–7
U. of Georgia	Athens, Ga.	
Agnes Scott Coll. and	Decatur, Ga.	Apr. 8–14
Georgia Inst. of Tech.	Atlanta, Ga.	
Bowling Green U.	Bowling Green, Ohio	Apr. 21–23
Dartmouth Coll.	Hanover, N. H.	Apr. 25–30

THE FIFTEENTH ANNUAL WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

The fifteenth annual William Lowell Putnam Mathematical Competition will be held on Saturday, March 5, 1955. This competition, made possible by the trustees of the William Lowell Putnam Intercollegiate Memorial Fund left by Mrs. Putnam in memory of her husband, is under the sponsorship of the Mathematical Association of America and is open to regularly enrolled undergraduate students in universities and colleges of the United States and Canada who have not received a college degree. The examination will consist of two parts of three hours each. The questions will be taken from the fields of calculus (elementary and advanced) with applications to geometry and mechanics not involving techniques beyond the usual applications, higher algebra (determinants and theory of equations), elementary differential equations, and geometry (advanced plane and solid analytic geometry). Any college or university wishing to enter a team or individual contestants may secure an application blank from Professor L. E. Bush, Box 30, Kent State University, Kent, Ohio, by a postcard request. Application blanks will be mailed out early in January. All applications must be filed with the Director not later than February 10, 1955. If three candidates are presented from a college or university, they are to constitute a team. If more than

three candidates are presented from any one college or university, the team of three must be named on the application. Fewer than three candidates from one college or university may compete as individual contestants.

The examination may be given at any place where a team, or at least three candidates, can be assembled. Exceptions to this rule may be made by the Director in cases where it would entail unusual inconvenience to the contestant. Sealed copies of the examinations will be sent to the supervisor of the examination in time for the examination day and are not to be opened before the hour set.

The prizes to be awarded to the departments of mathematics of the institutions with the winning teams are $400, $300, $200 and $100, in the order of their rank. In addition, there will be prizes of $40, $30, $20 and $10 awarded to the members of these teams according to the rank of the team; a prize of $50 to each of the five highest contestants and a prize of $20 to each of the succeeding five highest contestants. Each of the winners will receive a suitable medal. Honorable mention will be given to several teams next in order after the four winning teams and to several individuals next in order after the ten individual winners. For further encouragement of the Competition, there will be awarded at Harvard University (or at Radcliffe College in the case of a woman) an annual $2000 William Lowell Putnam Prize Scholarship to one of the first five contestants, this to be available either immediately or on completion of the student's undergraduate work.

Reports of the fourteen previous competitions and examinations will be found in this MONTHLY for May 1938, 1939, 1940, 1941, 1942, October 1946, August–September 1947, December 1948, August–September 1949, 1950, 1951, October 1952, 1953 and 1954.

ACKNOWLEDGMENT

The Editors wish to acknowledge the services of the following persons, not members of the editorial staff, who have assisted the Editors by refereeing manuscripts during the past year.

General Articles: R. P. Agnew, J. P. Ballantine, Lipman Bers, Z. W. Birnbaum, David Blackwell, R. P. Boas, Jr., L. Carlitz, S. S. Chern, N. A. Court, Paul Cross, G. B. Dantzig, D. B. Dekker, W. Feller, Harley Flanders, Lester Ford, G. E. Forsythe, J. S. Frame, Michael Goldberg, Marshall Hall, S. T. Hu, Burrowes Hunt, A. S. Householder, P. S. Jones, J. L. Kelley, V. L. Klee, N. H. Kuiper, R. B. Leipnik, A. E. Livingston, G. W. Mackey, N. H. McCoy, L. H. McFarlan, W. E. Milne, L. J. Mordell, C. V. Newsom, S. Perlis, G. B. Price, W. V. Quine, G. de B. Robinson, Hans Samelson, M. R. Spiegel, R. D. Stalley, B. M. Stewart, R. R. Stoll, R. F. Tate, T. Y. Thomas, C. A. Truesdell, R. M. Winger, H. S. Zuckerman.

Mathematical Notes: G. E. Albert, R. Bellman, L. Bers, R. P. Boas, A. Brauer, L. E. Bush, L. Carlitz, S. Chowla, C. J. Coe, N. A. Court, R. R. Coveyou, J. M. Dobbie, F. G. Dressel, W. Feller, N. J. Fine, L. R. Ford, J. S.

Frame, B. Friedman, I. E. Garrick, H. Geiringer, W. Givens, A. W. Goodman, Harold Grad, H. C. Griffith, Marshall Hall, O. G. Harrold, Fritz Herzog, A. S. Householder, J. R. Isbell, S. L. Jamison, Mark Kac, L. M. Kelly, P. W. Ketchum, H. L. Lee, J. H. Lehner, Lee Lorch, C. C. MacDuffee, M. H. Martin, C. Masaitis, D. D. Miller, R. Moller, F. D. Murnaghan, L. Nirenberg, O. Ore, W. V. Parker, Harry Pollard, R. F. Rinehart, H. Robbins, H. N. Shapiro, V. L. Shapiro, S. Sherman, M. F. Smiley, W. S. Snyder, C. E. Springer, R. R. Stoll, G. Szegö, H. P. Thielman, A. W. Tucker, H. S. Vandiver, T. L. Wade, R. J. Walker, Morgan Ward, D. V. Widder, A. Wilansky.

Classroom Notes: H. W. Brinkmann, I. W. Burr, P. A. Clement, S. H. Crandall, R. B. Davis, Philip Franklin, H. K. Fulmer, Harriet Griffin, E. A. Hedberg, F. B. Hildebrand, R. C. James, R. E. Langer, Warren Loud, C. I. Lubin, F. H. Miller, R. K. Morley, J. C. Polley, Robert Solow, O. E. Stanaitis, F. M. Stewart, D. J. Struik, R. M. Thrall, J. R. Vatnsdahl, R. W. Walker, R. C. Yates.

CALENDAR OF FUTURE MEETINGS

Thirty-eighth Annual Meeting, University of Pittsburgh, Pittsburgh, Pennsylvania, December 30, 1954.

Thirty-sixth Summer Meeting, University of Michigan, Ann Arbor, Michigan, August 29–30, 1955.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Pittsburgh, Pennsylvania, April 30, 1955.

ILLINOIS, Monmouth College, Monmouth, May 13–14, 1955.

INDIANA, Butler University, Indianapolis, May, 1955.

IOWA, St. Ambrose College, Davenport, April 15–16, 1955.

KANSAS, Fort Hays Kansas State College, Hays, March 26, 1955.

KENTUCKY

LOUISIANA-MISSISSIPPI, Buena Vista Hotel, Biloxi, Mississippi, February 18–19, 1955.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA

METROPOLITAN NEW YORK, Queens College, Flushing, New York, April 30, 1955.

MICHIGAN, Michigan State College, East Lansing, March 26, 1955.

MINNESOTA, College of St. Teresa, Winona, Minnesota, May, 1955.

MISSOURI, University of Kansas City, Spring, 1955.

NEBRASKA, University of Nebraska, Lincoln, April 23, 1955.

NORTHERN CALIFORNIA, University of California, Berkeley, January 15, 1955.

OHIO

OKLAHOMA

PACIFIC NORTHWEST, University of British Columbia, Vancouver, June 17, 1955.

PHILADELPHIA

ROCKY MOUNTAIN, University of Wyoming, Laramie, Spring, 1955.

SOUTHEASTERN, Tennessee Polytechnic Institute, Cookeville, March 11–12, 1955.

SOUTHERN CALIFORNIA, Santa Monica City College, March 12, 1955.

SOUTHWESTERN, University of New Mexico, Albuquerque, Spring, 1955.

TEXAS, Abilene Christian College, Abilene, April, 1955.

UPPER NEW YORK STATE, University of Buffalo, May 14, 1955.

WISCONSIN, Cardinal Stritch College, Milwaukee, May, 1955.

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN